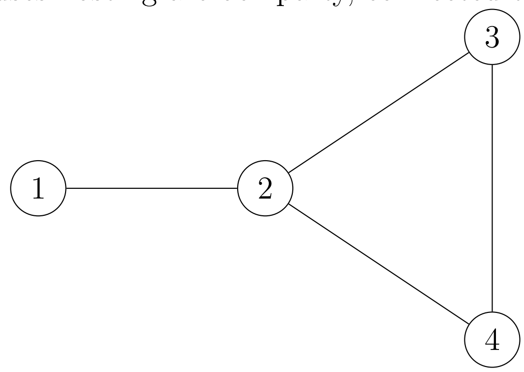
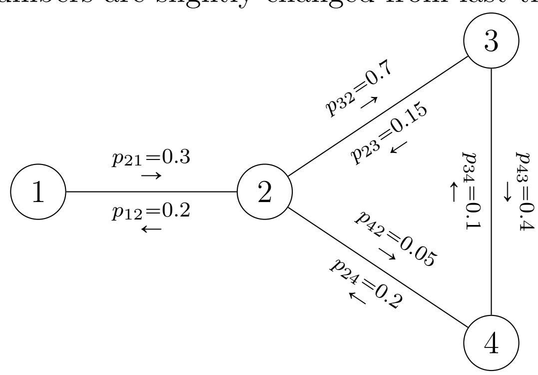
7.4 worksheet

Suppose there are four houses hosting a block party, connected to each other like so:



Every hour, some percent of people at one house leave and go to another house, as described by the following diagram (the numbers are slightly changed from last time)



So for instance, the value of $p_{21} = 0.3$ means that house 2 will get 30% of the people in house 1 every hour. This process is an example of a *Markov process*, also called a *Markov chain*.

1. What proportion of people at house 1 will stay at house 1 after an hour? We call this number p_{11}.

2. In general we let p_{ii} be the proportion of people in house i who decide to stay in house i when the hour changes. Find p_{22}, p_{33}, and p_{44} as well.

3. Let $x_i(t)$ be the number of people in house i, t hours after midnight. Let $\vec{x}(t)$ denote the vector $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$, and let $\vec{x}(0) = \begin{bmatrix} 100 \\ 300 \\ 500 \\ 200 \end{bmatrix}$. ($\vec{x}(t)$ is an example of a *vector-valued function*: its input is a number t and its output is a vector). Find $x_2(1)$.

4. Find a matrix A such that $\vec{x}(t+1) = A\vec{x}(t)$. This is called the *transition matrix* of the markov process.

5. Suppose there are x_1 people at house 1, x_2 people in house 2, etc. This configuration of people $\vec{x}_{\text{eq}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is called an *equillibrium state* if $A\vec{x}_{\text{eq}} = \vec{x}_{\text{eq}}$. Is there an equilibrium state to block party described above? Give an example of one. It may be helpful to know that the eigenvectors/eigenvalues of A are:

$$\vec{v}_1 = \begin{bmatrix} 0.184 \\ 0.276 \\ 0.716 \\ 1 \end{bmatrix}, \lambda_1 = 1 \qquad \vec{v}_2 = \begin{bmatrix} -0.925 \\ -0.222 \\ 0.148 \\ 1 \end{bmatrix}, \lambda_2 = 0.748$$

$$\vec{v}_3 = \begin{bmatrix} 0.053 \\ -0.102 \\ -0.951 \\ 1 \end{bmatrix}, \lambda_3 = 0.314 \qquad \vec{v}_4 = \begin{bmatrix} -0.416 \\ 1.798 \\ -2.381 \\ 1 \end{bmatrix}, \lambda_4 = -0.163$$

(so A is diagonalizable!)

6. Show that $\vec{x}(t)$ gets closer and closer to an equilibrium state $\vec{x}_{\text{eq}}$ as t gets bigger and bigger (hint: this is a lot like the foxes and hares problem from the homework! We know $x(t) = A^t x(0)$. Write $\vec{x}(0)$ as a linear combination of eigenvectors)

7. **Fact:** This always happens! Every[1] Markov chain has an *equilibrium state* $\vec{x}_{\text{eq}}$. Given any initial configuarion $\vec{x}(0)$, the vector $\vec{x}(t)$ will always get closer and closer to (a positive mutliple of) $\vec{x}_{\text{eq}}$.

   - The reason is the same as before:

   Let's solve the following problems using some similar ideas:

   For the matrices** A **and the vectors** $\vec{x}_0$ **in Exercises 13 through 19, find closed formulas for** $A^t\vec{x}_0$**, where** t **is an arbitrary positive integer. Follow the strategy outlined in Theorem 7.1.6 and illustrated in Example 1. In Exercises 16 through 19, feel free to use technology.

   13. $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

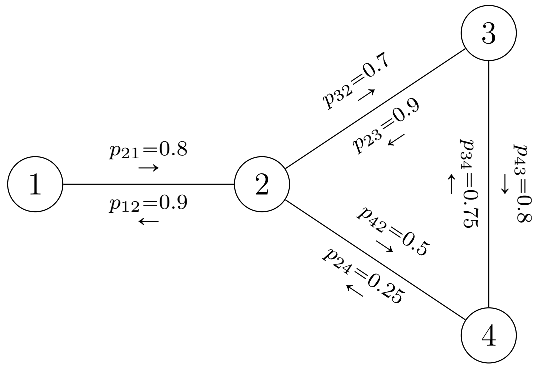
   For the matrices** A **and the vectors** $\vec{x}_0$ **in Exercises 25 through 29, find** $\lim_{t\to\infty}(A^t\vec{x}_0)$**. Feel free to use Theorem 7.4.1.

   25. $A = \begin{bmatrix} 0.3 & 1 \\ 0.7 & 0 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 0.64 \\ 0.36 \end{bmatrix}$

[1] Technically, we need this to be a *regular* Markov chain, meaning every node of the Markov chain is reachable from every other node

We can at least show that 1 is always an eigenvalue of A, and that the rest of the eigenvalues of A are < 1 in absolute value. Showing that $\text{gmult}(1) = 1$ and that the corresponding eigenvector can be chosen to have only positive entries is harder.

8. Albert takes some measurements of how many people are leaving each house, and gets the following diagram:



Does this diagram make physical sense? Why or why not?

9. Explain the following fact: no matter what numbers p_{ij} we have in our transition matrix A, the columns of A have to add up to 1.

10. Explain why $[1\ 1\ 1\ 1]^T$ is an eigenvector of A^T. What is its eigenvalue?

11. Show the following **General fact**: if B is any square matrix, then B^T and B have the same eigenvalues. (Hint: show they have the same characteristic polynomials!)

12. It follows that 1 is an eigenvalue of A!