Dynamic Matching and Bargaining Games: A General Approach

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Abstract

Dynamic matching and bargaining games provide models of decentralized markets with trading frictions. A central objective of the literature is to investigate how equilibrium outcomes depend on the level of the frictions. In particular, does the trading outcome become Walrasian (market clearing) when frictions become small? Existing specifications of such games provide divergent answers. To investigate what causes these differences, four conditions on trading outcomes are identified. These conditions are shown to be necessary and sufficient for the equilibrium outcomes of any game to become Walrasian when frictions are small. A parameterized example shows how the result can be used to understand the structural properties that drive or inhibit convergence to the Walrasian outcome in a variety of economic settings.

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1 Introduction

In a dynamic matching and bargaining game, a large population of traders interacts repeatedly in a decentralized market. Every trading period, traders are *matched* to form small groups where they *bargain* over the terms of trade. If they fail to reach an agreement, they can wait at some cost until the next period to be rematched into a new group. These waiting costs are the *frictions* of trading in the decentralized market. A major question in the literature concerns the trading outcome when frictions become small: Will the outcome become Walrasian? Ideally, one would like not only to find answers for particular trading institutions, but also to gain a general understanding of the conditions under which trading with vanishing frictions has this property and the conditions under which it does not. This paper examines this issue. Its primary purpose is to provide a general, "detail free" framework for the analysis of decentralized markets. Recent contributions that fall into the framework of this paper include work by Moreno and Wooders (2001), Satterthwaite and Shneyerov (2007), and De Fraja and Sakovics (2001).

As an illustration of the main result we use the following class of steady state dynamic matching and bargaining games, similar to the one used by Gale (1987): There is a continuum of buyers who have unit demand and valuations $v \in [0, 1]$ for an indivisible good, and there is a continuum of sellers who have unit capacity and costs $c \in [0, 1]$. These traders are matched into small groups. In these groups, they bargain, and, if they reach an agreement, they trade. The groups are connected to form a large market by allowing unsuccessful traders to be matched into new groups in the next period. Integration, however, is imperfect because there is a probability $\delta \in (0, 1)$ that a trader will die while waiting. These are the *frictions* of trading. Finally, at the end of each period, there is an exogenous inflow of new buyers and sellers.

This class of games is general with respect to both the matching technology and the bargaining protocol; that is, we do not specify how traders are matched into groups. Also, we do not specify how bargaining within the groups takes place and what information is released before and during bargaining. We will see how existing models in the literature differ in how they fill in these details. However, no matter how this is done, every specification of the model gives rise to an *outcome* that consists of (a) probabilities of trading for entering types and (b) expected equilibrium payoffs. Let $Q^S(c)$ denote the probability that a seller of type c sells his good, and let $Q^B(v)$ denote the probability that a buyer of type v gets the good. Similarly, let $V^S(c)$ and $V^B(v)$ be the payoffs to these types. Taken together, an outcome is a vector $A = \left[Q^S, Q^B, V^S, V^B\right]$. An outcome is called *feasible* if it corresponds to an allocation for the quasilinear economy defined by the distribution of buyer's valuations and seller's costs.

Now, suppose there is some sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$, which converges to zero, $\delta_k \to 0$. In addition, suppose for each δ_k, we take an equilibrium outcome of a specific trading game. This gives us a sequence of outcomes $\{A_k\}_{k=1}^{\infty}$ with $A_k = \left[Q_k^S, Q_k^B, V_k^S, V_k^B\right]$. We state four conditions on this sequence which are jointly necessary and sufficient for convergence to the competitive outcome. The first condition, *Monotonicity*, requires that trading probabilities are monotone – buyers with higher valuations are more likely to trade while sellers with higher costs are less likely to trade. The second condition, *No Rent Extraction*, requires that traders receive some part of the surplus they generate. Technically, this is a condition on the slope of the payoffs. The third condition, *Availability*, requires that a trader is matched frequently with those traders who do not trade with certainty and who remain in the stock for many periods. These traders are said to be available. The fourth condition, *Weak Pairwise Efficiency*, requires that for all pairs of buyers and sellers who are available (all pairs who are matched frequently) the sum of their expected payoffs is at least their private surplus, and $V^S(c) + V^B(v) \geq (v - c)$. In our illustrating example, the *Availability* condition relates to the matching technology, while the other conditions relate to the bargaining protocol.

Let A^W denote the Walrasian outcome at which all trade happens at the market clearing price. Our main result is this: A sequence of feasible outcomes converges to the Walrasian outcome A^W if and only if the four conditions hold.

The main result is illustrated by a parameterized dynamic matching and bargaining game that incorporates simplified versions of games from the literature. For each of our conditions, we show how to verify it. Importantly, we demonstrate that the conditions follow from basic equilibrium restriction onto outcomes which can be easily checked in many settings. It is not necessary to actually compute equilibria. We use simplifications of the models by Gale (1987) and Satterthwaite and Shneyerov (2008) to make this argument. For example, with asymmetric information, incentive compatibility restrictions imply the first two conditions, Monotonicity and No Rent Extraction. This observation suggests a surprising, pro-competitive effect of asymmetric information, discussed in Section 4.3.

Whenever convergence fails in some model at least one of our conditions must be violated. By pointing out exactly which conditions are violated, we show which assumptions of the model are the reasons for non-convergence. Thus, the conditions from this paper allow a classification of failures of convergence to the competitive outcome. In particular, we show that the failure in Lauermann (2008b) can be traced back to rent extraction (a failure of the second condition), the failure in Serrano (2002) to the failure of weak efficiency (the fourth condition), and the failure in De Fraja and Sakovics (2001) to a failure of a fundamental mass balance (feasibility) condition.

This paper makes two contributions. First, it provides a framework to discuss the existing literature. It suggests a common cause of convergence for those models in which the limit outcome is Walrasian. Furthermore, it allows a clarification of divergent results in the literature by relating them within a general structure. Second, the paper provides a methodology to address the following problem. Predictions from general equilibrium theory are made under the assumption that market clearance is a good approximation of trading outcomes *across* market institutions. However, it is difficult to provide a non-cooperative microfoundation for this robustness property since these microfoundations are inevitably made with respect to a *specific* institution (a specific game). The methodology proposed in this paper is to view games as mappings from an economy to outcomes and to find structural properties of these mappings that hold across games and which imply the market clearing prediction. Examples of such structural properties are those implied by incentive compatibility constraints under asymmetric information.

The main result gives conditions on endogenous equilibrium outcomes. The result is methodologically similar to the revenue equivalence theorem - namely, that the seller of an object obtains the same revenue in two auction formats if bidder's equilibrium trading probabilities and equilibrium payoffs satisfy certain conditions. The revenue equivalence theorem explains why the revenue of a seller is the same across a wide variety of auctions.

The rest of the paper is structured as follows. First, we introduce the general model. We define an economy and introduce the four conditions. Then, we state and provide the proof of the main result: A sequence of outcomes becomes Walrasian if and only if the four conditions hold. In the second part, we introduce a parameterized example of a dynamic matching and bargaining game that is general with respect to the matching technology and the bargaining protocol and that allows us to capture basic features of models in the literature. We use this example to discuss the four conditions in detail. In Section 6, we discuss how to extend the approach to a non-stationary dynamic matching and bargaining game with a finite number of players. We also indicate how to apply the theorem to large double auctions. Section 7 is the conclusion.

2 The Model

The trading environment consists of buyers and sellers who want to trade an indivisible good. Sellers each have one unit of the good, and their costs of trading are given by $c \in [0,1]$. Buyers each want to buy one unit of the good, and their valuations of the good are given by $v \in [0,1]$. If a seller trades with a buyer at a price p, the payoffs are $p - c$ and $v - p$, respectively. An abstract economy is characterized by two functions $G^S(c)$ and $G^B(v)$. These functions map $[0,1]$ into $[0,1]$, are zero at zero, and they are strictly increasing and continuously differentiable.

We can interpret these functions as defining a large static economy, for which $G^S(c)$ is the mass of sellers with costs below c and $G^B(v)$ is the mass of buyers with valuations below v.[1] (In Section 4, we interpret G^S and G^B as a constant exogeneous inflow of new traders into a *dynamic* economy.) Let p^w be defined as the Walrasian price such that the mass of sellers having costs below p^w is equal to the mass of buyers having a valuation above p^w, $G^S(p^w) = G^B(1) - G^B(p^w)$. Since G^S and G^B are strictly increasing, continuous functions, the market clearing price exists and is unique.

A trading outcome is a vector $A = \left[V^S(\cdot), V^B(\cdot), Q^S(\cdot), Q^B(\cdot)\right]$, where $V^S(c)$ and $V^B(v)$ are the expected payoffs, and $Q^S(c)$ and $Q^B(v)$ are the trading probabilities of the sellers and buyers. While an outcome A does not explicitly specify transfers, we can interpret the difference between the consumption value $vQ^B(v)$ and the payoff $V^B(v)$ as the transfer made by the buyer, and we can interpret the sum of the cost $cQ^S(c)$ and the payoff $V^S(c)$ as the transfer received by the seller. Let Σ denote the set of measurable functions $f : [0,1] \to [0,1]$. Any element of Σ^4 constitutes an outcome.

An outcome A defines an allocation for an economy given by $G^S(c)$ and $G^B(v)$ if (a) the total mass of buyers who trade equals the total mass of sellers who trade, that is, $\int_0^1 Q^S(c)\, dG^S(c) = \int_0^1 Q^B(v)\, dG^B(v)$ and if (b) the total transfers collectively made by buyers equals the total transfer received by sellers, that is, $\int_0^1 \left(v\, Q^B(v) - V^B(v)\right) dG^B(v) =$

[1] In general, $G^S(1)$ and $G^B(1)$ do not need to be one. Therefore, we can model large economies with a different mass of agents on each side. In Section 6 we assume $G^S(1) = G^B(1) = 1$. There, the functions are cumulative distribution functions that describe the distribution of types for a fixed number of buyers and sellers.

$\int_0^1 \left(V^S(c) + c\,Q^S(c)\right) dG^S(c)$. An outcome A that meets these two requirements is said to satisfy **mass balance.**[2]

Given an outcome A, trading surplus is defined as $S(A) \equiv \int_0^1 V^B(v)\, dG^B(v) + \int_0^1 V^S(c)\, dG^S(c)$. The surplus coincides with the ex ante expected payoffs. Our object of interest is the maximal surplus that can be realized subject to the mass balance constraint. We denote the maximal surplus by S^*. If mass balance holds then transfers cancel, and the surplus is determined solely by the allocation of the indivisible good given by the trading probabilities $Q = \left[Q^S, Q^B\right]$. Denote the Walrasian allocation by Q^W. It is straightforward to prove that an outcome is efficient if and only if the allocation is Walrasian (this is really the analogue of the First and Second Welfare Theorem for a quasilinear economy) and we state this without proof:[3]

Lemma 1 *For all outcomes that satisfy mass balance: $S(A) = S^*$ if and only if $Q \in \left[Q^W\right]$, where Q^W is the set of functions such that for sellers, $Q^S(c) = 1$ if $c < p^w$ and $Q^S(c) = 0$ if $c > p^w$ and for buyers, $Q^B(v) = 1$ if $v > p^w$ and $Q^B(v) = 0$ if $v < p^w$.*

Next, we derive a simple, sufficient condition for the efficiency of an outcome: An outcome is efficient if for any pair of types c and v, the sum of their interim expected payoffs $V^S(c) + V^B(v)$ is weakly larger than their private surplus $v - c$. Intuitively, all gains from trade are exhausted. Following Feldman (1973) such an outcome is called pairwise efficient:

Lemma 2 ***(Sufficiency)***. *Suppose an outcome A satisfies mass balance; Then $S(A) = S^*$ if $V^S(c) + V^B(v) \geq v - c$ for all v and c.*

Proof: Let $\bar{p} \equiv \inf_{c \leq p^w} \left(V^S(c) + c\right)$. Then $V^S(c) + V^B(v) \geq v - c$ for all v and c implies $V^B(v) \geq v - \inf_{c \leq p^w} \left(V^S(c) + c\right)$ for all v. By definition of S, $S(A) \geq \int_{p^w}^1 V^B(v)\, dG^B(v) + \int_0^{p^w} V^S(c)\, dG^S(c)$ and by definition of $\bar{p}$, $S(A) \geq \int_{p^w}^1 (v - \bar{p})\, dG^B(v) + \int_0^{p^w} (\bar{p} - c)\, dG^S(c)$. Using the definition of p^w, we obtain $S(A) \geq S^* + \bar{p}\left(G^S(p^w) - \left(G^B(1) - G^B(p^w)\right)\right)$ and thus $S(A) \geq S^*$. By the definition of S^*, $S(A) \leq S^*$ and therefore $S(A) = S^*$. *QED*

[2] Outcomes of some dynamic matching and bargaining games do not satisfy mass balance; see the discussion of models with cloning in Section 5.3.

[3] The surplus does not change if a zero measure of traders has trading probabilities different from Q^W. Therefore, we state the lemma for the equivalence class of Q^W, which is denoted by $\left[Q^W\right]$. Two functions are equivalent if the integral of their difference is zero.

3 The Main Result

3.1 Summary and Conditions

Let $\{A_k\}_{k=1}^{\infty}$ be some sequence of outcomes. In Section 4 we obtain such a sequence as the sequence of outcomes of equilibria of a dynamic matching and bargaining game when the exit rate converges to zero. We now define four conditions onto such sequences. Since we want to state conditions that are necessary for convergence to a Walrasian limit, these conditions are stated directly onto limits. A sequence of outcomes satisfies *Monotonicity* if limit trading probabilities are monotone. It satisfies *No Rent Extraction* if the absolute value of the slopes of limit payoffs is bounded by $[0, 1]$ and if the slope is bounded from below if some type trades with probability one. The sequence satisfies *Availability* if types who do not trade with probability one in the limit are available (to be defined below) and the sequence satisfies *Weak Pairwise Efficiency* if the sum of the expected payoffs of pairs of available traders is pairwise efficient. The main result is that a sequence of outcomes that has uniformly bounded variation and that satisfies Mass Balance becomes Walrasian if and only if these four conditions hold.

The assumption that the sequence has uniformly bounded variation ensures that a pointwise convergent subsequence exists (by Helley's selection theorem; see Kolmogorov and Fomin (1970)). A sufficient condition for a set of functions to have uniformly bounded variation is that the functions are monotone; see the discussion following Corollary 1. Let $\bar{A}$ be the limit of some convergent subsequence, $\bar{A} = \left(\bar{V}^S, \bar{V}^B, \bar{Q}^S, \bar{Q}^B\right)$. The following conditions are with respect to $\bar{A}$.

The first two conditions are closely related. They are both requirements with respect to the slope of the elements of limit outcomes. A sequence of outcomes satisfies **Monotonicity** (Condition 1) if given any limit $\left[\bar{Q}^S, \bar{Q}^B\right]$, $\bar{Q}^S(\cdot)$ is nonincreasing and $\bar{Q}^B(\cdot)$ is nondecreasing. Thus, a sequence satisfies the condition if limit trading probabilities of any pointwise convergent subsequence are monotone. Monotonicity holds universally in all games in the existing literature.

The **No Rent Extraction** (Condition 2) requires that $a - b \leq \bar{V}^S(b) - \bar{V}^S(a) \leq 0$ if $a \leq b$, and whenever $\bar{Q}^S(c_x) = 1$ for some c_x then $\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c)$ for all c. Thus, sellers' payoffs must be nonincreasing in costs and the difference between the payoffs of any two types must be no more than the difference between the types. Furthermore, whenever some seller type c_x trades with probability one, then all other sellers must receive at least the same payoff plus the difference in costs $(c_x - c)$. Similarly, for buyers $0 \leq \bar{V}^B(b) - \bar{V}^B(a) \leq b - a$ if $a \leq b$, and whenever $\bar{Q}^B(v_x) = 1$, then $\bar{V}^B(v) \geq \bar{V}^B(v_x) + (v - v_x)$.

In many games with asymmetric information the No Rent Extraction condition is a direct consequence of the envelope formula, see Section 4.3. The idea is this: If some seller c_x can trade with probability $Q^S(c_x)$ and if types are private, then every other seller c can *mimic* this type's strategy. Type's c payoff from mimicking c_x would be equal to the payoff of type c_x adjusted for the expected cost difference $Q^S(c_x)\ (c_x - c)$. Thus, in equilibrium, type c must receive at least a payoff of $V^S(c_x) + Q^S(c_x)(c_x - c)$, because otherwise mimicking c_x would be a profitable deviation. However, the No Rent Extraction condition is weaker than the implications of the envelope theorem. Equilibrium payoffs of symmetric information games (in particular, Gale (1987)) satisfy only this weaker condition.

The No Rent Extraction condition implies that whenever an outcome is efficient, it must be a competitive outcome. Thus, given No Rent Extraction, the question of whether an outcome is competitive reduces to the question of whether it is efficient. Intuitively, the condition ensures that every trader receives his full marginal contribution to the social surplus.

For the last two conditions, we introduce the concept of *availability*. Availability is formalized by two sequences of arbitrary functions $L_k^j(\cdot) : [0,1] \times \Sigma^4 \to [0,1]$ with $j \in \{B, S\}$. For the result, we require that the two conditions have to hold with respect to the same, fixed pair of functions. Given the sequence, let the limits be $\bar{L}^B(v) \equiv \liminf_{k\to\infty} L_k^B(v, A_k)$ and $\bar{L}^S(c) \equiv \liminf_{k\to\infty} L_k^S(c, A_k)$.

A sequence of outcomes satisfies **Availability** (Condition 3) relative to a pair of functions L_k^B and L_k^S if $\bar{Q}^B(v) < 1$ for all v below some v_x implies that $\bar{L}^B(v) = 1$ for all $v < v_x$ and if $\bar{Q}^S(c) < 1$ for all c above some c_x implies that $\bar{L}^S(c) = 1$ for all $c > c_x$. Thus, traders who do not trade with certainty are available in the limit, with availability defined as L^B or L^S converging to one.

A sequence of outcomes satisfies **Weak Pairwise Efficiency** (Condition 4) relative to a pair of functions L_k^B and L_k^S if $\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x$ for any pair of types c_x and v_x for which $\bar{L}^S(c_x) = 1$ and $\bar{L}^B(v_x) = 1$. Thus, for all pairs of traders v_x and c_x who are available, the sum of the expected payoffs exceeds the private surplus between the types.

The reader will notice that, mathematically, we split a single condition into two separate ones, using $\bar{L}^S$ and $\bar{L}^B$ as "indicator functions" to connect them. The reason for this split is that there is a natural economic interpretation of these functions and that, given this interpretation, these two conditions can fail separately, that is, economically, these conditions are separate.

In the application in Section 4, $L_k^B(v, A_k)$ is interpreted as the probability that a seller who is passively waiting in the stock is matched at least once with a buyer having a type larger than v before he has to exit, given the exit rate δ_k and the outcome A_k; see the definition of L_k^B on Page 16. With this interpretation, Availability is a property of the matching technology.

Suppose types c_x and v_x are *available*. The Weak Pairwise Efficiency condition then requires that the sum of their payoffs should not be below the surplus they could realize by trading with each other. This is motivated by the observation that otherwise it would become certain that (a) between these types there is "money left on the table," and (b), by availability, these types are certain to meet each other so that they can realize the money that is left. Whether or not available agents can realize money left on the table is a property of the bargaining protocol; see Section 4.4. Weak Pairwise Efficiency holds almost immediately for all those full information dynamic matching and bargaining games that specify a surplus sharing rule (like the generalized Nash bargaining solution).

Note that the last condition requires pairwise efficiency only with respect to those types that are available. Payoffs might be inefficient for those pairs for which one type is not available (because the type trades with probability one). Lauermann (2008b) is an example of a game in which the Weak Pairwise Efficiency condition holds but in which the limit outcome is not efficient.

3.2 Main Result

In this section, we state and prove our main result. Let V^W denote Walrasian payoffs, where $V^S = \max\{p^w - c, 0\}$ and $V^B = \max\{v - p^w, 0\}$ and $V^W = \left(V^S, V^B\right)$. Let A^W denote the set of Walrasian outcomes with $Q \in Q^W$ and $V = V^W$. The set Q^W specifies unique trading probabilities for all types except for $v = c = p^w$. We say a sequence of functions $\{Q_k\}$ ($\{A_k\}$) converges pointwise to the set Q^W (A^W) if it converges everywhere expect possibly at the point p^w. Our main result states that outcomes converge to the Walrasian outcome if and only if the four conditions hold:

Proposition 1 *Suppose some sequence* $\{A_k\}_{k=1}^{\infty}$ *satisfies Mass Balance and has uniformly bounded variation. Then the sequence converges pointwise to the Walrasian Outcome* A^W *if and only if* $\{A_k\}_{k=1}^{\infty}$ *satisfies Monotonicity, No Rent Extraction, and Availability and Weak Pairwise Efficiency relative to some pair of sequences of functions* L_k^B*,* L_k^S*.*

Proof of Proposition 1: As observed before, every sequence of functions with uniformly bounded variation has a pointwise convergent subsequence by Helley's selection theorem. Therefore, we can work with the limit of some convergent subsequence, $\bar{A}$. We first show that $\left(\bar{Q}^S, \bar{Q}^B\right)$ is in the set of Walrasian allocations Q^W for every such subsequence. This implies that every convergent subsequence becomes efficient and, thus, that the sequence itself becomes efficient. Given efficiency of the limit, we show that the limit outcome is indeed the Walrasian outcome A^W as claimed. Finally, we show that if the limit of the sequence is A^W, then the sequence satisfies the four conditions.

Given the limit $\bar{A}$, define cutoff types c_x and v_x as the lowest cost and highest valuation such that traders with these types do not trade with certainty. Let $c_x \equiv \inf\left\{c|\bar{Q}^S(c) < 1\right\}$ if $\bar{Q}^S(c) < 1$ for some c and $c_x = 1$ otherwise. Similarly, let $v_x \equiv \sup\left\{v|\bar{Q}^B(v) < 1\right\}$ if $\bar{Q}^B(v) < 1$ for some v and $v_x = 0$ otherwise. First, we show that the No Rent Extraction conditions implies

$$\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c) \quad \text{for all } c, \tag{1}$$

$$\text{and} \quad \bar{V}^B(v) \geq \bar{V}^B(v_x) + (v - v_x) \quad \text{for all } v. \tag{2}$$

For all types $c \in [c_x, 1]$, the first inequality follows directly by the No Rent Extraction condition, observing that $(c_x - c)$ is negative and V^S is decreasing at a slope bounded by -1. For types $c \in [0, c_x]$, the inequality is trivially true if $c_x = 0$; if $c_x > 0$, choose some $\varepsilon \in (0, c_x)$ and note that $\bar{Q}^S(c_x - \varepsilon) = 1$ by definition of c_x and by monotonicity of $\bar{Q}^S(\cdot)$. Hence, for all $c \leq c_x - \varepsilon$, the No Rent Extraction condition implies that $\bar{V}^S(c) \geq \bar{V}^S(c_x - \varepsilon) + (c_x - c) - \varepsilon$. Because ε is arbitrary and because $\bar{V}^S$ is continuous (the No Rent Extraction condition implies (Lipschitz-)continuity), we get $\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c)$. So the first inequality holds for all $c \in [0, 1]$. The second inequality follows for buyers by symmetric reasoning. Adding the two inequalities yields a lower bound on the sum of the payoffs of all types c and v:

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c + \bar{V}^S(c_x) + \bar{V}^B(v_x) - (v_x - c_x). \tag{3}$$

We use the Availability and the Weak Pairwise Efficiency conditions to show that the right hand side is at least $(v-c)$.

We consider two cases for the ordering of c_x and v_x. First, suppose $v_x - c_x > 0$. Take some $\varepsilon \in (0, v_x - c_x)$. By definition of c_x and v_x, and by monotonicity of $\bar{Q}^S(\cdot)$ and $\bar{Q}^B(\cdot)$, we have $\bar{Q}^S(c_x + 0.5\varepsilon) < 1$ and $\bar{Q}^B(v_x - 0.5\varepsilon) < 1$. Let L^S and L^B be the functions for which the third and fourth conditions hold. The availability condition implies that $\bar{L}^S(c_x+\varepsilon) = \bar{L}^B(v_x - \varepsilon) = 1$. Therefore, by the Weak Pairwise Efficiency condition, the sum of the expected payoffs of $c_x + \varepsilon$ and $v_x - \varepsilon$ must be such that $\bar{V}^S(c_x+\varepsilon) + \bar{V}^B(v_x - \varepsilon) \geq v_x - c_x - 2\varepsilon$. Since the sum $\bar{V}^S(\cdot) + \bar{V}^B(\cdot)$ is continuous and ε is arbitrary, we get $V^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x$ in the first case. Now consider the second case, $v_x - c_x \leq 0$. This case is trivial: Since $(v_x - c_x)$ is non-positive and payoffs are non-negative, we get $\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x$. So, for both possible orderings of c_x and v_x, the sum of the last three terms in (3) is positive. Hence, payoffs are pairwise efficient; that is, for all v and for all c, $\bar{V}^S(c) + \bar{V}^B(v) \geq v - c$.

By continuity of the integral operator, if each element A_k of the sequence satisfies Mass Balance, the limit $\bar{A}$ does so as well. According to Lemma 2, pairwise efficiency of the limit outcome is a sufficient condition for efficiency, $S(\bar{A}) = S^*$. Again by continuity of the integral operator, we therefore have $\lim_{k' \to \infty} S(A_{k'}) = S^*$ along the subsequence. Given efficiency of the limit outcome, Lemma 1 implies that limit trading probabilities must be Walrasian, that is, $(\bar{Q}^S, \bar{Q}^B)$ must be in the equivalence class $[Q^W]$. By Monotonicity of the limit functions, the limit must be exactly in Q^W. Because the choice of the subsequence is arbitrary, this implies that the limit of *every* convergent subsequence is in Q^W. This implies $\lim_{k\to\infty}(Q_k^S, Q_k^B) = Q^W$ for the original sequence. (Otherwise, there must be some subsequence that does not converge to Q^W; this subsequence must have a further subsubsequence which has a limit and, from our reasoning before, this limit must be in Q^W.)

$(\bar{Q}^S, \bar{Q}^B) \in Q^W$ implies that $c_x = v_x = p^w$. Substituting p^w into the bounds (1) and (2) - which are derived from the No Rent Extraction condition - and noting that payoffs are non-negative shows that all traders' individual payoffs are at least as large as their Walrasian payoffs, $\bar{V}^S(c) \geq \max\{p^w - c, 0\}$ and $\bar{V}^B(v) \geq \max\{v - p^w, 0\}$ for all c and v. So the limit surplus is at least equal to S^*. Since the limit outcome satisfies mass balance, S^* is also an upper bound on the payoffs. This implies that no type can expect strictly higher payoffs than its Walrasian payoffs: Otherwise, if for some type c,

$\bar{V}^S(c) > p^w - c$, the continuity of $\bar{V}^S$ would imply that payoffs are higher than $p^w - c$ for an open set of sellers' types, implying that $S(\bar{A}) > S^*$. Therefore, $\bar{V}^S(c) = \max\{p^w - c, 0\}$ for all c, and likewise, $\bar{V}^B(v) = \max\{v - p^w, 0\}$ for all v. Thus, we have proven that $\lim_{k\to\infty} A_k = A^W$.

Necessity of the four conditions is shown as follows. Suppose the sequence $\{A_k\}$ becomes Walrasian. *Monotonicity*: The limit trading probabilities of sellers are monotone because those sellers with costs below p^w trade with probability one, while those with costs above p^w trade with probability zero. (Trading probabilities at p^w are not determined.) The symmetric observation applies to buyers. *No Rent extraction*: Sellers' payoffs are decreasing at a slope equal to minus one if $c < p^w$ and payoffs have a slope of zero for all $c > p^w$. So the slope is bounded within $[-1, 0]$ and the slope is equal to -1 if $\bar{Q}^S(c) = 1$. Again, a symmetric observation applies to buyers. *Availability and Weak Pairwise Efficiency*: Weak Pairwise Efficiency holds relative to all functions L^S and L^B because the Walrasian outcome is such that $\bar{V}^S(c) + \bar{V}^B(v) \geq (v - p^w) + (p^w - c)$ for all v and c. There always exist functions such that the Availability condition holds, e.g., choosing the functions to be constants equal to one. *QED*

Intuition. The monotonicity condition makes it possible to define cut-off types c_x and v_x. If $v_x > c_x$, the outcome for the intermediate types (buyers with valuations below v_x and seller with costs above c_x) is (pairwise) efficient. These intermediate types trade with probability less than one and are therefore available and, given their availability, their payoffs must satisfy Weak Pairwise Efficiency. The No Rent Extraction allows us to extend this efficiency result to the extreme types (buyers with valuations above v_x and sellers with costs below c_x) who might not be available. This implies pairwise efficiency for all types. Using Lemma 2 we can conclude that the outcome must be efficient, so $S(A_k) \to S^*$. Finally, the No Rent Extraction condition implies that, whenever the outcome is efficient, it must be the competitive, Walrasian outcome A^W.

Our approach suggests that convergence to the competitive outcome is driven by the incentives to realize gains from trade. The significance of this can be illustrated by Rubinstein and Wolinsky (1985). Their paper analyzes the steady state of a dynamic matching and bargaining model with an economy with only one type of buyer (valuation $v = 1$) and only one type of seller (cost $c = 0$). The market clearing price depends on which side of the market is larger; for example, if there are more sellers than buyers, the market clearing price is zero. This economy violates our assumption that G^S and G^B are strictly increasing and, therefore, in this economy, the No Rent Extraction condition is vacuous: The efficiency of an outcome implies nothing about the prices at which the buyer and the seller trade. In fact, equilibrium outcomes are shown to be always efficient in their model but the price at which trade happens is typically far from being competitive.[4] Since trade at a non-competitive price is efficient, no unrealized gains provide incentives towards the competitive outcome.

4 An Application of the Main Theorem

We introduce a parameterized example of a steady state dynamic matching and bargaining game that illustrates how Theorem 1 can be applied. We demonstrate how the conditions can be checked easily in a variety of settings without calculating equilibrium outcomes explicitly. The example includes as special cases simplifications of models by Gale (1987) and by Satterthwaite and Shneyerov (2008). In Section 5 we discuss extensions.

4.1 Model

Traders interact repeatedly in a stationary market over infinitely many periods. At the beginning of each period, there is a *stock* of traders. This stock is characterized by the distribution of the types. $\Phi^S(c)$ is the mass of sellers with costs below c and $\Phi^B(v)$ is the mass of buyers with valuations below v. Within each period we have the following interaction:

1. Matching. Buyers and sellers from the stock are randomly matched into groups consisting of either one buyer and one seller or one buyer and two sellers, depending on a parameter ζ. The probability that a buyer is matched with one or two sellers is assumed to be $(1-\zeta)\frac{\Phi^S(1)}{M}$ and $\frac{\zeta}{2}\frac{\Phi^S(1)}{M}$, respectively, with $M = \max\left\{(1-\zeta)\Phi^S(1) + \frac{\zeta}{2}\Phi^S(1), \Phi^B(1)\right\}$. The probability that a seller is matched either alone with a buyer or together with another seller and a buyer is assumed to be $(1-\zeta)\frac{\Phi^B(1)}{M}$ and $\zeta\frac{\Phi^B(1)}{M}$, respectively. If $\zeta = 0$, all matches are in pairs of one buyer and one seller and if $\zeta = 1$ all matches are between one buyer and two sellers. The parameter ζ measures the degree of direct competition between sellers.[5] Matching is independent of the types so that the type of any given trader

[4] There are some subtleties involved in the definition of a competitive price, see the discussion in Gale (1987) and Osborne and Rubinstein (1990).

[5] These matching probabilities arise if first a share ζ of sellers are bound into pairs. The resulting mass of individual sellers and pairs of sellers is $(1-\zeta)\Phi^S(1) + \frac{\zeta}{2}\Phi^S(1)$. Then, all traders (individuals and pairs) of the short side of the market are matched randomly with the long side, while the longer side

in a match is distributed according to the distribution of types in the stock. The matching technology is similar to De Fraja and Sakovics (2001) and allows us to capture one-to-one, and two("many")-to-one matching. We assume that sellers do not know whether they have a competitor.

2a. Bargaining: Observation. Within each group the buyer observes the type(s) of the seller(s). Seller(s) observe a *signal* $\hat{v}$ about the buyer's type v, $\hat{v} = (1-\eta)\,v + \eta\varepsilon$: The parameter η measures how noisy the signal is, with noise ε being distributed according to the standard normal. If $\eta = 0$, the type is perfectly observed and we have symmetric information. If $\eta = 1$, nothing about the type is observed and we have a bargaining game with asymmetric information.

2b. Bargaining: Offers. Having observed types and signals, one market side is chosen to be the proposer of a price offer. With probability β the buyer makes a price offer and with probability $(1-\beta)$ the seller(s) make(s) a price offer. The other market side can either accept or reject the offer. If the buyer is chosen to propose and if there are two sellers and both accept the offer, each seller gets to trade with probability $\frac{1}{2}$. If there are two sellers and they are chosen to propose, the buyer can accept the lower of the two prices. The parameter β measures the bargaining power of the buyer.

3. Exit and Entry. After the bargaining stage, traders exit and enter the market: Those pairs of traders who reached an agreement leave the market and consume the good. Of those traders who did not reach an agreement, a share δ exits ("dies") and looses the possibility of trading. A share $(1-\delta)$ of these traders remains for the next period. Finally, there is entry by a mass $G^B(1)$ of buyers and a mass $G^S(1)$ of sellers with types distributed according to the functions $G^S(\cdot)$ and $G^B(\cdot)$, defined in Section 2.

The endogenous objects in this market are the distributions of types, Φ^S and Φ^B, and the actions in the bargaining stage. The actions are denoted by $a^S = \left[p^S(c,\hat{v}), r^S(c)\right]$ and $a^B = \left[p^B(v,c_1,c_2), r^B(v)\right]$, where p and r denote the price offer and acceptance (reservation price) strategies. For example, $p^B(v,c_1,c_2)$ is the offer of type v when facing two sellers with costs c_1 and c_2. We encode the price offer to a single seller by setting $c_2 = 2$ as $p^B(v,c_1,2)$. The buyer accepts a price offer p if and only if $p \leq r^B(v)$. We collect these endogenous objects in the *market constellation* $\sigma = \left[\Phi^S, \Phi^B, a^S, a^B\right]$.

For each market constellation σ we can calculate the payoffs of the traders. We denote by $q^S(c,a)$ the per period trading probability of a seller c who uses action a

is rationed.

given σ. We denote by $Q^S(c,a)$ the probability to trade at some time (rather than exiting), the so called lifetime trading probability. Let $P(c,a)$ denote the expected price conditional on trading. The seller's payoffs are denoted by $U^S(c,a)$ and equal to $U^S(c,a) = Q^S(c,a)(P(c,a) - c)$: Payoffs from action a are equal to the expected trading probability times the profit conditional on trade. (If a seller does not trade, the profit is zero.) We define q^B, Q^B, P, and $U^B(v,a)$ for the buyer analogously. Given a constellation σ, maximized payoffs are denoted by $V^B(v) = sup_a U^B(v,a)$ and $V^S(c) = sup_a U^S(c,a)$.

Steady State. We discuss the buyer's side: The stock of buyers at the beginning of a period is characterized by Φ^B. The mass of buyers at the end of the period is the sum of the entering buyers and the initial buyers who neither traded nor died. Φ^B is a steady-state stock if and only if the stock at the end of a period is the same as the stock in the beginning, that is,

$$G^B(v) + (1-\delta)\int_0^v \left(1 - q^B(\tau, a(\tau))\right) d\Phi^B(\tau) = \Phi^B(v).$$

A similar condition has to hold for the distribution of sellers' types.

Steady State Equilibrium. A market constellation σ^* constitutes an equilibrium if (a) the steady state conditions hold, if (b) the actions are mutually optimal and if (c) the acceptance decision is such that an offer is accepted if and only if it makes the receiver better off than continuation, $r^*(c) = (1-\delta)V^S(c) + c$ and $r^*(v) = v - (1-\delta)V^B(v)$. This latter requirement is a refinement that captures sequential rationality. Without this refinement, traders would be free to reject any off equilibrium price offer.

Examples. We obtain a model with symmetric information and pairwise matching as in Gale (1987) when $\beta \in (0,1)$ (interior bargaining power), $\eta = 0$ (no noise), and $\zeta = 0$ (no direct competition). We obtain a model with asymmetric information in which a random number of sellers competes for the buyer as in Satterthwaite and Shneyerov (2008) when $\beta = 0$ (only sellers make offers), $\eta = 1$ (no information is revealed), and $\zeta \in (0,1)$ (there is some direct competition).[6]

Result. Take a vanishing sequence of exit rates $\{\delta_k\}$ with $\delta_k \to 0$. We assume that there exists at least one equilibrium for each δ_k. Pick one equilibrium for each k and denote

[6] In Satterthwaite and Shneyerov (2008) the roles of buyers and sellers are reversed: A random number of buyers make price offers to a single seller. The seller cannot commit to a prior reservation price. Ex post, the seller can either accept the highest offered price or reject it.

the corresponding outcome by A_k. This gives a sequence of outcomes $\{A_k\}$. We argue in the remaining subsections that this sequence satisfies the conditions of the main theorem and that the limit of the sequence is therefore competitive in two cases, which roughly resemble the settings of Gale (1987) and Satterthwaite and Shneyerov (2008):

Corollary 1 *If $\{A_k\}$ is a sequence of outcomes generated by equilibria of the basic example for a vanishing sequence of exit rates $\{\delta_k\}$, then the sequence converges to the competitive outcome A^W if (i) information is asymmetric, $\beta = 0$, $\eta = 1$, $\zeta \in [0,1]$ or if (ii) information is symmetric $\eta = 0$, matching is pairwise, $\zeta = 0$, and the buyer has bargaining power, $\beta \in (0,1)$.*

In the next sections, we show why the four conditions hold. For the result above to be indeed a corollary to Theorem 1, we also need to show that A_k has a uniformly bounded variation and that it satisfies mass balance. Mass balance follows immediately from the steady state conditions; see for example Lauermann (2008a). The sequence has uniformly bounded variation because its elements are monotone along the sequence; see the remark in Section 4.3. More generally, if Monotonicity and No Rent Extraction hold for all elements of the sequence (trading probabilities are monotone and slopes of payoffs are within the unit interval), the sequence has uniformly bounded variation.

4.2 Availability

We want to argue that Availability holds for all parameter choices of ζ, β, and η. For this, we first observe that the steady state conditions imply that types who do not trade with probability one must make up a positive share of the stock. Our matching technology implies that whenever a set of types makes up a positive share of the stock, the probability to match with such a type is strictly positive and non-vanishing; this implies Availability.

The basic observation is the following: traders who are less likely to trade stay in the stock for a longer period of time and make up a larger share of it. The steady state condition can be rewritten to show that (see e.g., Lauermann (2008a)),

$$\Phi^B(v') - \Phi^B(v'') = \frac{1}{\delta} \int_{v''}^{v'} \left(1 - Q^B(\tau) + \delta Q^B(\tau)\right) dG^B(\tau). \tag{4}$$

The mass (density) of any given type is proportional to the probability of *not* being able to trade, $\left(1 - Q^B(\tau)\right)$ and the mass in the inflow, dG^B. This implies in particular that buyers who do not trade with probability one make up a positive, non-vanishing share of the stock of traders: By (4) their mass is proportional to $\delta^{-1}\left(1-Q^B\right)\left(G^B\left(v'\right)-G^B\left(v''\right)\right)$ while the total mass of buyer (and seller) is at most $\delta^{-1}G^B(1)$ (by taking the integral from 0 to 1 at $Q^B=0$). The relation between the probability of not trading and the share in the stock is independent of the specific matching technology and follows mechanically from the steady state conditions.

The probability to be matched in any given period with a buyer with type at least as large as v is denoted by $X^B(v)$ and the probability to be matched with a seller with type at most c is denoted by $X^S(c)$. For example, according to our matching technology the probability to be matched with a buyer from the set $[v,1]$ is $X^B(v)=\left(\Phi^B(1)-\Phi^B(v)\right)M^{-1}$. The probability for a seller to be matched some time during his lifetime with a buyer who has a type at least as large as v is denoted by L^B,[7]

$$L^B(v)=X^B(v)+\left(1-X^B(v)\right)(1-\delta)\,L^B(v)\,. \tag{5}$$

As apparent from the definition, when $\delta_k\to 0$ the probability $L_k^B(v)$ converges to one if and only if the per period matching probability is large relative to the exit rate,

$$L_k^B(v)\to 1 \quad\Leftrightarrow\quad \frac{X_k^B(v)}{\delta_k}\to\infty. \tag{6}$$

Now we show that the Availability condition holds. Suppose there is some v_x such that the limit trading probability is smaller than one for all types below. Take any v'' and v' below v_x to define an interval $[v'',v']$ below v_x for which the probability of not trading is strictly positive, $\left(1-\bar{Q}(v)\right)>0$. Using (4) we have argued that the share of these types in the stock must be strictly positive and non vanishing in the limit. Thus, given our matching technology, the matching probability per period is strictly positive and $\liminf X_k^B(v'')>0$. Hence, (6) holds and types $v\geq v''$ become available, $L_k^B(v'')\to 1$. We have now demonstrated that the Availability condition holds in our model relative to L^B as defined in (5) for all parameters ζ,β,η. The availability condition for the seller's side relative to an analogously defined function L^S follows from the same logic.

We can extend the arguments from before to other matching technologies. First, the relation between the trading probability and the share of types in the stock follows

[7] Note that the trading probabilities Q_k^B and the exit rate δ_k uniquely determine the stock Φ^B via (4) and therefore the matching rate $X_k^B(v)$. Thus, for given δ_k, we can define $L_k(v,A_k)$ as a function of A_k only, as required for the application of the conditions.

solely from the steady state conditions. Therefore, it is sufficient for Availability that the matching technology is such that there is a strictly positive, non-vanishing probability to be matched with any set of types that make up a positive share of the stock. Importantly, it is not necessary to calculate an equilibrium to check whether or not any given matching technology implies a positive matching probability with types having a positive share in the stock. In the current example, this property follows immediately from inspection of the matching function.

Failure of Availability with Entry. Availability does not hold in models with an entry stage: Agents who do not enter are not available even though they trade with probability zero. Entry is discussed in Section 5.2.

4.3 Monotonicity and No Rent Extraction

We want to argue that the Monotonicity and the No Rent Extraction conditions are immediate whenever bargaining takes place under asymmetric information. Bargaining is said to be under asymmetric information if sellers' signals about the buyers' willingness to pay are not informative, $\eta = 1$ and if buyers never make offers, $\beta = 0$ (since, by assumption, the buyer observes the type(s) of the seller(s)). For these parameters, the trading probability and the expected price of an agent depend only on his action chosen in the bargaining game but not on his type.

We discuss the seller's side. In equilibrium, the seller's action a maximizes payoffs,

$$V^S(c) = \max_a Q^S(a, c) \quad (P(c, a) - c).$$

With asymmetric information, the Monotonicity and the No Rent Extraction condition follow from standard reasoning about maximizing a payoff function that satisfies the single crossing condition (see, for example, Mas-Colell, Whinston, and Green (1995)). The allocation must be monotone and payoffs must satisfy the envelope formula:

$$\begin{aligned} &Q^S(a(c), c) \text{ is nonincreasing in } c \\ &V^S(c) = V^S(c_x) + \int_c^{c_x} Q^S(a(\tau), \tau)\, d\tau \end{aligned}$$

Symmetric reasoning applies to the buyer's side. Furthermore, since trading probabilities and payoffs are monotone for *all* δ_k, the sequence A_k has uniformly bounded variation. Importantly, we do not have to calculate equilibria in order to check whether or not the conditions hold. Instead, the conditions follow directly from incentive compatibility restriction on equilibrium objects, see also the discussion in Section 5.4.

Failure of No Rent Extraction with Symmetric Information. According to the discussion before, the No Rent Extraction condition is most likely to fail in a situation with *symmetric* information. Consider the basic example with symmetric information in which only sellers make price offers ($\beta = 0, \eta = 0, \zeta = 0$). This case is analyzed in Lauermann (2008b). It is shown that No Rent Extraction fails: Since sellers have all the bargaining power, they receive the whole trading surplus and buyers' payoffs are zero, independent of their type, $\bar{V}^B(v) \equiv 0$, that is, the *rent* of the buyers is *extracted*. Since there must be some interior type of buyer who trades with probability one in the limit, this implies that the No Rent Extraction condition is violated. The other three conditions continue to hold. As shown in Lauermann (2008b), the limit outcome is inefficient because sellers receive a strictly positive profit; sellers are therefore not willing to trade with buyers who have a marginal willingness to pay. Since the limit outcome is inefficient, the limit is in particular not Walrasian.

Interior Bargaining Power. In Gale's (1987) original model, buyers can make offers as well, that is, β is strictly positive. In this case, the basic example with symmetric information has a property that makes it similar to a game with asymmetric information. Suppose $\eta = 0, \zeta = 0$, and $\beta \in (0, 1)$. Although it is still true that a trader of type v does not need to **receive** the same offers as a trader of type v_x, he can now **make** the same offers when chosen as the proposer. Importantly, in equilibrium, payoffs depend only on the offers made as a proposer. (If a trader is chosen to respond to an offer, this offer is such that the responding trader is just indifferent between accepting and rejecting.) Therefore, a buyer of type v can mimic the strategy of another type v_x just as he can mimic the actions of this type with asymmetric information. This is sufficient to restore No Rent Extraction.[8] Since the other conditions hold as well, this implies that the bargaining game with symmetric information is Walrasian if $\beta \in (0, 1)$. Thus, interior bargaining power and private information play a similar role in ensuring that the No Rent Extraction condition is satisfied.

[8] Given some equilibrium σ^*, let $P^P(v)$ and $Q^P(v)$ be the expected price and trading probabilities of a buyer having type v who rejects all offers when chosen to respond but who makes optimal offers when chosen to propose. By the reasoning in the text, equilibrium payoffs depend only on the own offers. Therefore, $V^B(v) = Q^P\left(v - P^P\right)$. This implies that for any two types v and v', $V^B(v) \geq V^B(v') + Q^P(v')(v - v')$. Together with the observation that $Q_k^P(v)$ converges to one along any sequence of equilibria for which $Q_k(v)$ converges to one, Monotonicity and No Rent Extraction follow.

The Role of Information. Intuition derived from the Myerson-Satterthwaite impossibility theorem suggests that asymmetric information is detrimental to efficiency. However, this is not the case here. Asymmetric information directly implies that two of the four conditions hold. To further illustrate this point, let us fix parameters in the example such that sellers have all the bargaining power and face no direct competition, $\beta = 0$ and $\zeta = 0$. Then the limit outcome is efficient if information is asymmetric ($\eta = 0$) but the limit is inefficient if information is symmetric ($\eta = 1$) as discussed in this section. Moreno and Wooders (2002) make a similar observation about the positive implications of asymmetric information in a trading situation; our analysis suggests that their observation is not a coincidence. We can use our framework to interpret the counterintuitive findings about the role of private information by pointing to its role in ensuring the No Rent Extraction condition, that is, in ensuring that traders receive their marginal contribution to the social surplus.

4.4 Weak Pairwise Efficiency

Bargaining games with symmetric information that specify a surplus sharing rule, like Nash Bargaining, satisfy Weak Pairwise Efficiency since the total expected surplus can be realized. In general, it is critical that the bargaining protocol is not "too inefficient," so that whenever there is a positive expected surplus between two traders, at least one of the traders can realize a positive, non-vanishing fraction of this surplus. Conversely, if the sum of the expected payoffs for two types of traders is zero despite the existence of a positive expected surplus for each of them, the condition does not hold.

Consider an exit rate δ and a constellation σ. Given a pair of types (v, c) we define

$$\begin{aligned} \Delta(v, c) &= v - c - (1-\delta)\left(V^S(c) + V^B(v)\right) \\ x(v, c) &= \min\left\{X^B(v), X^S(c)\right\}, \end{aligned}$$

where Δ is the surplus that is available between the types and x is the minimum of the probabilities that the seller is matched with a buyer of type at least v and the probability that the buyer is matched with a seller with cost at most c. When V^S and V^B are monotone, Δ is increasing in v and decreasing in c. Then, if $x(v, c)\Delta(v, c)$ is positive, there is surplus available in expectation for each type.

We can write payoffs in a recursive fashion. Let π^S be the expected net gain conditional on trading, then $V^S = q^S\left(\pi^S + (1-\delta)V^S\right) + \left(1 - q^S\right)(1-\delta)V^S$. Reordering terms yields $\delta V^S = q^S\pi^S$. Given some specification of the game, let γ^S be a uniform lower bound such that $\delta V^S \geq \gamma^S x\Delta$ for all types c, all exit rates δ, and all equilibria. The parameter γ^S measures how much of the expected surplus is realized by the seller. Let γ^B be an analogous uniform bound for buyers so that $\delta V^B \geq \gamma^B x\Delta$. Such bounds exists trivially since $\gamma^S = \gamma^B = 0$ suffices. As we see below for most standard bargaining protocols there are also bounds that are not trivial and a positive share of the surplus can be realized by at least one side of the market.

If a positive share of the surplus can be realized, Weak Pairwise efficiency has to hold: Take a sequence σ_k and suppose types v_x and c_x are available. By definition of γ^S and γ^B, the sum of their payoffs is bounded from below by

$$V_k^S(c_x) + V_k^B(v_x) \geq \left(\gamma^S + \gamma^B\right)\frac{x_k}{\delta_k}\Delta_k. \tag{7}$$

Suppose there are non-trivial lower bounds such that $\left(\gamma^S + \gamma^B\right) > 0$. Since the types v_x and c_x are available by hypothesis, $\frac{x_k}{\delta_k} \to \infty$, from (6). Therefore, it must be the case that the surplus between c_x and v_x becomes zero, $\Delta_k \to 0$, for otherwise the right hand side would become infinite. The surplus is zero between v_x and c_x if and only if $\bar{V}^S + \bar{V}^B \geq v_x - c_x$. Thus, Weak Pairwise Efficiency holds whenever nontrivial uniform lower bounds exist, that is, whenever $\left(\gamma^S + \gamma^B\right) > 0$.

Let us consider a model similar to Gale (1987) with symmetric information, $\eta = 0$, no competition, $\zeta = 0$, and $\beta \in (0,1)$. As said before, the parameter β measures the distribution of bargaining power. Indeed, it is straightforward to verify that the buyer and the seller can expect a share $\gamma^B = \beta$ and $\gamma^S = (1-\beta)$, respectively. We verify this for the seller. The seller's payoffs are

$$\delta V^S(c_x) = (1-\beta)\int_{v:\Delta(v,c_x)\geq 0} \Delta(c_x, v)\, dX^B(v)$$

that is, whenever a seller is matched with a type v and chosen to be the proposer, he makes an acceptable offer that captures the entire surplus $\Delta(c_x, v)$. When chosen to be the responder, he captures nothing over and above his continuation value. Observing that in equilibrium the surplus $\Delta(c_x, v)$ is nondecreasing in v, the above formula implies in particular that $\delta V^S(c_x) \geq (1-\beta)\, x(v, c_x)\, \Delta(v, c_x)$ for all v and therefore $\gamma^S = (1-\beta)$ is indeed a uniform lower bound on the seller's share of the expected surplus $x\Delta$. The bound is independent of the type, the exit rate, and the equilibrium.

Now let us consider a model similar to Satterthwaite and Shneyerov (2008) or Lauermann

(2008a): All offers are made by sellers, $\beta = 0$ and information is asymmetric, $\eta = 1$. There may be competition $\zeta \in [0,1)$. With asymmetric information, bargaining cannot be efficient. Still, traders can expect to realize a non-zero share of the expected surplus. In fact, for these parameters, $\gamma^S = (1-\zeta)$. To see why, suppose the seller offers a price equal to the maximal willingness to pay of v_x, $p_x = r(v_x)$ with $r(v_x) = v_x - (1-\delta) V^B(v_x)$ by the equilibrium requirement. In equilibrium, $r(\cdot)$ is increasing so that buyers with types $v \geq v_x$ accept the offer. The probability to be matched with such a buyer is $X^B(v_x)$. The probability to have no competitor is $(1-\zeta)$. Therefore, the price offer is accepted with a probability of at least $(1-\zeta) X^B(v_x)$. The payoff from offering p_x is a lower bound on the equilibrium payoff,

$$\begin{aligned} \delta V^S(c_x) &\geq x(1-\zeta)\left(p_x - c_x - (1-\delta) V^S(c_x)\right) \\ &= x(1-\zeta)\left(v_x - (1-\delta) V^B(v_x) - c_x - (1-\delta) V^S(c_x)\right) = (1-\zeta) x\Delta \end{aligned}$$

and so $\gamma^S = (1-\zeta)$ is indeed a uniform lower bound of the seller's share of the expected surplus.[9]

As we have demonstrated, we do not need to calculate the equilibrium in order to check whether equilibrium payoffs admit a nontrivial lower bound. It is sufficient to show that in equilibrium individual agents have actions available that ensure a minimal payoff. We did use an equilibrium requirement on reservation prices that implies that the other agents accept an offer whenever accepting the offer is individually rational. The example below demonstrates the significance of this requirement.

Failure of Weak Pairwise Efficiency with Simultaneous Auctions. Serrano (2002) specifies the bargaining protocol as a simultaneous double auction.[10] He shows that equilibrium outcomes do not need to become competitive. We can replicate main features of his bargaining protocol in our basic example by dropping the equilibrium requirement (c) on reservation prices and by analyzing the larger set of "Nash equilibria". Suppose matching is pairwise and sellers are always chosen to propose, $\zeta = 0$ and $\beta = 0$. The following action profile with trading at an arbitrary price $\bar{p}$ constitutes a mutual best response for all δ_k: Sellers offer the price $\bar{p}$ if $c \leq \bar{p}$ and they offer $p = 1$ otherwise. Buyers choose a reservation price $r = \bar{p}$ if $v \geq \bar{p}$ and $r = 0$ otherwise. If $\bar{p} \neq p^w$, the limit of the corresponding sequence of outcomes is not competitive. While the sequence of outcomes

[9] It can be verified that $\gamma^S + \gamma^B > 0$ if $\zeta = 1$ and sellers face competitors with certainty.

[10] His interest stems from the prior use of simultaneous auctions in dynamic matching and bargaining games in the context of common values.

satisfies Monotonicity, No Rent Extraction, and Availability, Weak Pairwise Efficiency fails. Suppose $\bar{p} = 1$ for concreteness and pick some pair $(v,\ c)$ with $v > c$. Since no trade takes place, trading probabilities are zero and the types are Available. However, the private trading surplus $(v - c)$ is strictly larger than the sum of their limit payoff, which is 0. Note that depending on $\bar{p}$ the equilibrium outcome is not Pareto dominated by the competitive outcome. Therefore, one can interpret an outcome with $p^w < \bar{p} < 1$ as stemming from collusion among sellers. Rubinstein and Wolinsky (1990) show how to construct collusive equilibria that are subgame perfect in a model with a finite number of agents and observable actions, see the discussion in Section 6.1.

5 Discussion of Parameterized Model

We use this section to discuss how the general approach can be modified and used to understand economic situation that are similar to the one modeled in the basic example. We discuss extensions that go beyond the realm of the basic example in Section 6.

5.1 Discounting and Infinitely Lived Agents

In the basic example, agents exit with an exogeneous rate δ and do not discount payoffs. Many models in the literature assume no exogeneous exit, so that traders are "infinitely lived" and can exit the stock only through trading, see, e.g., Gale (1987) and Satterthwaite and Shneyerov (2007). Time preferences are introduced by assuming the presence of a discount factor. If we want to allow for discounting we have to modify our approach. The problem is that our definition of an allocation is no longer appropriate: With discounting, traders care about the *discounted* trading outcome while feasibility constraints are defined with respect to *undiscounted* outcomes. As shown in Lauermann (2006), this implies, for example, that we need to take care of the problem of Ponzi Schemes. Because of such schemes, the expected payoff of entering traders can in principle be much higher than the surplus in the Walrasian outcome: By shifting the timing of sellers' trades, discounting implies that costs are diminished. Nevertheless, since sellers are infinitely lived, a mere shift in the timing of their trading does not influence how many buyers can trade – a shift does not affect the feasibility conditions. In the extreme, when shifting the timing of trade for all sellers to "infinity," all costs are discounted to zero. Lauermann (2006) establishes conditions which ensure that equilibrium outcomes cannot involve such Ponzi Schemes. These conditions hold in all standard bargaining protocols and it can be shown that that our approach remains valid in settings with discounting and infinitely lived agents.

5.2 Entry

A restriction of the current paper relative to the literature is that we do not have an entry stage. In terms of the general approach, adding an entry stage to our example leads to a failure of Availability: Traders who choose not to enter are not available. Lauermann (2006) shows how the general approach can be modified to include an entry stage (by extending the definition of an outcome and including an indicator variable for the entry decision). In addition, the conditions are modified: the Availability condition is weakened so that only those traders who choose to enter the market need to be available. To compensate for the weaker availability condition, the fourth condition (Weak Pairwise Efficiency) is strengthened. It requires a *particular* split of the surplus between available types and not just any efficient realization of the surplus. This requirement rules out an extreme distribution of the bargaining power. In the basic example, it rules out the case in which it is always the seller making offers without direct competition. Under these modified conditions, every sequence of non-trivial outcomes (those in which at least some trade happens in the limit) becomes Walrasian. However, our approach cannot be used to prove existence of such sequences of non-trivial outcomes. Existence of equilibrium can only be proven for explicit games.

Intuitively, an entry stage can imply that an individual seller faces a vertical demand curve at some price because of the absence of certain types in the stock. This reduces the incentives towards competitive prices and can lead to non-convergence. There are two interpretations of an entry stage.[11] First, an entry stage might literally correspond to an economic situation in which both sides of the market have to search actively. In particular, one-sided communication of offers (formal advertising or informal transmission in a social network) is not possible. Second, an entry stage might be a reduced model of an *investment decision* in a production economy that is prior to the interaction in the market. Cole, Mailath, and Postlewaite (2001) analyze a model in which traders have to take non-contractible, complementary investment decisions prior to trading in an exchange economy. They show that even if the trading outcome is competitive (stable) on the second stage, inefficiencies may persist on the prior investment stage.

[11] Having an entry stage is not a modeling choice but a technical necessity in steady state models that do not have an exit rate, like, e.g., the model in Gale (1987). Without an exit rate, the number of agents who do not trade becomes infinite. In order to keep agents who do not trade outside of the relevant stock, it is assumed that these agents do not enter.

5.3 Cloning

Cloning refers to the assumption that every trader who leaves the market is replaced by an exact copy of his type, a *clone*. With this assumption, the inflow depends on the trading outcome and is *endogenous*. The stock of traders, however, does not change over time and is exogenous. A model with cloning has recently been used by De Fraja and Sakovics (2001). They show that payoffs are often not Walrasian with respect to the economy defined by the stock of traders. This is not necessarily bad for welfare: It can be shown that with cloning, trading outcomes can yield a surplus strictly *above* S^*. Intuitively, with cloning, high valuation buyers and low cost sellers are not scarce. As shown in Lauermann (2006), it is for example possible to construct (limit) equilibria in which almost all types of buyers trade with probability one with sellers having costs close to zero. While such an outcome clearly violates mass balance, one can verify that sequence of equilibrium outcomes of De Fraja and Sakovics (2001) satisfy the four main conditions. This suggests that the reason for the non-competitive limit outcome in De Fraja and Sakovics (2001) is the cloning assumption. For further discussions of the implications of fixing the stock rather than fixing the flow, see also Gale (1987) and Osborne and Rubinstein (1990).

5.4 A General Matching Technology and Bargaining Protocol

The ideas from the basic example can be generalized. In an earlier version of this paper, a class of dynamic matching and bargaining games is analyzed that uses the same steady state framework but that uses a very general matching technology and bargaining protocol. As in the example, at the beginning of each period, there is a stock of traders, characterized by the distributions Φ^S and Φ^B. Then, buyers and sellers from the stock are matched into small groups according to a general *matching technology*: A matching technology is a mapping $\mathcal{X}$ that defines for a given stock the probability to be matched with $0, 1, 2, ...$ traders from the other side of the market. (In the basic example, a buyer's probability to match with one or two sellers is $(1 - \zeta) \frac{\Phi^S(1)}{M}$ and $\frac{\zeta}{2} \frac{\Phi^S(1)}{M}$, respectively). For a given matching technology, within each group, buyers and sellers bargain according to some general *bargaining protocol*. A bargaining protocol is a mapping Γ that maps action profiles into bargaining outcomes (trading probabilities and transfers conditional on trading) for a given period. For given continuation values V, a solution $E(V)$ for the bargaining stage is specified. After the matching stage, agents exit and enter as in the basic example. Three lemmas describe properties of $\mathcal{X}$, Γ, and E that are sufficient to ensure that the conditions of the main theorem hold. Given these properties, if a sequence of outcomes

$\{A_k\}$ is generated by equilibria of a game from the class $\langle \Gamma, \mathcal{X} \rangle$ for a vanishing sequence $\{\delta_k\}$, then the limit of the outcomes is competitive. These three lemmas are closely related to the arguments we gave before for the parameterized example. Let us sketch the results here.

1) Availability holds if the matching technology $\mathcal{X}$ is "non-vanishing," meaning that for every set of types who have a positive share in the stock, there is a positive, non-vanishing probability to meet some type from this set. (This lemma uses the fact that types of agents who do not trade with probability one make up a positive share in the stock.)
2) Monotonicity and *No Rent Extraction* hold if the bargaining protocol Γ specifies a game with asymmetric information, meaning that the trading outcome for any given trader depends only on the chosen action but not on the type. (This lemma uses our observation that incentive compatibility requires that the allocation is monotone and that payoffs can be characterized by the envelope formula.)
3) Weak Pairwise Efficiency holds if the bargaining protocol Γ together with its solution is "not too inefficient," meaning that, whenever there is a positive expected surplus between two traders, traders realize at least a positive fraction of this surplus. (This lemma uses the observation from equation (7) that the surplus Δ between available types must be zero when the sum of the fractions $\gamma^S + \gamma^B$ is positive)

6 Two Extensions

6.1 Finite Number of Traders

In the parameterized example, we have analyzed the steady state of a large economy with a continuum of traders. In the example, a single agent cannot affect the overall market (the aggregate outcome) by assumption. Here we discuss how to use Proposition 1 to analyze a non-stationary dynamic matching and bargaining game with a finite number of agents on each side of the market as, e.g., in Gale (2000) or in Rubinstein and Wolinsky (1990). This analysis allows for single agents to affect the overall market. We informally go over the line of reasoning suggested by our proposition. Sketch a game as follows: Suppose M_k agents enter in the first period on each side having costs and valuations drawn independently from distributions G^S and G^B . No further entry takes place. At the beginning of each period $t \in \{1, 2, ..\}$ there is a stock of remaining traders. Each seller has a chance $\alpha \in (0, 1)$ to be matched with a randomly drawn trader from the stock. In each match between a seller and a buyer, the seller makes a price offer and the buyer accepts or rejects the offer. Pairs of traders who reach an agreement leave the market, while those who do not reach an agreement have a chance $(1 - \delta_k)$ to remain in the stock until the beginning of the next period. For each M_k and δ_k, an equilibrium of the game sketched before leads to an outcome A_k, which is defined with respect to the expected payoffs and trading probabilities of agents in the first period.

For a variant of this game with *complete information* (actions and types are observable) and with a *fixed number of players* M_k, Rubinstein and Wolinsky (1990) have shown that there exist a sequence of equilibrium outcomes $\{A_k\}$ that does not converge to the competitive outcome when δ_k converges to zero. As discussed in Section 4.4, this non-convergence can be attributed to a failure of Weak Pairwise Efficiency: Traders do not have an incentive to deviate from the equilibrium in order to realize sure gains from trade because of the impact of their deviation on their future trading opportunities. In effect, for small enough δ_k, a fixed number of traders describes a small economy in which traders interact frequently and non-competitive outcomes can be sustained through threats and rewards, analogously to repeated games.

If, however, actions and types are unobservable *and* the number M_k grows large, then our result suggests that the equilibrium outcome becomes Walrasian when δ_k converges to zero and M_k grows to infinity fast enough at the same time: First, since types are private information, the Monotonicity condition and the No Rent Extraction condition have to hold. Second, if some types of agents do not trade with certainty, they become available. Finally, traders have an incentive to deviate from an equilibrium in order to realize gains from trade: when M_k is large relative to δ_k, deviations cannot trigger a substantial change in continuation payoffs since an agent will almost never meet the same partner again (or the partner's partner or the partner's partner's partner...). Thus, according to our analysis, whether or not a market is large enough must be determined relative to the interaction frequency given by δ_k.

6.2 Double Auctions

Apart from dynamic matching and bargaining games, the analysis of double auctions with large number of bidders is a prominent strand of the literature that provides microfoundations for general equilibrium predictions. For example, Satterthwaite and Williams (1989) model a double auction as follows: There are M_k bidders on each side of the market having costs and valuations distributed according to G^S and G^B. All traders submit bids. The $(M_k)\,th$ highest bid is chosen as the market clearing bid. For each M_k, an equilibrium defines an outcome A_k, that is, interim expected payoffs and trading probabilities for all types. When the number of traders becomes large, the trading outcome becomes Walrasian, $A_k \to A^W$. We can apply our result to this sequence of outcomes to answer the following question: Is convergence in a double auction with a large number of traders and the microfoundation provided thereby different from the convergence in a dynamic matching and bargaining game?

To see the relationship between double auctions and dynamic matching and bargaining games, let us indicate how to use the theorem here, using to observations. First, bids of buyers and sellers must be strictly increasing. If there were a positive probability of a tie, a trader would have an incentive to change his bid. Second, attention is usually restricted to equilibria in undominated strategies. As in a second price auction, this implies here that one can restrict attention to equilibria in which buyers bid truthfully; $b(v) = v$. Now, let us check the conditions. The first two conditions are straightforward: Since bidders' types are private information, outcomes must satisfy Monotonicity and No Rent Extraction. To apply the other two conditions, we need to define availability in an auction. Let us say a buyer v is *available* if the bid of the buyer is below the market clearing bid. In this event, a seller could make a bid that is low enough and be sure that he trades with a buyer with a valuation of at least v (using monotonicity of bids). With this definition of Availability in place, one can show that Availability and Weak Pairwise Efficiency hold for any sequence of equilibrium outcomes A_k when the number of traders grows large.

7 Concluding Remarks

We have introduced a new approach to the analysis of decentralized markets with vanishing frictions. By directly characterizing sequences of trading outcomes we have shown which conditions imply convergence to the competitive outcome. Then we have applied this approach to a variety of models to understand which economic forces drive or inhibit convergence when frictions become small. The incentives to realize gains from trade have been discussed as being crucial to ensure the competitive outcome.

The general approach is not confined to the analysis of dynamic matching and bargaining games. The approach covers games that map economies into trading outcomes. Within this unifying framework we uncovered structural connection between the proofs of convergence in different types of institutions. For example, we have discussed how to apply the approach to large double auctions when the number of bidders grows large. In general, the dependence of the analysis on the details of a trading institution is one of the fundamental problems when providing non-cooperative microfoundations for the competitive model. Our approach addresses this problem and provides a meaningful way of talking about convergence across different trading institutions. At the same time, our approach is firmly rooted in the non-cooperative literature and we have verified our approach for a class of well-defined games.

There are many other ways in which the approach might be used to contribute to our understanding of decentralized markets. For example, in our example we assumed that traders are rational and that they know the equilibrium. Each of these assumptions might be considered restrictive. Could we also get convergence when players are not perfectly rational or purely self interested?[12] The general approach of this paper allows us to check whether specific behavioral rules would lead to a competitive outcome. As long as the behavioral rules imply that the four conditions hold, the limit outcome will be Walrasian. Thereby, future research could prove the robustness of the Walrasian outcome.

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