

Dynamic Matching and Bargaining Games: A General Approach

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Abstract

Dynamic matching and bargaining games provide models of decentralized markets with trading frictions. A central objective of the literature is to investigate how equilibrium outcomes depend on the size of the frictions. In particular, will the outcome become efficient when frictions become small? Existing specifications of such games give different answers. To investigate what causes these differences, we identify four simple conditions on trading outcomes. We show that for every game which satisfies these conditions, the equilibrium outcome must become efficient when frictions are small. We demonstrate that our conditions hold under several specifications in the literature, suggesting a common cause behind their convergence results. These specifications include, for example, the recent contribution by Satterthwaite and Shneyerov (*Econometrica*, forthcoming.) For those specifications in the literature for which outcomes do not become efficient, we show exactly which of our conditions do not hold. These specifications include, for example, Serrano (2002, *JME*) and DeFraja and Sakovics (2001, *JPE*).

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1 Introduction

In a dynamic matching and bargaining game, a large population of traders interacts repeatedly in a decentralized market. Every trading period, traders are *matched* to form small groups where they *bargain* over the terms of trade. If they fail to reach an agreement, at some cost they can wait until the next period to be rematched into a new group. These waiting costs are the *frictions* of trading in the decentralized market. A major question in the literature concerns the trading outcome when frictions become small: Will the outcome become efficient? Ideally, one would like not only to find answers for particular trading institutions but also to gain a general understanding of the conditions under which trading with vanishing frictions has this property and the conditions under which it does not. This is the task of this paper. Its primary goal is to provide a general, "detail free" framework for the analysis of decentralized markets. Recent contributions that fall into the framework of this paper include papers by Moreno and Wooders (2001), Satterthwaite and Shneyerov (2006), and De Fraja and Sakovics (2001).

As a basic setup we use the following dynamic matching and bargaining environment, similar to the one used by Gale (1987):¹ There is a continuum of buyers who have unit demand and valuations $v \in [0, 1]$ for an indivisible good, and there is a continuum of sellers who have unit capacity and costs $c \in [0, 1]$. These traders are matched into small groups. In these groups they bargain, and if they reach an agreement, they trade. The groups are connected to form a large market by allowing unsuccessful traders to be matched into new groups in the next period. Integration, however, is imperfect because there is a probability $\delta \in (0, 1)$ that a trader will die while waiting. These are the *frictions* of trading. Finally, at the end of each period, there is an exogenous inflow of new buyers and sellers.

This framework is general with respect to both the matching technology and the bargaining protocol, i.e., we do not specify how traders are matched into groups. Also, we do not specify how bargaining within the groups takes place and what information is released before and during bargaining. We will see how existing models in the literature differ in how they fill in these details. But no matter how this is done, every specification of the model will give rise to an *outcome* which consists of (a) probabilities of trading for entering types and (b) expected equilibrium payoffs. Let $Q^S(c)$ denote the probability that a seller of type c sells his good, and let $Q^B(v)$ denote the probability that a buyer of type v gets the good. Similarly, let $V^S(c)$ and $V^B(v)$ be the payoffs to these types. Taken together, an outcome is a vector $A = [Q^S, Q^B, V^S, V^B]$.

Now, suppose there is some sequence of exit rates $\{\delta_k\}_{k=1}^\infty$, which converges to zero, $\delta_k \rightarrow 0$. In addition, suppose for each δ_k , we take an equilibrium outcome of a specific trading game. This gives us a sequence of outcomes $\{A_k\}_{k=1}^\infty$, with $A_k =$

¹The main difference is our assumption that traders have a finite life expectancy; see Section 6.2 for the case of infinitely lived traders.

$[Q_k^S, Q_k^B, V_k^S, V_k^B]$. We state four conditions on this sequence that will ensure that its limit is efficient. The first condition, *monotonicity*, requires that trading probabilities are monotone, i.e., buyers with higher valuations are more likely to trade while sellers with higher costs are less likely to trade. The second condition, *no rent extraction*, requires that traders receive some part of the surplus that they generate. Technically, this is a condition on the slope of the payoffs. The third condition, *availability*, requires that a trader is matched frequently with those traders who do not trade with certainty and who remain in the pool for many periods. These traders are said to be available. The fourth condition, *weak pairwise efficiency*, requires that for all pairs of buyers and sellers who are available, i.e., for all pairs who are matched frequently, the joint surplus is at least their private surplus and $V^S(c) + V^B(v) \geq (v - c)$. Note that the third condition relates to the matching technology, while the other conditions relate to the bargaining protocol. As we will see, these conditions hold for a wide range of models.

Next, let $S(A_k)$ be the surplus of an outcome A_k , and let S^* be its maximum over the set of all outcomes satisfying a mass balance condition (see Section 3.2). Our main result is this: Every sequence of outcomes $\{A_k\}_{k=1}^\infty$ which satisfies the four conditions becomes efficient when δ becomes small, i.e.,

$$\lim_{k \rightarrow \infty} S(A_k) = S^*.$$

In the second part of the paper, we discuss specific dynamic matching and bargaining games. In particular, we use these games to discuss the economic meaning of our conditions. To keep this part consistent, we introduce the games by varying a *basic model* which we take from Lauermaun (2006a). This model is particularly simple: Groups consist of just one seller and one buyer. Bargaining takes place by the seller making a price offer to the buyer, which the buyer can either accept or reject. The seller cannot observe the valuation of the buyer, i.e., information is asymmetric in the basic model. The setup of this model is introduced in the next section - and before the general framework - to allow the reader to familiarize himself with the environment.

We introduce the first variant of our model to show that our conditions do not only hold with asymmetric information, but also in a model similar to Douglas Gale's own specification with *symmetric* information. We then provide some intuition that the analysis is not confined to steady states by considering the non-steady-state model of Moreno and Wooders (2001), where traders enter only in the first period. We also provide some intuition for the case of bargaining between one seller and many buyers, assuming that the seller holds a second price auction.

Auctions are also used to specify the bargaining protocol in the model of Satterthwaite and Shneyerov (2006). But, like Gale (1987), they include an entry stage and assume infinitely lived traders. To show that our approach is also valid with these additional complications, we extend our general framework by including an entry decision

and by considering the case where the exit rate δ is set equal to zero. In this new framework, we need stronger conditions to ensure convergence to efficiency. This is discussed in detail in Section 6.

Whenever convergence to efficiency fails in some model, at least one of our conditions must be violated. By pointing out exactly which conditions are violated, we show which assumptions of the model are the reasons for the non-convergence results. In particular, we show that the failure in Lauer mann (2006b) can be traced back to rent extraction (a failure of the second condition), the failure in Serrano (2002), to the failure of weak efficiency (the fourth condition), and the failure in De Fraja and Sakovics (2001), to a failure of a fundamental mass balance condition.

The rest of the paper is structured as follows. First, we introduce the basic model as an example in Section 2. Then we provide the general framework in Section 3.1, and in Section 3.2 we discuss necessary and sufficient conditions for outcomes to be efficient. In Section 3.3 we introduce the four conditions on outcomes. We prove our main result in Section 4: every sequence of outcomes that meets the four conditions becomes efficient. In Section 5 we demonstrate that the conditions are met in some examples. We introduce some variations of the general framework by adding an entry stage (Section 6.1) and assuming that traders are infinitely lived (Section 6.2). Failures of convergence to efficiency are discussed in Section 7.

2 The Basic Model - An Example of a Dynamic Matching and Bargaining Game

In this section, we introduce a specific dynamic matching and bargaining game. We will use this game to motivate and illustrate the general framework.

We assume that there is a continuum of buyers and sellers who interact in a repeated market over an infinite number of periods, with time running from minus to plus infinity. Sellers have one unit of an indivisible good, and their costs of trading are $c \in [0, 1]$. Buyers want to acquire one unit of the good, and their valuation for the good is $v \in [0, 1]$. At the beginning of each period, there is some pool of buyers and sellers. The traders from this pool are matched into pairs consisting of one seller and one buyer. Within each pair, the seller announces a price offer $p \in [0, 1]$ and the buyer announces whether he rejects or accepts the offer. If he accepts, the seller receives $p - c$, while the buyer receives $v - p$. Next, all buyers and sellers who have traded exit the pool. Likewise, a share δ of all those traders who failed to trade exits. Finally, new players enter the market and the period ends. The next period starts, using the same rules. We will look at the steady-state equilibrium of this market.

The inflow of buyers and the inflow of sellers each have mass one. The distribution of valuations among buyers in the inflow is exogeneously given by some c.d.f. $G^B(\cdot)$ and,

similarly, the distribution of costs is given by some distribution $G^S(\cdot)$. We assume that $G^B(\cdot)$ and $G^S(\cdot)$ have continuous and strictly positive densities. Let p^w be the price at which the mass of sellers in the inflow with costs below p^w is exactly equal to the mass of buyers with valuations above p^w :

$$G^S(p^w) = 1 - G^B(p^w). \quad (1)$$

Since the left hand side is strictly increasing while the right hand side is strictly decreasing, the solution to (1) is unique. The function $G^S(\cdot)$ can be interpreted as *supply*, and $1 - G^B(p^w)$ can be interpreted as *demand*. So p^w is the price at which supply equals demand, i.e., p^w is the *Walrasian* market clearing price relative to the inflow.

The market constellation is given by a vector $\sigma = [p(\cdot), r(\cdot), \Phi^S(\cdot), \Phi^B(\cdot), M]$, where $p(c) \in [0, 1]$ is the price offered by a seller of type c , $r(v) \in [0, 1]$ is the highest price accepted by a buyer of type v , $\Phi^S(\cdot)$ is the cumulative distribution function of costs in the pool of sellers, $\Phi^B(\cdot)$ is the corresponding distribution function for buyers, and M is the total mass of buyers in the pool, which is equal to the total mass of sellers in a steady state. For the analysis, we assume that all functions under consideration are measurable. With Σ_M being the set of measurable functions $f : [0, 1] \rightarrow [0, 1]$, σ is an element of $\Sigma \equiv \Sigma_M^4 \times \mathbb{R}$.

We say that a vector σ constitutes an *equilibrium* if strategies are mutually optimal given the distribution of types and if the distribution of types in the pool is consistent with the trading strategies and the exogeneous inflow. These conditions are now spelled out in detail.

First we turn to the sellers. If the seller offers a price p , let us denote by $D(p|\sigma)$ the probability that the buyer will accept the offer. Buyers accept a price p if $p \leq r(v)$ (see below), so $D(p|\sigma)$ is

$$D(p|\sigma) \equiv \int_{\{v|p \leq r(v)\}} d\Phi^B(v). \quad (2)$$

Let $q^S(p|\sigma, \delta)$ be the probability that a seller can trade some time during his lifetime

$$q^S(p|\sigma, \delta) \equiv \frac{D(p|\sigma)}{1 - (1 - D(p|\sigma))(1 - \delta)}, \quad (3)$$

which we also call the *lifetime trading probability*, and which is derived from the recursive formula

$$q^S(p|\sigma, \delta) = D(p|\sigma) + (1 - D(p|\sigma))(1 - \delta)q^S(p|\sigma, \delta).$$

The expected payoff to a seller when offering a price p is his trading probability times his profit, i.e.,

$$U^S(p, c|\sigma, \delta) \equiv q^S(p|\sigma, \delta)(p - c),$$

and we require that $p(c) \in \arg \max U^S(\cdot, c|\sigma, \delta)$ for all c in equilibrium.

For buyers, let $S(r|\sigma)$ denote the probability of receiving an offer $p \leq r$ in any period. Again, we define the lifetime trading probability by

$$q^B(r|\sigma, \delta) \equiv \frac{S(r|\sigma)}{1 - (1 - S(r|\sigma))(1 - \delta)}.$$

The expected price offer conditional on $p \leq r$ is denoted by $E[p|p \leq r, \sigma]$.² Payoffs when accepting all $p \leq r$ are given by

$$U^B(r, v|\sigma, \delta) \equiv q^B(r|\sigma, \delta)(v - E[p|p \leq r, \sigma]). \quad (4)$$

Let $V^B(v|\sigma) \equiv \sup_r U^B(r, v|\sigma, \delta)$. Suppose that the following condition holds

$$r(v) = v - (1 - \delta)V^B(v|\sigma, \delta). \quad (5)$$

Then a buyer who receives an offer $p = r(v)$ is just indifferent about accepting and rejecting the offer: his payoff if accepting the offer, $v - r(v)$, is equal to his payoff if rejecting it, which is the continuation payoff $(1 - \delta)V^B(v|\sigma, \delta)$. If $p < r(v)$, the buyer is strictly better off when accepting the offer, and when $p > r(v)$, the buyer is strictly better off rejecting the offer. Hence, it is optimal for a buyer to accept a price if it is below $r(v)$ and to reject the price otherwise.³

We restrict attention to steady-state equilibria in which the pool does not change over time. If the distribution at the beginning of a period is given by $\Phi_t^S(\cdot)$ and the trading strategies are $r(\cdot)$ and $p(\cdot)$, then the distribution of sellers at the end of the period is the sum of the entering and the initial sellers who have neither traded nor died:

$$\Phi_{t+1}^S(c|\sigma) = G^S(c) + (1 - \delta) \int_0^c (1 - D(p(\tau))) d\Phi_t^S(\tau).$$

The pool of traders is in a steady state if and only if the distribution of types does not change over time. For sellers it is necessary that $\Phi_{t+1}^S(c|\sigma) = \Phi_t^S(c) = \Phi^S(c)$ for all c . This condition can be written as⁴

$$\Phi^S(c) = \int_0^c \frac{dG^S(\tau)}{M(D(p(\tau)|\sigma) + \delta(1 - D(p(\tau)|\sigma)))} \quad \text{for all } c. \quad (6)$$

²If $Q^B(r) = 0$, then $E[p|p \leq r] \equiv r$.

³In general, reservation price strategies are the unique optimal sequentially rational strategies when sampling without recall from a known stationary distribution of prices; see McMillan and Rothschild (1994). In our model, we nevertheless simply *assume* that buyers use reservation prices to avoid unnecessary notation.

⁴We get this by an algebraic manipulation of $\Phi^S(c) = \Phi_{t+1}^S(c)$, by observing that for all c , $\int_0^c d\Phi^S(\tau) = \int_0^c dG^S(\tau) + \int_0^c (1 - \delta)(1 - D(p(\tau))) d\Phi^S(c)$ and then $\int_0^c d\Phi^S(\tau) - \frac{dG^S(\tau)}{1 - (1 - \delta)(1 - D(p(\tau)))} = 0$.

A similar condition can be obtained for buyers:

$$\Phi^B(v) = \int_0^v \frac{dG^B(\tau)}{M(S(r(\tau)|\sigma) + \delta(1 - S(r(\tau)|\sigma)))} \quad \text{for all } v. \quad (7)$$

Summing up, we say σ^* is an equilibrium if it satisfies the above conditions:

Definition 1 *A steady-state equilibrium vector $\sigma^* \in \Sigma$ consists of an optimal pair of strategies and a corresponding steady-state pool, i.e., σ^* is a vector $[p(\cdot), r(\cdot), \Phi^S(\cdot), \Phi^B(\cdot), M]$ for which*

- $p(c) \in \arg \max U^S(p, c|\sigma^*, \delta)$ for all c
- $r(v) = v - (1 - \delta)V^B(v|\sigma^*, \delta)$ for all v
- $\Phi^S(\cdot), \Phi^B(\cdot)$, and M satisfy the steady-state conditions (6), (7).

Every equilibrium σ^* and exit rate δ , gives rise to a trading *outcome* as follows. Let $V^S(c|\sigma^*, \delta) \equiv U^S(p(c), c|\sigma^*, \delta)$ and $V^B(v|\sigma^*, \delta) \equiv U^B(r(v), v|\sigma^*, \delta)$ be the equilibrium payoffs and let $Q^S(c|\sigma^*, \delta) = q^S(p(c)|\sigma^*, \delta)$ and $Q^B(v|\sigma^*, \delta) = q^B(r(v)|\sigma^*, \delta)$ be the equilibrium trading probabilities. Then the outcome $A = [V^S, V^B, Q^S, Q^B]$ of the equilibrium σ^* is given by the mapping $A(\cdot, \cdot) : \Sigma \times [0, 1] \rightarrow \Sigma_M^4$, i.e., $A(\sigma^*, \delta) = [V^S(\cdot|\sigma^*, \delta), V^B(\cdot|\sigma^*, \delta), Q^S(\cdot|\sigma^*, \delta), Q^B(\cdot|\sigma^*, \delta)]$. As we will see in Section 5, outcomes can be identified across a wide range of specifications. Therefore, we now turn to a general discussion of outcomes.

3 The General Approach

For the general approach, we introduce the basic notation and make some preliminary observations in the first subsection. In the following subsection, we show that outcomes are efficient if they are "Walrasian", and we derive a sufficient condition for efficiency. Finally, we introduce the four conditions that we want to use to characterize outcomes.

3.1 Outcomes

An *outcome* is a vector $A = [V^S(\cdot), V^B(\cdot), Q^S(\cdot), Q^B(\cdot)]$, where $V^S(c)$ is the expected payoff of an entering seller of type c and $Q^S(c)$ is his (lifetime) trading probability. Similarly, $V^B(v)$ is the expected payoff of an entering buyer of type v and $Q^B(v)$ is his (lifetime) trading probability. We define $T^S(\cdot)$ and $T^B(\cdot)$ implicitly by

$$V^S(c) = T^S(c) - cQ^S(c) \quad \text{and} \quad V^B(v) = vQ^B(v) - T^B(v). \quad (8)$$

Because we assume that there is no discounting,⁵ $T^S(\cdot)$ and $T^B(\cdot)$ can be interpreted as expected transfers. Given an outcome A , the surplus of entering traders is defined as

$$S(A|G^S(\cdot), G^B(\cdot)) \equiv \int_0^1 V^B(v) dG^B(v) + \int_0^1 V^S(c) dG^S(c).$$

The distribution of types $G^S(\cdot)$ and $G^B(\cdot)$, together with the size of the friction $\delta \in (0, 1)$, describe our economy.⁶ We assume that both distributions are smooth and strictly increasing. In particular, we assume that $G^S(\cdot)$ and $G^B(\cdot)$ have continuous and strictly positive densities. This assumption is identical to the one made in the basic model, and it ensures that there is a unique Walrasian price p^w which satisfies $G^S(p^w) = 1 - G^B(p^w)$. In the sequel, we take $G^S(\cdot)$ and $G^B(\cdot)$ as fixed and drop them from the argument.

For our analysis, we assume that all components of A are elements of the set of measurable functions, i.e., $A \in \Sigma_M^4$ and $S(\cdot) : \Sigma_M^4 \rightarrow \mathbb{R}$. With the Lebesgue integral, we can define a *distance* between two functions, $d(\cdot, \cdot) : \Sigma_M^2 \rightarrow [0, 1]$ with $d(f_1, f_2) = \int_0^1 |f_1(x) - f_2(x)| dx$.⁷ We use $d(\cdot, \cdot)$ to define convergence in Σ_M . In many cases, we can find conditions that ensure that the set of functions is restricted to the set of monotone functions. This will turn out to be helpful because every sequence of monotone functions has a convergent subsequence, i.e., sets of monotone functions are sequentially compact.⁸ For future references, let $\Sigma_+ \subset \Sigma_M$ be the subset of weakly increasing functions and let $\Sigma_- \subset \Sigma_M$ be the subset of weakly decreasing functions.

A natural consistency requirement on an outcome is that total transfers collectively made by all buyers are equal to total transfers received by all sellers, $\int_0^1 T^B(v) dG^B(v) = \int_0^1 T^S(c) dG^S(c)$. From (8), it follows that this is equivalent to the following condition on A :

$$\int_0^1 (v Q^B(v) - V^B(v)) dG^B(v) = \int_0^1 (V^S(c) + c Q^S(c)) dG^S(c). \quad (9)$$

Define

$$S_Q(A) \equiv \int_0^1 v Q^B(v) dG^B(v) - \int_0^1 c Q^S(c) dG^S(c).$$

⁵When we consider infinitely lived agents in Section 6.2, we also discuss the implications of a model with discounting. We show that the basic insights still apply.

⁶The parameters δ and G^S and G^B can be interpreted more broadly. In Section 6.2 the exit rate δ is replaced by a discount factor β , and in Section 5.3 we suggest that one can interpret G^S and G^B as the population at the beginning of a non-steady state dynamic matching and bargaining game that has no further entry.

⁷Note that $d(\cdot, \cdot)$ is only a semimetric: $d(f_1, f_2) = 0$ does not imply $f_1 = f_2$. Still $d(f_1, f_2)$ is non-negative and symmetric, and it satisfies $d(f_1, f_1) = 0$ and the triangle inequality. We endow Σ_M with the semimetric topology (see Aliprantis, Border (1994, p. 23)), defined in the usual way by using open ε -balls $B_\varepsilon(f_1) = \{f \in \Sigma_M | d(f_1, f) < \varepsilon\}$, to define open sets just as in a metric space.

⁸According to Helly's selection theorem (see Kolmogorov, Fomin (1970, p. 372)), every sequence $\{f_N\}_{N=1}^\infty$ of monotone functions has a pointwise convergent subsequence $\{f_{N'}\}_{N'=1}^\infty$. Lebesgue's bounded convergence theorem implies $d(f_{N'}, \bar{f}) \rightarrow 0$ for some \bar{f} . The limit \bar{f} is clearly monotone itself.

Condition (9) is equivalent to

$$S(A) = S_Q(A). \quad (10)$$

This equality reflects the idea that, for the purpose of welfare analysis, only the allocation of the good matters while transfers cancel.

Similar to the balance of transfers, the total mass of buyers who trade is required to be equal to the total mass of sellers who trade:

$$\int_0^1 Q^S(c) dG^S(c) = \int_0^1 Q^B(v) dG^B(v). \quad (11)$$

Economically, this condition corresponds to the scarcity of the good: For every buyer who enjoys consumption, there must be some seller who incurs costs. We define the set \hat{Q} of all trading outcomes satisfying the balance of total trades:

$$\hat{Q} \equiv \{Q^S(\cdot), Q^B(\cdot) \in \Sigma_M^2 \mid \text{condition (11) holds}\}.$$

An outcome A satisfies *mass balance* if it satisfies the two consistency conditions:

Definition 2 Mass balance. *An outcome $A = [V^S(\cdot), V^B(\cdot), Q^S(\cdot), Q^B(\cdot)]$ is said to satisfy mass balance if*

$$A \in \hat{A} \equiv \left\{ A \mid Q \in \hat{Q} \text{ and } S(A) = S_Q(A) \right\}.$$

We say that a *sequence* of outcomes $\{A_k\}_{k=1}^\infty$ satisfies mass balance if each of its members A_k is in \hat{A} .

3.2 Efficiency

Our object of interest is the maximal surplus that can be reached subject to the resource constraint $Q \in \hat{Q}$:

$$S^* \equiv \sup_{A \in \hat{A}} S_Q(\cdot).$$

Basic economic intuition suggests that the optimal allocation is the following: All buyers with valuations above the market clearing price p^w get the good, while all sellers with costs below p^w sell theirs; buyers with lower valuations and sellers with higher costs do not trade. Let Q^W be the set of *Walrasian* allocations of the good that are equivalent⁹ to this rule:

$$Q^W \equiv \left\{ Q \in \hat{Q} \mid \int_0^1 |Q^S(c) - 1_{c \leq p^w}(c)| dc = 0, \int_0^1 |Q^B(v) - 1_{v \geq p^w}(v)| dv = 0 \right\}. \quad (12)$$

⁹Two functions Q_1 and Q_2 are equivalent if $d(Q_1^s, Q_2^s) = d(Q_1^b, Q_2^b) = 0$.

It is straightforward to prove that indeed an outcome is efficient if and only if it is in Q^W (see the appendix for details)¹⁰:

Lemma 1 *For all outcomes that satisfy mass balance, i.e., for all $A \in \hat{A}$: $S(A) = S^*$ if and only if $Q \in Q^W$.*

Accordingly, the maximal surplus S^* is given by:

$$S^* = \int_{p^w}^1 v dG^B(v) - \int_0^{p^w} c dG^S(c). \quad (13)$$

Let \hat{Q}_+ be the set of trading probabilities which are monotone and which satisfy mass balance of trades i.e., $\hat{Q}_+ \equiv \{Q \in \hat{Q} \mid Q^S \in \Sigma_-, Q^B \in \Sigma_+\}$. Because \hat{Q}_+ is sequentially compact, we can show that the former lemma also holds in the limit: a sequence of outcomes $\{Q_k\}_{k=1}^\infty$ becomes efficient if and only if it converges to the set Q^W . (We say that a sequence $\{Q_k\}_{k=1}^\infty$ converges to Q^W if its distance to any element of Q^W becomes zero in every component.) The proof of the following lemma is relegated to the appendix:

Lemma 2 *For every sequence $\{A_k\}_{k=1}^\infty$ with $A_k \in \hat{A}$ and with $Q_k \in \hat{Q}_+$:*

$$\lim_{k \rightarrow \infty} S_Q(A_k) = S^* \quad \text{if and only if} \quad \lim_{k \rightarrow \infty} Q_k = Q^W.$$

Next, we derive a simple sufficient condition for the efficiency of an outcome: Suppose an outcome A is such that, for any cost c and for any valuation v , joint payoffs $V^S(c) + V^B(v)$ are weakly larger than the private surplus $v - c$. It is easy to show that this implies that A is efficient, i.e., $S(A) = S^*$:

Lemma 3 Sufficiency. *If some outcome satisfies mass balance, i.e., if $A \in \hat{A}$ and if for all v and c , $V^S(c) + V^B(v) \geq v - c$, then $S(A) = S^*$.*

Proof: Let $\bar{p} \equiv \inf_{c \leq p^w} (V^S(c) + c)$. Then $V^S(c) + V^B(v) \geq v - c$ for all v and c implies $V^B(v) \geq v - \inf_{c \leq p^w} (V^S(c) + c)$ for all v . Together with the definition of \bar{p} , we use this to bound $S(A)$:

$$\begin{aligned} S(A) &\geq \int_{p^w}^1 V^B(v) dG^B(v) + \int_0^{p^w} V^S(c) dG^S(c) \\ &\geq \int_{p^w}^1 (v - \bar{p}) dG^B(v) + \int_0^{p^w} (\bar{p} - c) dG^S(c) \\ &= S^* + \bar{p} (G^S(p^w) - (1 - G^B(p^w))) = S^*, \end{aligned}$$

¹⁰Note, however, that in many models of the literature, this relation between efficiency and the Walrasian allocation is less straightforward: With infinitely lived agents, as, e.g., in Gale (1987) we need to take care of "Ponzi-Schemes" and with cloning, as e.g., in the model by DeFraja and Sakovics (2001) the definition of surplus itself becomes problematic (see Section 6.2 and 7.3, respectively).

where the last line follows from the definition of p^w . By the restriction $A \in \hat{A}$ and by the definition of S^* , $S(A) \leq S^*$. Therefore, $S(A) \geq S^*$ implies $S(A) = S^*$. ■

By continuity of $S(\cdot)$, the last lemma carries over to sequences (see the appendix for details). For technical reasons, we restrict the elements of A_k to be in the set of outcomes which satisfy mass balance and which are monotone in each component, $\hat{A}_+ \equiv \hat{A} \cap [\Sigma_- \times \Sigma_+ \times \Sigma_- \times \Sigma_+]$:

Lemma 4 *For every sequence $\{A_k\}_{k=1}^\infty$ with $A_k \in \hat{A}_+$*

$$\lim_{k \rightarrow \infty} S(A_k) = S^* \quad \text{if} \quad \liminf_{k \rightarrow \infty} [V_k^S(c) + V_k^B(v)] \geq v - c \quad \text{for all } v, c.$$

3.3 General Conditions

We take some sequence of exit rates $\{\delta_k\}_{k=1}^\infty$, and for each exit rate δ_k we take some outcome A_k . This gives us a sequence $\{A_k\}_{k=1}^\infty$. We now define four conditions for this sequence. In the next section, we show that if the sequence satisfies these conditions, then its limit is efficient.

In the following, we denote pointwise limits by upper bars. For sequences of trading probabilities, we define

$$\bar{Q}^S(c) \equiv \lim_{k \rightarrow \infty} Q_k^S(c), \text{ and } \bar{Q}^B(v) \equiv \lim_{k \rightarrow \infty} Q_k^B(v),$$

whenever these limits exist. For sequences of payoffs, we define analogously

$$\bar{V}^S(c) \equiv \lim_{k \rightarrow \infty} V_k^S(c) \text{ and } \bar{V}^B(v) \equiv \lim_{k \rightarrow \infty} V_k^B(v).$$

We motivate the first two conditions by the trading situation with asymmetric information in the basic model. While we provide here only a sketch of the idea, we prove in Section 5.1 in detail that the conditions hold. In Section 5.2 we show that the conditions also hold with symmetric information when bargaining power is intermediate (see also remark 3).¹¹

The main observation for the basic model is that, with asymmetric information, the revelation principle requires that the trading outcome is incentive compatible. Intuitively, a type c can *mimic* the strategy of another type c_x . If he does so, he receives a transfer $T_k^S(c_x)$ and trades with probability $Q_k^S(c_x)$. Thus, for $V_k^S(c)$ to be the equilibrium payoff for type c , $V_k^S(c)$ must be at least as large as $T_k^S(c_x) - c Q_k^S(c_x)$. The same observations apply to buyers.

It is standard to verify that incentive compatibility requires that trading probabilities are monotone (for details, see Section 5.1):

¹¹See also Remark 7 in Section 7.1 for the case of noisy information.

Condition 1 Monotonicity. A sequence $\{A_k\}_{k=1}^\infty$ satisfies monotonicity if every member A_k has monotone trading probabilities:

$$Q_k^S(\cdot) \in \Sigma_- \quad \text{and} \quad Q_k^B(\cdot) \in \Sigma_+.$$

Also, by standard reasoning, incentive compatibility imposes a restriction on the slopes of payoff functions: The difference between the payoffs of two types cannot be too large for otherwise one of these types would have an incentive to mimic the other (see Section 5.1, equation (18) for details). This is reflected in the following condition, which requires that the slope is bounded between zero and one. We require a tighter bound if a type c_x trades with certainty in the limit. In this case, every other type could mimic him and receive at least the same revenue. The payoff difference would be entirely due to the difference in their costs, i.e., payoffs from mimicking the type c_x change with a slope of one.

Condition 2 No Rent Extraction. A sequence $\{A_k\}_{k=1}^\infty$ satisfies no rent extraction if for every member A_k of the sequence $\{A_k\}_{k=1}^\infty$ and for every $c, c_x \in [0, 1]$ and $v, v_x \in [0, 1]$ there is some $a \in [0, 1]$ such that

$$V_k^S(c) \geq V_k^S(c_x) + a(c_x - c) \quad \text{and} \quad V_k^B(v) \geq V_k^B(v_x) + a(v - v_x).$$

In addition, whenever $\bar{Q}^S(c_x)$ and $\bar{V}^S(c_x)$ exist and $\bar{Q}^S(c_x) = 1$, then $\liminf V_k^S(c) \geq \bar{V}^S(c_x) + (c_x - c)$ for all c . Symmetrically, whenever $\bar{Q}^B(v_x)$ and $\bar{V}^B(v_x)$ exist and $\bar{Q}^B(v_x) = 1$ then $\liminf V_k^B(v) \geq \bar{V}^B(v_x) + (v - v_x)$ for all v .

The no rent extraction property implies monotonicity and continuity of the payoff functions, something we will utilize in the proof. In particular, monotonicity and continuity carry over to the limiting functions \bar{V}^S and \bar{V}^B .¹²

Remark 1 With asymmetric information, it is well known from the proof of the envelope theorem (see, e.g., Milgrom and Segal (2002)) that we can state the bound more tightly as $V_k^S(c) \geq V_k^S(c_x) + Q_k^S(c_x)(c_x - c)$ (see inequality (18)). The same applies to buyers, of course. However, here we want to find conditions which are just strong enough to imply the convergence result, but still weak enough so that they hold in a wide range of models. In particular, we want to include the possibility of symmetric information, and, in this case, we can only require the weaker bounds that we stated in the condition.

For the next two conditions, we introduce the concept of *availability*. This is formalized by the introduction of an operator $L^B(\cdot) : [0, 1]^2 \times \Sigma_M^4 \rightarrow [0, 1]$. $L^B(v'', \delta_k, A_k)$ is

¹² According to the condition, all payoff functions must be Lipschitz continuous with Lipschitz constant 1, since $|V^S(c) - V^S(c_x)| \leq |c - c_x|$. Therefore, every sequence of such functions is equicontinuous; hence, its limit must be continuous whenever it exists (see Kolmogorov, Fomin (1970, p. 102)).

interpreted as the probability that a seller who is just passively waiting in the pool will be matched at least once with a buyer of type $v \geq v''$ before he dies, given the exit rate δ_k and the outcome A_k .¹³ Let $L_k^B(v) \equiv L^B(v, \delta_k, A_k)$ and let $\bar{L}^B(v) = \liminf_{k \rightarrow \infty} L_k^B(v)$. As we will demonstrate in the basic model, whenever some set of buyers does not trade with certainty in the limit, then this set is available, i.e., $L^B = 1$. Introducing a similar function $L^S(\cdot)$ for sellers, we state:

Condition 3 Availability. *A sequence $\{A_k\}_{k=1}^\infty$ satisfies availability relative to some pair of functions L^B and L^S if, whenever $\bar{Q}^B(v')$ exists for some v' and $\bar{Q}^B(v') < 1$, then $\bar{L}^B(v'')$ exists and $\bar{L}^B(v'') = 1$ for all $v'' < v'$; And if $\bar{Q}^S(c')$ exists for some c' and if $\bar{Q}^S(c') = 1$, then $\bar{L}^S(c'')$ exists and $\bar{L}^S(c'') < 1$ for all $c'' > c'$.*

Now, suppose it is commonly known that types c_x and v_x are *available*, i.e., buyers and sellers are mutually sure to meet some $c \leq c_x$ and some $v \geq v_x$, respectively. Then, intuitively, their joint payoffs should be ex ante pairwise efficient. Otherwise, their joint payoffs is below the surplus they could realize by trading. So it becomes certain that (a) between these types there is still "money left on the table", and (b) these types are certain to meet each other so that they can realize this additional surplus. This observation motivates the final condition:

Condition 4 Weak pairwise efficiency. *A sequence $\{A_k\}_{k=1}^\infty$ satisfies weak pairwise efficiency relative to some pair of functions L^B and L^S , if $\bar{L}^S(c_x) = 1$ and $\bar{L}^B(v_x) = 1$ for any pair of types c_x and v_x implies*

$$\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x.$$

But note well that the condition requires pairwise efficiency only with respect to those types which do not trade with certainty and which are available. Payoffs might still be inefficient for those pairs of types which trade with certainty and which are not available.

Remark 2 *Instead of giving two conditions, we could have stated a single condition that requires that whenever for some pair v_x and c_x , $\bar{Q}^B(v_x) < 1$ and $\bar{Q}^S(c_x) < 1$, then for all pairs v and c with $v < v_x$ and $c > c_x$, $V^S(c) + \bar{V}^B(v) \geq v - c$. Mathematically, the functions L^S and L^B are just arbitrary indicator functions that connect the two conditions. However, we stated them separately, since in the applications the first of the two conditions can be formulated as a condition on the matching technology while the second condition refers to the bilateral bargaining outcomes (see Section 5 and the definition of the functions L^S and L^B in (17) and (16)). As discussed in Section 7.2, these two condition can fail separately, i.e., economically, they are separate.*

¹³For an example, see the definition of $L^B(\cdot)$ in the basic model in equation (16).

4 Main Result

In this section, we state and prove our main result: Suppose there is a pair of functions L^S, L^B and a sequence of exit rates $\{\delta_k\}_{k=1}^\infty$ such that a given sequence of outcomes $\{A_k\}_{k=1}^\infty$ satisfies the conditions stated before. Then outcomes along this sequence become efficient:

Proposition 1 *Suppose some sequence $\{A_k\}_{k=1}^\infty$ satisfies mass balance, monotonicity, no rent extraction, and suppose it satisfies availability and weak pairwise efficiency relative to some sequence of frictions $\{\delta_k\}_{k=1}^\infty$ and to some pair of functions L^B, L^S . Then the outcome becomes efficient, i.e.,*

$$\lim_{k \rightarrow \infty} S(A_k) = S^*.$$

Proof: The monotonicity condition and the no rent extraction condition require that all components of each element A_k are monotone, i.e., $A_k \in \hat{A}_+$. By Helly's selection principle (see Kolmogorov, Fomin 1970), we can find a pointwise convergent subsequence $\{A_{k'}\}_{k'=1}^\infty$. Let its limit be denoted by $(\bar{V}^S, \bar{V}^B, \bar{Q}^S, \bar{Q}^B)$. We first show that (\bar{Q}^S, \bar{Q}^B) is in the set of Walrasian allocations Q^W for every such subsequence. Then we show that this is sufficient for $\lim_{k \rightarrow \infty} S_Q(A_k) = S^*$ for the sequence itself.

Given the subsequence $\{A_{k'}\}_{k'=1}^\infty$, define cutoff types c_x and v_x as the lowest cost and highest valuation, such that traders with these types do not trade with certainty in the limit, i.e.,

$$c_x \equiv \inf \{c, 1 | \bar{Q}^S(c) < 1\} \quad \text{and} \quad v_x \equiv \sup \{v, 0 | \bar{Q}^B(v) < 1\}.$$

First, we show that the no rent extraction conditions implies

$$\begin{aligned} \bar{V}^S(c) &\geq \bar{V}^S(c_x) + (c_x - c) && \text{for all } c, \\ \text{and} \quad \bar{V}^B(v) &\geq \bar{V}^B(v_x) + (v - v_x) && \text{for all } v. \end{aligned}$$

So the payoffs to all types can be bounded from below once we know the payoffs of the cutoff types. The first inequality follows directly for all types $c \in [c_x, 1]$ by the no rent extraction condition, observing that $(c_x - c)$ is negative. For types $[0, c_x]$, the inequality is trivially true if $c_x = 0$; if $c_x > 0$, choose some $\varepsilon \in (0, c_x)$ and note that $\bar{Q}^S(c_x - \varepsilon) = 1$ by definition of c_x and by monotonicity of $\bar{Q}(\cdot)$ (which is implied by the monotonicity of each element $Q_{k'}$). Hence, for all $c \leq c_x - \varepsilon$, the no rent extraction condition implies that $\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c) - \varepsilon$. Because $\bar{V}(\cdot)$ is continuous (see the statements following the no rent extraction condition), and because ε was chosen arbitrary, we get $\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c)$. So the first inequality holds for all $c \in [0, 1]$. The second inequality follows for all buyers by symmetric reasoning.

Adding the two inequalities yields a lower bound on the joint surplus of all c and v :

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c + \bar{V}^S(c_x) + \bar{V}^B(v_x) - (v_x - c_x). \quad (14)$$

We use the availability and the weak efficiency conditions to show that the right hand side is at least $(v - c)$:

We consider two cases for the ordering of c_x and v_x . First, suppose $c_x < v_x$. Take some $\varepsilon \in (0, v_x - c_x)$. By definition of c_x and v_x , and by monotonicity of $\bar{Q}^S(\cdot)$ and $\bar{Q}^B(\cdot)$, we have $\bar{Q}^S(c_x + 0.5\varepsilon) < 1$ and $\bar{Q}^B(v_x - 0.5\varepsilon) < 1$. The availability condition implies that $\bar{L}^S(c_x + \varepsilon) = \bar{L}^B(v_x - \varepsilon) = 1$. By the weak efficiency condition:

$$\bar{V}^S(c_x + \varepsilon) + \bar{V}^B(v_x - \varepsilon) \geq v_x - c_x - 2\varepsilon.$$

Since the sum $\bar{V}^S(\cdot) + \bar{V}^B(\cdot)$ is continuous and ε is arbitrary:

$$\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x.$$

Now consider the case $v_x \leq c_x$. Since $(v_x - c_x)$ is non-positive and since payoffs are non-negative, we get

$$\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x.$$

So for both possible orderings of c_x and v_x , the sum of the last four terms in (14) is positive. Hence, payoffs are pairwise efficient, i.e., for all v and for all c :

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c.$$

According to Lemma 3, pairwise efficiency is a sufficient condition for the subsequence to become efficient since payoffs V_k^S and V_k^B are monotone. Therefore $\lim_{k \rightarrow \infty} S(A_{k'}) = S^*$ along the subsequence. Hence, the Lemma 2 implies that limiting trading probabilities must necessarily be Walrasian, i.e., (\bar{Q}^S, \bar{Q}^B) must be in Q^W .

Because the choice of the subsequence was arbitrary, this implies that the limit of *every* convergent subsequence is in Q^W . Because \hat{A} is sequentially compact, this implies $\lim_{k \rightarrow \infty} (Q_k^S, Q_k^B) = Q^W$ for the original sequence.¹⁴ According to Lemma 2 this is sufficient for the sequence to become efficient and $\lim_{k \rightarrow \infty} S(A_k) = S^*$, as claimed. ■

5 Application of the Main Result

In this section we discuss four specifications of dynamic matching and bargaining games to show how to apply and check our conditions: First, we show that the conditions hold

¹⁴In a sequentially compact space, if all convergent subsequences of some sequence have a common limit, then the sequence itself converges to that limit (see also Lemma 7 in the Appendix.)

in the basic example introduced already in Section 2. Second, we consider a variant of this specification where we assume symmetric information and intermediate bargaining power. This variant is essentially a version of the steady-state model in Gale (1987). The last two specifications are only briefly sketched: We consider a variant where entry occurs only in the first period and another variant where sellers conduct auctions. These variants similar to the setups in Moreno and Wooders (2002) and in Satterthwaite and Shneyerov (2006), respectively.

5.1 Basic Model

Take a decreasing sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$ with $\delta_k \rightarrow 0$. As shown in Lauer mann (2006a), for every k there exists an equilibrium σ_k^* . Fixing one equilibrium for each k yields a sequence $\{\sigma_k^*\}_{k=0}^{\infty}$. With every equilibrium σ_k^* , we associate an outcome A_k , using the map $A(\cdot, \cdot) : \Sigma \times [0, 1] \rightarrow \Sigma_M^4$ defined as in Section 2.

Now, we want to derive the functions L^B and L^S . For this, we first show how trading probabilities relate to the distributions of types in the pool; then we show how the distribution of types in the pool translates into the matching probabilities L^B and L^S . First, we substitute Q_k^B into the steady-state condition (7), so that we can write the distribution function Φ^B as a function of A_k :

$$\Phi^B(v|\delta_k, A_k) = \int_0^v \frac{1 - Q_k^B(\tau) + \delta Q_k^B(\tau)}{M_k \delta_k} dG^B(\tau), \quad (15)$$

where M_k is derived from $\Phi^B(1|\delta_k, A_k) = 1$. $\Phi^S(c|\delta_k, A_k)$ can be defined similarly. Now, we define L^S as the solution to the recursive matching formula:

$$L^S(c|\delta_k, A_k) = \Phi^S(c|\delta_k, A_k) + (1 - \delta_k) (1 - \Phi^S(c|\delta_k, A_k)) L^S(c|\delta_k, A_k)$$

and together with the equivalent formula for L^B we get:

$$L^B(v|\delta_k, A_k) = \frac{1 - \Phi^B(v|\delta_k, A_k)}{1 - \Phi^B(v|\delta_k, A_k) (1 - \delta_k)} \quad (16)$$

$$\text{and } L^S(c|\delta_k, A_k) = \frac{\Phi^S(c|\delta_k, A_k)}{1 - (1 - \Phi^S(c|\delta_k, A_k)) (1 - \delta_k)}. \quad (17)$$

Now we prove that our conditions hold:

Lemma 5 *Given any sequence of exits rates $\{\delta_k\}_{k=1}^{\infty}$ with $\delta_k \rightarrow 0$, every sequence of equilibrium outcomes $\{A_k\}_{k=0}^{\infty}$ satisfies mass balance, monotonicity, no rent extraction, and it satisfies availability and weak pairwise efficiency with respect to L^S and L^B as defined in (17) and (16).*

Proof:

Mass Balance: Proven in Lauer mann (2006a, Lemma 2).

Monotonicity. For $Q_k^S(\cdot)$: Suppose the function is not monotone decreasing for some k and for some $c_H > c_L$, $Q_k^S(c_H) \equiv Q_H > Q_L \equiv Q_k^S(c_L)$ by optimality. Then with $p_L \equiv p_k(c_L)$ and $p_H \equiv p_k(c_H)$, it must be that $U^S(p_H, c_H | \sigma_k, \delta_k) \geq U^S(p_L, c_H | \sigma_k, \delta_k)$. This is equivalent to

$$Q_H(p_H - c_H) \geq Q_L(p_L - c_H),$$

and this implies that for costs $c_L < c_H$

$$Q_H(p_H - c_L) > Q_L(p_L - c_L),$$

and thus, $U^S(p_H, c_L | \sigma_k, \delta_k) > U^S(p_L, c_L | \sigma_k, \delta_k)$; This contradicts the optimality of $p_L \equiv p_k(c_L)$ for c_L . Similar reasoning holds for $Q_k^S(\cdot)$.

No Rent Extraction. For $V_k^S(\cdot)$: Again, we use a direct implication of optimality:

$$V_k^S(c) - V_k^S(c_x) \geq U^S(p_k(c_x), c | \sigma_k, \delta_k) - U^S(p_k(c_x), c_x | \sigma_k, \delta_k),$$

which implies that for all c the condition holds, since by definition of $U^S(\cdot, \cdot | \sigma_k, \delta_k)$, the above inequality is equivalent to

$$V_k^S(c) \geq V_k^S(c_x) + q^S(p_k(c_x) | \delta_k, \tilde{\sigma}_k^*)(c_x - c), \quad (18)$$

and similarly for V_k^B .

Availability. For $\{L_k^B\}_{k=1}^\infty \equiv \{L^B(\cdot | \delta_k, A_k)\}_{k=1}^\infty$. Evaluating the steady-state condition (15) at $\Phi_k^B(1 | \delta_k, A_k)$ shows $M_k \delta_k \leq 1$. Choosing any $v' < v$, we get a lower bound on $1 - \Phi^B(v' | \delta_k, A_k)$:

$$1 - \Phi^B(v' | \delta_k, A_k) \geq \int_{v'}^v [1 - Q_k^B(\tau)] dG^B(\tau).$$

Since Q_k^B is monotone, $Q_k^B(v') \leq Q_k^B(v)$ for all $v' < v$. By assumption, the density $dG^B(\tau)$ is continuous and strictly positive, so there is some $g_L > 0$ such that $dG^B(v) \geq g_L$ for all v . Together:

$$1 - \Phi^B(v' | \delta_k, A_k) \geq (1 - Q_k^B(v)) (v - v') g_L,$$

and so for all sequences of $Q_k^B(v)$ with a limit $\bar{Q}^B(v) < 1$:

$$\liminf_{k \rightarrow \infty} L_k^B(v') \geq \frac{(1 - \bar{Q}^B(v)) (v - v') g_L}{1 - (1 - \bar{Q}^B(v)) (v - v') g_L} = 1,$$

and similarly for $\{L_k^S\}_{k=1}^\infty$.

Weak Efficiency: Suppose for some c_x and v_x , $\bar{L}^S(c_x) = \bar{L}^B(v_x) = 1$. By the no rent extraction condition, $V_k^B(\cdot)$ is increasing with a slope in $[0, 1]$. Thus, $r_k(v)$ is

increasing by $r_k(v) = v - (1 - \delta_k)V_k(v)$. Therefore, the set of types accepting a price $p = r_k(v_x)$ is at least the set $[v_x, 1]$. Therefore, the trading probability $D(r_k(v_x) | \sigma_k)$ is at least $1 - \Phi^B(v_x)$. By definition of L_k^B and q^S :

$$q_k^S(r_k(v_x) | \delta_k, \tilde{\sigma}_k^*) \geq L_k^B(v_x),$$

and therefore

$$\begin{aligned} U_k^S(r_k(v_x), c_x | \delta_k, \tilde{\sigma}_k^*) &\geq L_k^B(v_x)(r_k(v_x) - c_x) \\ &= L_k^B(v_x)(v_x - (1 - \delta_k)V_k^B(v_x) - c_x), \end{aligned}$$

where the last line follows from the equilibrium condition for $r_k(v_x)$. Given the equilibrium conditions, $V_k^S(c_x) \geq U_k^S(r_k(v_x), c_x | \delta_k, \tilde{\sigma}_k^*)$ for all k . Therefore,

$$\begin{aligned} \liminf_{k \rightarrow \infty} V_k^S(c_x) &\geq \liminf_{k \rightarrow \infty} L_k^B(v_x)(v_x - (1 - \delta_k)V_k^B(v_x) - c_x) \\ &= (v_x - \bar{V}_k^B(v_x) - c_x), \end{aligned}$$

which implies $\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x$ (whenever they exist), as claimed. ■

So $\{A_k\}_{k=1}^\infty$ satisfies our conditions and thus we have:

Corollary 1 *For every sequence of exit rates $\{\delta_k\}_{k=1}^\infty$ with $\delta_k \rightarrow 0$ and for every sequence of associated equilibrium outcomes $\{A_k\}_{k=1}^\infty$ of the basic model:*

$$\lim_{k \rightarrow \infty} S(A_k) = S^*.$$

5.2 Symmetric Information and Intermediate Bargaining Power

We change the basic model by assuming that traders in each match observe what type of player they are up against and that both of them, the buyer and the seller, have a chance to propose a price. Let $\alpha \in (0, 1)$ be the probability that the seller is chosen to propose a price, and let $(1 - \alpha)$ be the probability that the buyer is chosen. This is similar to the analysis in Gale (1987).¹⁵ We will first derive the formal setup and provide an equilibrium definition. Then we sketch why our conditions hold in this setup. A remark at the end of this section summarizes the intuition.

Strategies now account for the role of the trader and for the type of the opponent. Let Σ_{M^2} be the set of measurable functions $f : [0, 1]^2 \rightarrow [0, 1]$. Strategies are $[p^S, p^B, r^S, r^B] \in \Sigma_{M^2}^2 \times \Sigma_M^2$, where $p^B(v, c)$ is the price proposed by a buyer of type v to a seller of type c , and $r^S(c)$ is the reservation price of a seller c . $p^S(c, v)$ and $r^B(v)$ are the corresponding proposals and reservation prices of sellers and buyers. A market

¹⁵Different from Gale we consider a continuum of types. He also assumes that traders are infinitely lived and that (therefore) there is an entry stage. In Section 6 we cover the latter cases.

constellation is a vector $\sigma_F \in \Sigma_F$ with $\Sigma_F \equiv \Sigma_{M^2}^2 \times \Sigma_M^4 \times \mathbb{R}_+$ and with a typical element $\sigma_F = [p^S, p^B, r^S, r^B, \Phi^S, \Phi^B, M]$.

Let $P^S(p^S, c|\sigma_F)$ be the probability that a seller who is chosen as a proposer will trade in a given period when using $p^S = p^S(\cdot, \cdot)$, defined as

$$P^S(p^S, c|\sigma_F) \equiv \int_{v|r^B(v) \geq p^S(c, v)} d\Phi^B(v),$$

and let $R^S(r^S, c|\sigma_F)$ be the probability that the seller will trade when chosen to respond:

$$R^S(r^S, c|\sigma_F) \equiv \int_{v|p^B(v, c) \geq r^S(c)} d\Phi^B(v);$$

then the per period probability of trading is $D^S(p^S, r^S, c|\sigma_F) = \alpha P^S(p^S, c|\sigma_F) + (1 - \alpha) R^S(r^S, c|\sigma_F)$. Let $E^{RS}[p|p \geq r^S(c), c, \sigma_F]$ be the expected price conditional on trade when responding, and let $E^{PS}[p|p \leq r^B(v), \sigma_F]$ be the expected price conditional on trade when proposing. Expected payoffs are implicitly defined via

$$U^S(p^S, r^S, c|\sigma_F) = \alpha P^S(p^S, c|\sigma_F) (E^{PS}[p] - c) + (1 - \alpha) R^S(r^S, c|\sigma_F) (E^{RS}[p] - c) + (1 - \delta) (1 - D^S(p^S, r^S, c|\sigma_F)) U^S(p^S, r^S, c), \quad (19)$$

with $E^{PS}[p] = E^{PS}[p|p \leq r^B(v), \sigma_F]$ and $E^{RS}[p] = E^{RS}[p|p \geq r^S(c), c, \sigma_F]$. Let $U^{PS}(p, v|p^S, r^S, c, \sigma_F)$ be the payoff when matched with a type v , proposing p and continuing according to (p^S, r^S) :

$$U^{PS}(p, v, c|p^S, r^S, \sigma_F) = \begin{cases} p - c & \text{if } p \leq r^B(v), \\ (1 - \delta) U^S(p^S, r^S, c|\sigma_F) & \text{otherwise.} \end{cases}$$

We define the corresponding functions for buyers analogously.

The steady-state conditions do not change. They are

$$\Phi^S(c) = \int_0^c \frac{dG^S(\tau)}{M(D^S(p^S, r^S, \tau|\sigma_F) + \delta(1 - D^S(p^S, r^S, \tau|\sigma_F)))} \quad \forall c \quad (20)$$

$$\text{and } \Phi^B(v) = \int_0^v \frac{dG^B(\tau)}{M(D^B(p^B, r^B, \tau|\sigma_F) + \delta(1 - D^B(p^B, r^B, \tau|\sigma_F)))} \quad \forall v. \quad (21)$$

We define an equilibrium, with x and y denoting types of traders. We require that the price offered by the proposer must be optimal for every possible type of responder, and we require that the reservation price has the same properties as derived in the basic model. These requirements incorporate the idea of sequential rationality:

Definition 3 *A steady-state equilibrium vector with full information, $\sigma_F^* \in \Sigma_F$, consists of an optimal pair of strategies and a corresponding steady-state pool such that*

1. $(p^j, r^j) \in \arg \max U^j(\cdot, \cdot, x | \sigma_F) \quad \forall x \text{ and } j \in \{B, S\}$
2. $p^j(x) \in \arg \max U^{Pj}(\cdot, x, y | p^j, r^j, \sigma_F) \quad \forall x, y \text{ and } j \in \{B, S\}$
3. $r^B(v) = v - (1 - \delta) U^B(p^B, r^B, v | \sigma_F)$ and $r^S(c) = (1 - \delta) U^S(p^S, r^S, c | \sigma_F) + c$
 $\forall v, c$
4. $\Phi^S(\cdot), \Phi^B(\cdot), M$ satisfy the steady-state conditions (20), (21).

We show that payoffs can be rewritten very compactly. First, the optimal price offer of a buyer v to a seller of type c is clearly never strictly above $r^S(c)$, but is either equal to the reservation price or equal to some unacceptable price below, $p < r^S(c)$. Hence, the expected price offer to the seller, conditional upon acceptance, is $E^{RS} [p | p \geq r^S(c), c, \sigma_F] = r^S(c)$. This also applies to buyers. This implies in particular that a responder is indifferent about accepting or rejecting an offer. Therefore, expected payoffs do not change if a trader plans to simply reject *all* offers. Thus, payoffs depend only on the price offers a trader makes when he is a proposer. To derive this payoff, let q^{PS} be the lifetime trading probability conditional on trading only as a proposer and using the offer strategy p^S . We can derive $q^{PS}(\cdot, \cdot)$ as the solution to

$$q^{PS}(p^S, c | \sigma_F) = \alpha P^S(p^S, c) + (1 - \delta) (1 - \alpha P^S(p^S, c)) q^{PS}(p^S, r^S, c | \sigma_F).$$

where $P^S(p^S, c) = P^S(p^S, c | \sigma_F)$. Rewriting the payoff definition (19), using q^{PS} and the observation that $E^{RS} [p | p \geq r^S(c), c, \sigma_F] = r^S(c)$, yields

$$U^S(p^S, r^S, c | \sigma_F) = q^{PS}(p^S, c | \sigma_F) (E^{PS} [p | p \leq r^B(v), \sigma_F] - c), \quad (22)$$

and similarly for buyers,

$$U^B(p^B, r^B, v | \sigma_F) = q^{PB}(p^B, c | \sigma_F) (v - E^{PB} [p | p \geq r^S(c), \sigma_F]). \quad (23)$$

Now take a sequence of exit rates $\{\delta_k\}_{k=1}^\infty$, with $\delta_k \rightarrow 0$, as before, and assume that for every δ_k , there is some equilibrium. Let this be σ_{Fk} , which gives us a sequence $\{\sigma_{Fk}\}_{k=1}^\infty$. Let A_k be the outcome of equilibrium σ_{Fk} , with $A_k = A(\sigma_k^F, \delta_k)$ defined in the obvious way. We check only the no rent Extraction condition, because the other conditions are immediate. For this, let $U^S(p_k^S(\cdot, \cdot), 1, c | \sigma_{Fk}, \delta_k)$ be the payoff to a seller of type c if offering a price $p_k^S(\cdot, c)$ when chosen to propose, while rejecting any price offer if chosen to respond. From (22):

$$V_k^S(c) = U^S(p_k^S(\cdot, \cdot), 1, c | \sigma_{Fk}, \delta_k),$$

and from optimality

$$\begin{aligned} V_k^S(c_x) - V_k^S(c) &\geq U^S(p_k^S(\cdot, c), 1, c_x | \sigma_{Fk}, \delta_k) - U^S(p_k^S(\cdot, c), 1, c | \sigma_{Fk}, \delta_k) \\ &\geq q^{PS}(p_k^S(\cdot, \cdot), c | \sigma_F)(c - c_x), \end{aligned} \quad (24)$$

and together with symmetric reasoning for buyers, the first parts of the condition hold. For the limiting part, we show that if the lifetime trading probability Q_k^S converges to one, then $q^{PS}(p_k^S(\cdot, \cdot), c | \sigma_F)$ converges to one as well. This is proven in detail in the appendix, Section A.5. Therefore, (24) implies that whenever $Q_k^S \rightarrow 1$, we get $V_k^S(c_x) - V_k^S(c) \geq (c - c_x)$.

Now the other conditions follow, and we sketch out the idea: Given the no rent extraction condition, payoffs $V^S(\cdot)$ and $V^B(\cdot)$ are monotone. From the equilibrium conditions it follows that two matched traders v_x and c_x trade if and only if their joint trading surplus $v_x - c_x$ is larger than their joint continuation payoff $(1 - \delta)[V^S(c_x) + V^B(v_x)]$. This, together with $V^B(\cdot)$ being increasing at a rate smaller than one (from the no rent extraction condition), implies that a buyer with a higher valuation trades with a larger set of sellers, and hence, the trading probability $Q_k^B(\cdot)$ is monotone increasing in v . Analogous reasoning implies the same for sellers. Weak pairwise efficiency is a direct implication of the above observation. Finally, availability follows by the same reasoning as in the basic model, because we are using exactly the same matching technology. Hence:

Corollary 2 *For every sequence of exit rates $\{\delta_k\}_{k=1}^\infty$ and equilibrium outcomes $\{A_k\}_{k=1}^\infty$ of the full information model with intermediate bargaining power $\alpha \in (0, 1)$,*

$$\lim_{k \rightarrow \infty} S(A_k) = S^*.$$

Remark 3 *The crucial step for proving convergence with symmetric information is the following observation: Although it is true that a trader of type c does not need to **receive** the same offers as a trader of type c_x , he can **make** the same offers when chosen as the proposer. Even more to the point: As we have seen in (22) and (23), payoffs depend **only** on the offers made as a proposer. Therefore, a seller of type c can mimic the strategy of another type c_x in much the same way as a seller in our basic model can mimic the pricing strategy of another seller. When $\alpha = 1$, i.e., when buyers are never chosen to be the proposer, this reasoning breaks down. In Lauer mann (2006b), I look at this case and I show that convergence to efficiency does not hold, see also Section 7.1.*

5.3 Further Applications

Two more variants of the basic model that can be analyzed as before include one-time entry and second-price auctions with reservation prices. Suppose, in the basic model, we assume that time runs from $t = 0$ to infinity. In period zero, a unit mass of buyers and a unit mass of sellers arrive with types distributed according to distribution functions

G^S and G^B , with the same properties as in the basic model. There is no further inflow in the subsequent periods. Thus, the pool in $t \geq 1$ consists only of those who did not trade before and who did not die before. So the pool depletes over time.¹⁶ Otherwise, we assume that matching is pairwise, information is asymmetric and sellers make price offers, just as in the basic model.¹⁷ We can characterize outcomes by considering the *ex ante* trading probabilities and payoffs to the entering traders in the first period. Their joint expected surplus is the natural welfare criterion. Clearly, mass balance should hold with respect to the ex ante outcome. With Q_0^S, Q_0^B denoting the first period expected lifetime trading probabilities, $Q_0^S, Q_0^B \in Q^W$ is a necessary and sufficient condition for efficiency, with Q^W as defined in (12). For this model, one can show that our conditions hold: By asymmetric information, traders can mimic each other. Just as in the basic model, this implies that trading probabilities are monotone and ex ante payoffs have a bounded slope. For the availability condition, note that if the ex ante trading probability of some buyers is not one, then these buyers will stay in the market for many periods. One can show that this implies that a seller is certain to be matched with them some time, i.e., availability holds. Finally, weak efficiency holds by similar reasoning to that found in the basic model. Thus, our main result applies even to non-steady-state markets, and the outcome will become efficient with δ converging to 0.

We can include auctions similar to Satterthwaite and Shneyerov (2006) in the basic model as follows: Suppose matches consist of one seller and a random number of buyers, where the number of buyers ("bidders") per seller is Poisson distributed with parameter one. Further, suppose the seller conducts a second price auction among the bidders: Upon observing the number of buyers in his match, he announces a reservation price p . Then the buyers submit their bids r . Restricting attention to equilibria in dominant strategies, these bids are equal to the reservation prices derived before. This allows a simple characterization of the equilibrium. Suppose we keep the basic model otherwise - that is, we retain the assumption that there is an exogenous inflow and that there is some death rate δ . Our conditions hold in this model as well: Monotonicity and no rent extraction follow from asymmetric information, and availability and weak efficiency follow by reasoning familiar from the basic model. Therefore, if sellers can use auctions to sell their goods, with vanishing δ , the outcome becomes efficient.

6 Extensions

To show that our analysis also extends to the original setups by Gale (1987) and Satterthwaite and Shneyerov (2006), we first include an entry stage in the next subsection and then we assume that traders are infinitely lived.

¹⁶The pool cannot deplete fully if prices are individually rational.

¹⁷This model with one-time entry would be different in two aspects: First, instead of a stationary pool, the pool would depend on the time via some law-of-motion condition. Second, price offers and reservation prices would depend on time.

6.1 Including an Entry Stage

Suppose we include an entry stage into the basic model, i.e., suppose that new traders must decide whether they want to enter the pool or not. If they enter the pool, they must pay some entry costs $e \in (0, 1)$. Let $Z^S(\cdot) : [0, 1] \rightarrow \{0, 1\}$ denote the entry decision, with $Z^S(c) = 1$ indicating the decision of type c to become active. Let $V^S(\cdot)$ denote the expected payoffs to a seller if he enters, gross of e . ($V^S(\cdot)$ is also calculated for those who do not actually become active.) Let $Z^B(\cdot)$ and $V^B(\cdot)$ be the corresponding functions for buyers.

We assume that sellers enter whenever this is profitable, i.e., $Z^S(c) = 1$ whenever $V^S(c) \geq e$, and we assume $Z^S(c) = 0$ otherwise. For buyers, we assume the same: $Z^B(v) = 1$ whenever $V^B(v) \geq e$. Let c_0^e be the highest type of a seller for whom entry is profitable, $c_0^e \equiv \sup \{c, 0 | V^S(c) \geq e\}$, and let v_0^e be the lowest type of a buyer for whom entry is profitable, $v_0^e \equiv \inf \{v, 1 | V^B(v) \geq e\}$. By this definition, types $c > c_0^e$ or $v < v_0^e$ do not enter.¹⁸

Given the entry stage, the matching technology of the basic model has to be changed to account for the possibility that the masses of the two sides of the market are not identical. But no matter how this is done, types who do not enter are not available. Therefore, the probability to match some set of buyers might be zero, even though the lifetime trading probability of these types is strictly below one. One can show that this failure of availability leads to a failure of convergence to efficiency in the basic model (see Section 7.) Therefore, stronger forces towards efficiency are needed. In the models by Gale (1987), as well as in Satterthwaite and Shneyerov (2006), these forces come from curtailing the bargaining power of the seller. Formally, these models satisfy a stronger condition than Condition 4 (weak efficiency). Sequences of trading outcomes that satisfy this stronger condition converge to efficiency even though they satisfy only a weaker availability condition, due to the entry stage.

An outcome A^E of a model with an entry stage is a vector $[V^S, V^B, Q^S, Q^B, Z^S, Z^B]$. We assume that all components are measurable, i.e., $A^E \subset \Sigma_M^6$. Surplus conditional on (Q^S, Q^B, Z^S, Z^B) and gross of entry costs (which will become zero) is

$$S_Q^E(A^E) = \int_0^1 v Z Q^B(v) dG^B(v) - \int_0^1 c Z Q^S(c) dG^S(c),$$

with $ZB^B(v) \equiv Z^B(v) Q^B(v)$ and $ZQ^S(c) \equiv Z^S(c) Q^S(c)$. These latter functions are the *effective* trading probabilities, and we work with them throughout this section. Mass balance with entry is satisfied if the transfers collectively made by all buyers are equal to the expected transfers collectively received by all entering sellers. Equivalently, the mass of sellers who trade must be equal to the mass of buyers who trade:

¹⁸If payoffs are monotone, all types c below c_0 and all type v above v_0 enter.

Definition 4 Mass Balance with Entry. An outcome A^E satisfies mass balance with entry if

$$S^E(A^E) = \int_0^1 Z^S(c) V^S(c) dG^S(c) + \int_0^1 Z^B(v) V^B(v) dG^B(v) = S_Q^E(A^E), \quad (25)$$

and if

$$\int_0^1 ZQ^S(c) dG^S(c) = \int_0^1 ZQ^B(v) dG^B(v). \quad (26)$$

We say that an outcome A^E is *Walrasian* if the effective trading probabilities are in Q^W , i.e., if $(ZQ^S, ZQ^B) \in Q^W$. Reasoning analogously to the case without entry, we find that an outcome is efficient if and only if it is Walrasian:

Lemma 6 For all outcomes that satisfy mass balance with entry, $S^E(A^E) = S^*$ if and only if $ZQ \in Q^W$. For every sequence $\{A_k^E\}_{k=1}^\infty$ which satisfies mass balance with entry and which has monotone trading probabilities, $ZQ \in \Sigma_- \times \Sigma_+$:

$$\lim_{k \rightarrow \infty} S_Q^E(A_k^E) = S^* \quad \text{if and only if} \quad \lim_{k \rightarrow \infty} (ZQ_k) = Q^W.$$

Take any sequence of outcomes $\{A_k^E\}_{k=1}^\infty$ and a sequence of frictions $\{\delta_k, e_k\}_{k=1}^\infty$ with $(\delta_k, e_k) \rightarrow (0, 0)$. If the limits of effective trading probabilities exist, we denote them by \overline{ZQ}^S and \overline{ZQ}^B , and if the limits of the cutoff types $c_0^{e_k}$ and $v_0^{e_k}$ exist, we denote them by c_0 and v_0 . Now we restate the conditions. The monotonicity condition becomes a condition regarding effective trading probabilities:

Condition 5 Monotonicity with Entry. For every member A_k^E ,

$$\overline{ZQ}_k^S \in \Sigma_- \quad \text{and} \quad \overline{ZQ}_k^B \in \Sigma_+.$$

The no rent extraction condition remains unchanged. But as said in the introduction, we weaken availability and we assume that it holds only for those types below (above) the cutoffs, i.e., for those $c \leq c_0$ and $v \geq v_0$. With $L_E^j : [0, 1]^2 \times \Sigma_M^6 \rightarrow [0, 1]$, $j \in \{S, B\}$:

Condition 6 Weak Availability. If $\overline{ZQ}^S(c')$ and c_0 exist, and if $\overline{ZQ}^S(c') < 1$ for some $c' < c_0$, then

$$\bar{L}_E^S(c) = 1 \quad \text{for all } c \in (c', c_0).$$

If $\overline{ZQ}^B(v')$ and v_0 exist, and if $\overline{ZQ}^B(v') < 1$ for some $v' > v_0$, then

$$\bar{L}_E^S(v) = 1 \quad \text{for all } v \in (v_0, v').$$

We strengthen weak pairwise efficiency by requiring availability only on one side of the market. But it has to hold only for pairs involving either v_0 or c_0 . As we will see,

the limiting payoffs of these cutoff types are zero. Therefore, $\bar{V}^S(c') + \bar{V}^B(v_0) \geq v_0 - c'$ implies $\bar{V}^S(c') \geq v_0 - c'$. The following condition is formulated such that it is met by the models of Satterthwaite and Shneyerov and by the model of Gale:

Condition 7 Strong Pairwise Efficiency. *If $\bar{L}^S(c') = 1$ for some c' and if v_0 exists, then*

$$\bar{V}^S(c') \geq v_0 - c'.$$

If $\bar{L}^B(v') = 1$ for some v' and if c_0 exists, then

$$\bar{V}^B(v') \geq v - c_0.$$

Remark 4 *In the basic model, the first part of this condition does not hold: Suppose there is some cutoff $p^N > p^w$ such that all buyers with $v \geq p^N$ and all sellers with $c \leq p^N$ enter and no one else. Suppose in addition that all sellers offer the common price p^N . Since there are more sellers with costs below p^N than there are buyers with valuations above p^N , sellers must be rationed and they do not trade with certainty, i.e., $\bar{ZQ}^S(c') < 1$ and $\bar{L}^S(c') = 1$ for $c' \leq v_0 = p^N$. Payoffs for sellers become $\bar{V}^S(c') = \bar{ZQ}^S(c')(p^N - c') < p^N - c' = v_0 - c'$. Nevertheless, they have no incentive to decrease their offers since they cannot increase their revenue if $Z^B(v) = 0$ for all $v < p^N$. Therefore, strong pairwise efficiency fails in the basic model.*

Note that for all models with entry, there exists an equilibrium in which no trader enters. If a sequence of outcomes includes such outcomes as subsequence, its limit cannot become efficient. Hence, we restrict attention to *non-trivial sequences*, where entry does not vanish along any subsequence, i.e.,

$$\limsup_{k \rightarrow \infty} v_0^{e_k} < 1 \quad \text{and} \quad \liminf_{k \rightarrow \infty} c_o^{e_k} > 0.$$

Under the stronger efficiency condition, we can state:

Proposition 2 *Suppose some non-trivial sequence $\{A_k^E\}_{k=1}^\infty$ satisfies mass balance and monotonicity with entry, no rent extraction, weak availability and strong pairwise efficiency for some pair of functions L^B and L^S and for some sequence $\{\delta_k\}_{k=1}^\infty$ and $\{e_k\}_{k=1}^\infty$ with $e_k \rightarrow 0$. Then the outcome becomes efficient, i.e.,*

$$\lim_{k \rightarrow \infty} S^E(A_k^E) = S^*.$$

Proof: As before, we take some convergent subsequence of outcomes and denote the limit by $(\bar{V}^S, \bar{V}^B, \bar{ZQ}^S, \bar{ZQ}^B, \bar{Q}^S, \bar{Q}^B)$. Let v_x be the lowest valuation and c_x the highest cost that does not trade for sure in the limit:

$$v_x = \sup \left\{ v, 0 | \bar{ZQ}^B(v) < 1 \right\} \quad \text{and} \quad c_x = \inf \left\{ c, 1 | \bar{ZQ}^S(c) < 1 \right\}.$$

Let us take a further subsubsequence indexed by k' such that the cutoffs $v_0^{e_{k'}}$ and $c_0^{e_{k'}}$ converge to some v_0 and c_0 . Now we want to show that $(ZQ_{k'}) \rightarrow Q^W$. We will argue at the end of the proof that this implies $S(A_k^E) \rightarrow S^*$ for the sequence itself.

Noting that $\lim Q_{k'}^S(c) = 1$ whenever $\overline{ZQ}^S(c) = 1$, and, symmetrically, $\lim Q_{k'}^B(v) = 1$ whenever $\overline{ZQ}^S(v) = 1$, the no rent extraction condition has the same implication as in the proof of the main result, i.e.,

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c + \bar{V}^S(c_x) + \bar{V}^B(v_x) - (v_x - c_x). \quad (27)$$

Now we want to derive again a lower bound on the joint payoff of c_x and v_x by showing that $\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x$:

If $v_x \leq c_x$, the desired inequality follows immediately because payoffs are non-negative. So suppose $v_x > c_x$. We consider three subcases for the relation between c_x, c_0, v_0, v_x . Subcase 1 is $c_x < c_0 < v_0 < v_x$. Then, for all $\varepsilon \in (0, \min\{c_0 - c_x; v_x - v_0\})$, by definition of c_x , $\overline{ZQ}^S(c_x + 0.5\varepsilon) < 1$. Thus, $\bar{L}^S(c_x + \varepsilon) = 1$ by weak availability and, by symmetric reasoning, $\bar{L}^B(v_x - \varepsilon) = 1$. Therefore, strong efficiency implies

$$\bar{V}^S(c_x + \varepsilon) \geq v_0 - c_x - \varepsilon \quad \text{and} \quad \bar{V}^B(v_x - \varepsilon) \geq v_x - c_0 - \varepsilon.$$

Since payoffs are continuous, ε is arbitrary, and $v_0 \geq c_0$, we get

$$\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x + v_0 - c_0 \geq v_x - c_x,$$

as claimed.

Subcase 2a: $c_x = c_0$ and $v_0 < v_x$. By definition of v_x , $\overline{ZQ}^B(v_x - 0.5\varepsilon) < 1$, for all $\varepsilon \in (0, v_x - v_0)$. Then, the availability condition requires that $\bar{L}^B(v_x - \varepsilon) = 1$, and the strong efficiency condition implies

$$\bar{V}^B(v_x - \varepsilon) \geq v_x - c_0 - \varepsilon$$

Since, again, payoffs are continuous, and since $c_x = c_0$, this implies $\bar{V}^B(v_x) \geq v_x - c_x$. Hence, by $\bar{V}^S(c_x) \geq 0$, we get the desired inequality $\bar{V}^B(v_x) + \bar{V}^S(c_x) \geq v_x - c_x$. Subcase 2b: $c_x < c_0$ and $v_0 = v_x$. By analogous reasoning: $\bar{V}^B(v_x) + \bar{V}^S(c_x) \geq v_x - c_x$.

Subcase 3: $c_x = c_0$ and $v_0 = v_x$. Note first that marginal types must make zero profits in the limit: If $\lim_{k \rightarrow \infty} \sup V_k^S(c_o^{e_k}) > 0$, the (Lipschitz-) continuity of payoffs implies that for some ε small enough, $\lim_{k \rightarrow \infty} \sup V_k^S(c_o^{e_k} + \varepsilon) > 0$. This contradicts the definition of the marginal type. Hence $\lim_{k \rightarrow \infty} V_k^S(c_o^{e_k}) = 0$ and, symmetrically, $\lim_{k \rightarrow \infty} V_k^B(v_o^{e_k}) = 0$. With this observation, we show that this subcase leads to a contradiction: If $c_x = c_0$ and $v_0 = v_x$, with $c_x < v_x$, then the mass of sellers who trade becomes

$$\lim_{k' \rightarrow \infty} \int_0^1 ZQ_{k'}^S(c) dG^S(c) = \int_0^{c_x} dG^S(c) = G^S(c_x),$$

and, similarly, the mass of buyers who trade becomes

$$\lim_{k' \rightarrow \infty} \int_0^1 ZQ_{k'}^B(v) dG^B(v) = \int_{v_x}^1 dG^B(v) = 1 - G^B(c_x).$$

The mass balance of total trades, see (26), requires therefore that the mass of entering sellers becomes equal to the mass of entering buyers, i.e., $G^S(c_x) = 1 - (G^B(v_x))$. Furthermore, since $\bar{Q}^S(c) = 1$ for all $c < c_x = c_0$, no rent extraction requires that

$$\bar{V}^S(c_0) \geq \bar{V}^S(c) + (c - c_0),$$

and thus $\bar{V}^S(c) \leq \bar{V}^S(c_0) + (c_0 - c)$ for $c < c_0$. From before, we know that $\bar{V}^S(c_0) = 0$, so together we have $\bar{V}^S(c) \leq c_0 - c$. By symmetric reasoning, $\bar{V}^B(v) \leq v - v_0$. We use this to get an upper bound on the limit of $S(A_{k'}^E)$:

$$\begin{aligned} \liminf_{k' \rightarrow \infty} S^E(A_{k'}^E) &\leq \int_0^{c_x} (c_0 - c) dG^S(c) + \int_{v_x}^1 (v - v_0) dG^B(v) \\ &\leq \int_{v_x}^1 v dG^B(v) - \int_0^{c_x} c dG^S(c) - G^S(c_x)(v_x - c_x) \\ &< \int_{v_x}^1 v dG^B(v) - \int_0^{c_x} c dG^S(c) = \lim_{k' \rightarrow \infty} S_Q^E(A_{k'}^E), \end{aligned}$$

where we use that $G^S(c_x)$ is equal to $1 - (G^B(v_x))$ for the second line and the hypothesis of the subcase, $(v_x - c_x) > 0$, for the third line. Since $S_Q^E(A_{k'}^E)$ has a limit different from $S^E(A_{k'}^E)$, the mass balance identity (25), $S^E(A_{k'}^E) = S_Q^E(Q^S(\cdot), Q^B(\cdot))$ is violated for k' large enough. As a result, this subcase is impossible, since by choice of the (sub-)sequence $\{A_{k'}^E\}$, each of its elements does satisfy mass balance. (Note, that this subcase is the only place where we need $e_k \rightarrow 0$.)

Hence, $\bar{V}^B(v_x) + \bar{V}^S(c_x) \geq v_x - c_x$ in all possible cases. Thus, inequality (27) implies that limiting payoffs are pairwise efficient for all types c and v :

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c. \quad (28)$$

By reasoning analogously to the second part of the main result, this implies that the outcomes of the original sequence become efficient and

$$\lim_{k \rightarrow \infty} S^E(A_k^E) = S^*. \quad \blacksquare$$

Remark 5 *Since Gale (1987) assumes infinitely lived agents, he cannot let the entry fee converge to zero simultaneously with δ_k for technical reasons (see the next section.) Note however, that we need the assumption of vanishing entry fees only in Subcase 3 in the proof. As the reader can immediately verify, if entry costs remain constant, i.e., if $e_k = e$, this would imply that cutoff types might be separated by a wedge of size $2e$, i.e.,*

$(c_x - v_x) \in (0, 2e)$. In this case, the inequality (27) would imply that $\bar{V}^S(c) + \bar{V}^B(v) \geq v - c - 2e$ for all v and c . Hence, with e being small, the outcome is close to being pairwise efficient. By continuity of $S(\cdot)$, this implies that when e becomes small, the outcome becomes efficient. Satterthwaite and Shneyerov (2006) assume participation costs κ_k per period instead of one-time entry costs e_k . It can be easily verified that whenever the lifetime trading probability converges to one, accumulated lifetime participation costs become zero. In Subcase 3 this is the case for all entering traders (by definition of c_x and v_x). Thus, absolute entry costs are zero, and Subcase 3 is impossible in the case of participation costs as well.

6.2 Infinitely Lived Traders and Ponzi Schemes

Suppose the exit rate δ is equal to zero. This is a common assumption in the literature, e.g., it is used in Gale (1987) and in Satterthwaite and Shneyerov (2006). We want to know whether our approach is still valid. In the case of $\delta = 0$, traders are "infinitely lived" and they can exit the pool only through trading. Time preferences are introduced by assuming the presence of a discount factor $\beta \in (0, 1)$. Discounting then implies the existence of search costs and $(1 - \beta)$ corresponds to the size of the friction. Again the question is whether the trading outcome becomes efficient when frictions vanish and the discount factor becomes one, $(1 - \beta) \rightarrow 0$.

As we will see, we can indeed apply our approach to such a setup but we first need to take care of the problem of *Ponzi Schemes*. Because of such schemes, the expected payoff to entering traders can in principle be much higher than S^* (as defined in (13)): By shifting the timing of the sellers' trade, discounting implies that costs are diminished. Nevertheless, since sellers are infinitely lived, a mere shift in the timing of their trading does not influence how many buyers can trade, i.e., a shift does not influence the feasibility conditions. In the extreme, when shifting the timing of trade for all sellers to "infinity", all costs are discounted to zero. For the general problems of Ponzi schemes in economies with infinitely lived agent (or dynasties) and discounting, see for example Diamond (1965) and the subsequent literature.

Nevertheless, we show that Ponzi schemes are not part of *equilibrium* outcomes if two conditions hold: The first condition is that transfers are made through prices (condition "*prices only*"). The second condition is that a seller who trades receives a price above his cost, while a buyer who trades pays a price below his valuation (condition "*individual rationality*"). In outcomes which satisfy these conditions, Ponzi schemes are ruled out and the maximal surplus that can be attained subject to the conditions is S^* . Finally, since in Gale (1987) and in Satterthwaite and Shneyerov (2006) transfers are actually made through prices and since traders can reject to trade at unfavorable prices, equilibrium outcomes of their games satisfy these conditions.

Now, we go into the details. With infinitely lived agents, every trader who enters the market must ultimately trade. Otherwise a steady state with a finite pool is impossible.

This makes the inclusion of an entry stage necessary. As in Section 6.1, let $Z^j(\cdot) \in \{0, 1\}$ denote the entry decision, with $Z^S(c) = 1$ and $Z^B(v) = 1$ indicating the decision of types c and v to become active. Let $T_\infty^S(\cdot)$ be a measurable function, mapping $[0, 1]$ into \mathbb{R}_+ , where $T_\infty^S(c)$ denotes the undiscounted expected payments received by a seller of type c . Similarly, let $T_\infty^B(v)$ denote the undiscounted payment made by a buyer of type v . The undiscounted trading probabilities are $Q_\infty^S(\cdot) \in \Sigma_M$ and $Q_\infty^B(\cdot) \in \Sigma_M$. *Discounted* transfers and trading probabilities are denoted by $T_\beta^S(\cdot), T_\beta^B(\cdot)$ and by $Q_\beta^S(\cdot), Q_\beta^B(\cdot)$, respectively.¹⁹ Expected payoffs are given by

$$V^S(c) = T_\beta^S(c) - cQ_\beta^S(c) \quad \text{and} \quad V^B(v) = vQ_\beta^B(v) - T_\beta^B(v). \quad (29)$$

An outcome is given by $A^\infty = [V^S, V^B, Q_\infty^S, Q_\infty^B, T_\infty^S, T_\infty^B, Z^S, Z^B, Q_\beta^S, Q_\beta^B]$ and surplus is

$$S(A^\infty) = \int_0^1 V^S(c) dG^S(c) + \int_0^1 V^B(v) dG^B(v).$$

The mass balance condition becomes

Condition 8 *Mass Balance with Infinitely Lived Players.* *An outcome A^∞ satisfies mass balance if*

$$\int_0^1 Z^S(c) Q_\infty^S(c) dG^S(c) = \int_0^1 Z^B(v) Q_\infty^B(v) dG^B(v) \quad (30)$$

$$\text{and} \quad \int_0^1 Z^S(c) T_\infty^S(c) dG^S(c) = \int_0^1 Z^B(v) T_\infty^B(v) dG^B(v). \quad (31)$$

A surplus maximizing outcome A' which satisfies mass balance is the following: transfers are zero, all sellers and all buyers enter, and discounted trading probabilities are one for buyers and zero for sellers. Then we have

$$S(A') = \int_0^1 v dG^B(v) > S^*.$$

Note that this corresponds to the extreme case of a Ponzi scheme discussed in the beginning of this section: Since in expectation, sellers trade "infinitely" many periods after their entry, costs are discounted to zero.

To rule out outcomes like that, we introduce two conditions that are satisfied by the existing models. First, transfers are made only through prices:

Condition 9 *Prices only.* *There are functions $p^S(\cdot) \in \Sigma_M$ and $p^B(\cdot) \in \Sigma_M$ such that*

¹⁹Suppose $D(p)$ is the probability of trading per period, then the discounted trading probability is $Q^S(p) = \frac{D(p)}{1-\beta(1-D(p))}$. Similarly, if $t(p)$ is the expected transfer per period, then discounted expected transfers are $\frac{t(p)}{1-\beta(1-D(p))}$.

for all c and for all v :

$$\begin{aligned} T_\beta^S(c) &= Q_\beta^S(c) p^S(c) & \text{and} & & T_\beta^B(v) &= Q_\beta^B(v) p^B(v) \\ T_\infty^S(c) &= Q_\infty^S(c) p^S(c) & \text{and} & & T_\infty^B(v) &= Q_\infty^B(v) p^B(v). \end{aligned} \quad (32)$$

Secondly, we require that for all entering types, prices are *individually rational*:

Condition 10 *Individual Rationality.* An outcome is *individually rational* if

$$\begin{aligned} \forall c \text{ st. } Z^S(c) &= 1: & p^S(c) &\geq c, \\ \forall v \text{ st. } Z^B(v) &= 1: & p^B(v) &\leq v. \end{aligned}$$

Let \hat{A}^{IR} be the set of outcomes which satisfy *mass balance*, *prices only*, and *individual rationality*. Together with the definition of payoffs in (29), surplus for any $A \in \hat{A}^{IR}$ is given by

$$S^\infty(A) = \int_0^1 Z^S(c) Q_\beta^S(c) (p^S(c) - c) dG^S(c) + \int_0^1 Z^B(v) Q_\beta^B(v) (v - p^B(v)) dG^B(v).$$

Now we demonstrate that S^* , as defined in (13), is the constrained maximum. First, note that the terms in the integral are positive, i.e., for all c such that $Z^S(c) = 1$, we have $(p^S(c) - c) \geq 0$; the same applies to buyers. Hence, in order to maximize $Z^\infty(\cdot)$, all entering traders must trade immediately, i.e., $Q_\beta^S(c) = 1$ for all c st. $Z^S(c) = 1$, and similarly for buyers. In addition, mass balance (31) requires $\int_0^1 Z^S(c) p^S(c) dG^S(c) = \int_0^1 Z^B(v) p^B(v) dG^B(v)$. Together, a necessary condition for an outcome A to be in $\arg \max_{A \in \hat{A}^{IR}} S^\infty(\cdot)$ is that

$$S^\infty(A) = \int_0^1 v Z^B(v) dG^B(v) - \int_0^1 c Z^S(c) dG^S(c).$$

Now the problem of maximizing $S^\infty(\cdot)$ is similar to our original problem in Section 3.2. Indeed, let Q^{EW} be the set of "Walrasian" outcomes,

$$A^{EW} \equiv \left\{ A \mid \int_0^1 |Q_\beta^S(c) Z^S(c) - 1_{c \leq p^w(c)}| dc, \int_0^1 |Q_\beta^B(v) Z^B(v) - 1_{v \geq p^w(v)}| dv = 0 \right\},$$

then by reasoning analogously to Lemma 1, A^{EW} is the set of the maximizers of the surplus, $A^{EW} \equiv \arg \max_{A \in \hat{A}^{IR}} S^\infty(\cdot)$. Thus

$$S^* = \sup_{A \in \hat{A}^{IR}} S^\infty(\cdot).$$

As mentioned in the beginning of this section, in the models by Gale (1987) and Satterthwaite and Shneyerov (2006) traders are restricted to use prices and bids, respectively. In addition, trade is voluntary so that no seller would agree to trade at a price

below costs and no buyer would agree to trade at a price above his valuation. Thus, the set of equilibrium outcomes is a subset of \hat{A}^{IR} , and our approach is valid.

To apply our approach to specifications with infinitely lived traders and entry, we need to rewrite the conditions of Section 6.1 by simply substituting Q_β^j for Q^j .²⁰ Then, the proof in Section 6.1 would imply that $S^\infty(A_k^\infty) \rightarrow S^*$ for all sequences $\{A_k^\infty\}_{k=1}^\infty$ that satisfy the four conditions and that contain only elements from \hat{A}^{IR} . To check whether our conditions actually hold in the models by Gale (1987) and Satterthwaite and Shneyerov (2006), note that *monotonicity*, *no rent extraction*, and *strong efficiency* are immediate.

The *weak availability condition*, however, is somewhat more subtle with infinitely lived agents. In particular, if the trading probability for some set of types converges to zero, then this set might *flood* the market. And even if for some other set of types the limiting trading probability is below one, this other set might make up only a vanishing fraction of the total pool. To avoid this problem, i.e., to avoid the existence of a set of types that trade with a probability approaching zero, both papers include a variant of nonvanishing absolute search costs. The idea is that whenever the limiting trading probability becomes zero for some types, their expected trading revenues become zero and therefore they cannot recover any positive entry costs. Thus, lifetime trading probabilities must stay strictly positive for all entering types. Specifically, Gale (1987) assumes that even as the discount factor converges to one, the entry cost $e \in (0, 1)$ remains constant. Therefore, it is not profitable for agents to enter if they trade only with a probability close to zero. Satterthwaite and Shneyerov (2006) assume a participation cost κ per period that converges to zero at the same rate as the discount factor. One can verify that the accumulated lifetime participation costs are strictly positive whenever the limiting discounted lifetime trading probability becomes zero. Again, this implies that entry is unprofitable when trading probabilities are close to zero.

7 Failures

In this section we demonstrate how to use our approach to understand why convergence to efficiency fails in some specifications of dynamic matching and bargaining games. In the first subsection, we discuss how the failure of convergence with symmetric information can be attributed to the failure of the **no rent extraction** condition. In the following section, we discuss how the simultaneity of decisions in double auctions can lead to the failure of **weak efficiency**. Finally, we discuss a model with cloning and show that the **mass balance** condition does not hold in this case.

We do not provide a specification in which the **monotonicity** condition is the only condition that fails, because there is no such model in the literature. The failure of

²⁰Monotonicity would be required of \bar{Q}_β^j ; no rent extraction would be a condition on the slope of $V^j(\cdot)$; weak availability would require that $\bar{L}^j(x) = 1$ whenever $\bar{Q}_\beta^j(x) < 1$ and $Z^j(x) = 1$; and weak efficiency would still be a condition on the joint surplus of available types.

availability with an entry stage is discussed at the end of the second subsection and interpreted as a coordination failure when traders have to decide simultaneously whether to enter the market.

7.1 No Rent Extraction fails with Full Information and Asymmetric Bargaining Power

Suppose sellers in the basic model can observe the valuation of the buyer prior to making an offer. Clearly, this makes trading *within* each pair efficient: They trade whenever their trading surplus $(v - c)$ is larger than their joint continuation payoff $(1 - \delta)(V^S(c) + V^B(v))$. But as we will see, overall efficiency of trading in the market as a whole decreases: with $\delta \rightarrow 0$, the limiting trading outcome is no longer efficient. Here, we want to show which of our conditions is violated to explain why convergence to efficiency fails. A full discussion of the model can be found in the note by Lauer mann (2006b).²¹

For illustration, we use the setup of Section 5.2: There, traders in each pair can mutually observe their valuations and costs. With probability α , the seller is chosen to be the proposer of a price offer, while with probability $(1 - \alpha)$, the buyer is chosen. While in Section 5.2 we assume that α must be interior, i.e., $\alpha \in (0, 1)$, here we assume that the seller has all the bargaining power, i.e., $\alpha = 1$. Let $\{A_k^F\}_{k=1}^\infty$ be a sequence of equilibrium outcomes of the model of Section 5.2, with α set equal to 1. We can characterize the outcomes by two observations. First, sellers appropriate all the trading surplus: no buyer receives strictly positive payoffs and $V^B(v) \equiv 0$. The price offer to a buyer is either equal to his type or too high to be acceptable, i.e., $p^S(c, v) \geq v$ for all c, v .²² Second, the limiting outcome can be described by a some cutoff $\bar{v} \in (0, 1)$ such that the limiting lifetime trading probabilities of a buyer is zero if $v < \bar{v}$ and one if $v > \bar{v}$, i.e., $\bar{Q}^B = 1_{v > \bar{v}}$.²³

While the sequence $\{A_k^F\}_{k=1}^\infty$ can be shown to satisfy monotonicity, availability, and weak efficiency,²⁴ the no rent extraction condition fails: Since $\bar{Q}^B(v_x) = 1$ for any $v_x > \bar{v}$, the condition requires that payoffs increase with a slope of one, i.e., for types $v' > v_x$, it must be that $\bar{V}^B(v') \geq \bar{V}^B(v_x) + (v' - v_x) > 0$. However, the payoff to any such type

²¹In Lauer mann (2006b) sellers have homogeneous costs $c \equiv 0$ to ease exposition. Here, sellers are heterogeneous to retain the consistency of the underlying economy across specifications.

²²Price offers are always larger than or equal to reservation prices, as argued in Section 5.2, i.e., $p^S(c, v) \geq r^B(v)$. By definition, $v - r^B(v) = (1 - \delta)V^B(v)$, and by $V^B(v) \leq v - r^B(v)$, $v - r^B(v) = 0$.

²³Suppose not. Because trading probabilities $\bar{Q}^B(\cdot)$ can be shown to be monotone, this would imply that for some interval (a, b) , $\bar{Q}^B(v) \in (0, 1)$ for all $v \in (0, 1)$ ($\bar{Q}^B(v) \equiv 0$ (or $\equiv 1$) for all v is never an equilibrium outcome). Then, for any $v' \in (a, b)$, types $v \geq v'$ are *available* and a seller $c = 0$ who trades only with $v \geq v'$ at prices $p^S(0, v) = \max\{v', v\}$ would trade with certainty and receive a payoff $\lim_{k \rightarrow \infty} U^S(0, p^S) \geq v' > a$. This is a contradiction.

²⁴Weak efficiency is immediate with symmetric information; availability holds because the matching technology is unchanged to the case of $\alpha \in (0, 1)$; monotonicity holds essentially because sellers' profits satisfy the strict single crossing condition, i.e., sellers with lower costs prefer to trade with a higher probability at a lower price.

v' is still zero, and his rent $(v' - v_x)$ is *extracted*: Part of this rent will go to the sellers but part of it is wasted. Because of this, the equilibrium outcome is not efficient in the limit.

Remark 6 *Prices with symmetric information are "monopolistic," i.e., $p^S(c, v) \geq v$, by the same reasoning as in Diamond (1971): Sellers can use the waiting costs $\delta \in (0, 1)$ to "hold-up" buyers. However, in the models that are used to derive the familiar Diamond paradox, this outcome is still efficient because buyers and sellers are assumed to be homogeneous.²⁵ Here, with heterogeneous types, inefficiencies first stem from the fact that sellers rather incur rationing than trading at low prices with low valuation buyers, and second they stem from the possibility of trading for sellers who have costs above p^w and who should not trade.*

Remark 7 *One possible way to restore the no rent extraction property while leaving the bargaining power with sellers ($\alpha = 1$) is to assume that buyers' valuations are not perfectly observable (i.e., buyers have some "privacy"): the appendix of Lauer mann (2006b) contains an extension where sellers receive only a signal about the valuation of the buyer and where this signal contains noise. With $\delta \rightarrow 0$, buyers can patiently wait until their type is misconceived as being very low so that they receive a low price offer. In particular, suppose it becomes certain that some buyer of type v_x can trade at an expected price $p \leq v_x$ in the limit. Then any type $v' > v_x$ can wait until he receives the same offers and he can trade at an expected price $p \leq v_x$ as well. The payoff to v' is therefore at least $(v' - v_x)$ larger than the payoff to v_x . Thus, the no rent extraction condition holds, and the outcome becomes efficient in the limit.*

7.2 Weak Efficiency fails without Sequential Rationality

Serrano (2002) is the first to specify the bargaining protocol as a simultaneous double auction.²⁶ He shows that equilibrium outcomes do not need to become efficient. Without going into the details, we can replicate his result in our framework: Suppose we assume in the basic model that the buyer and the seller *simultaneously* announce a reservation price r and price offer p , respectively. Trade happens at the price p whenever the reservation price is below the price offer. If we leave the rest of the model unchanged, the following is an equilibrium for every δ_k : $p(c) \equiv 1$ and $r(v) \equiv 0$. In the corresponding equilibrium outcome A_k , trading probabilities are zero for all types and $S(A_k) = 0$ for all k .

While the sequence of outcomes satisfies monotonicity, no rent extraction, and availability, weak efficiency fails: For any pair (v, c) with $v > c$, the trading surplus $(v - c)$ is strictly larger than their joint limiting payoffs, $\liminf (V_k^S(c) + V_k^B(v))$, which is 0.

²⁵In the original model, individual buyers have elastic demand for multiple units. Sellers, however, are restricted to offer linear prices. Therefore, they distort the trading quantity downwards. This inefficiency disappears once the restriction to linear prices is dropped.

²⁶His interest, however, stems from the prior use of simultaneous auctions in dynamic matching and bargaining games in the context of *common values*, see e.g., Wolinsky (1990).

Bargaining is inefficient because of a "coordination failure" between the traders. As observed by Serrano, this failure occurs because we cannot use sequential rationality to rule out such equilibria.²⁷

Note the similarity to the failure of convergence with an entry stage: Setting a price above the highest valuation (and setting a reservation price below the lowest cost) is similar to deciding to not become an active trader. And just as it is a best response not to take an interior bargaining position if no other trader does so, it is a best response not to become active if no other trader does. But note also that just in the same way as we can restore sequential rationality by introducing "trembles" to the price setting decisions we can restore equilibria with trading when traders tremble at the entry decision stage.²⁸

7.3 Mass Balance fails with Cloning

Cloning refers to the assumption that every trader who leaves the market is replaced by an exact copy of his type, a *clone*. With this assumption, the inflow depends on the trading outcome and is *endogenous*. The pool of traders, however, does not change over time and is *exogenous*. A model with cloning has been recently used by De Fraja and Sakovics (2001). They use this model to argue that trading outcomes depend sensitively on the exact specifications of the bargaining protocol. Since this contrasts with the view taken in our paper, we want to understand their result. Throughout the first part of this section, we will follow De Fraja and Sakovics and take the exogenously given pool (the "stocks") as the fundamental of our model, i.e. we define the "Walrasian" price and the surplus both with respect to this distribution. As we will see, with cloning, *equilibrium* outcomes generically yield a surplus strictly above S^* . This is the analogue to prices not being Walrasian, which is what De Fraja and Sakovics concentrate on.

Gale (1987) argued that one should define the Walrasian price and, analogously, the surplus, with respect to the inflows (see his critique of the model by Rubinstein and Wolinsky (1985) who use a cloning assumption). We provide a short comment on how to evaluate the surplus with respect to the inflow at the end.

To understand how it is feasible with cloning that the surplus of an outcome exceeds S^* we first sketch the idea in the following example. In this example, the full consumer surplus for all $v > 0$ is realized while expected costs are zero. (Note the similarity with the Ponzi scheme with infinitely lived traders): Suppose all buyers who are matched with a seller with costs $c \leq \varepsilon$ can trade at a price ε . If ε is close to zero and if all buyers can be certain to be matched with such a seller, then indeed the full consumer surplus for all $v > 0$ is realized while costs are zero. Cloning makes it possible: Because of cloning, the share of sellers in the pool who have costs $c \in [0, \varepsilon]$ is exogenously fixed and strictly positive. Therefore, buyers have a strictly positive chance to be matched

²⁷In our basic model, sequential rationality enter via the assumption that buyers use a reservation price that is equal to the continuation payoff.

²⁸See Gale (1987, p. 30), who argues that equilibria without entry are not stable.

with such a seller in every single period, and with $\delta \rightarrow 0$, they become certain to be able to trade with such a seller in the limit.²⁹

To understand the result in more depth, we use the symmetric information model of Section 5.2.³⁰ To recall the model: All traders from the pool are matched into pairs. In each pair they observe each others' valuation v and cost c . Then, with probability $\alpha \in (0, 1)$, the seller is chosen to be the *proposer* of a price while with probability $(1 - \alpha)$ the buyer is chosen to be the proposer. The other trader, the *responder*, can either accept or reject the offer. Afterwards, all those pairs in which the responder accepts the offer, trade and leave the pool. An additional share δ of those who did not trade leaves (dies). Now the new traders enter. But different from the model in Section 5.2, the inflow consists of exact clones of the leaving traders. Therefore, independent of who actually traded, the distribution of traders in the pool at the end of the period is always equal to the distribution in the beginning. Let these distributions be $G^S(\cdot)$ and $G^B(\cdot)$.

For every δ_k and for α , we fix an equilibrium outcome $A_k^C(\alpha) = [V_k^S, V_k^B, Q_k^S, Q_k^B]$ of the cloning variant. Since we are using exactly the same matching and bargaining technology as in Section 5.2, our four conditions still hold. Therefore, from the first part of the proof of the main Proposition 1, we know that the outcome must become pairwise efficient for every convergent subsequence, i.e.,

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c. \quad (33)$$

Actually, the limiting outcome can be fully characterized quite easily: It is standard to verify³¹ that there is a price $p^N(\alpha)$ such that, for $\delta_k \rightarrow 0$, limiting payoffs become $\bar{V}^S(c) = \max\{p^N(\alpha) - c, 0\}$ and $\bar{V}^B(v) = \max\{v - p^N(\alpha), 0\}$. Limiting trading probabilities become $\bar{Q}^S(\cdot) = 1_{c < p^N(\alpha)}$ and $\bar{Q}^B(\cdot) = 1_{v > p^N(\alpha)}$. The price $p^N(\alpha)$ itself is given as the unique solution to the following condition:

$$(1 - \alpha) \int_0^{p^N(\alpha)} (p^N(\alpha) - c) dG^S(c) = \alpha \int_{p^N(\alpha)}^1 (v - p^N(\alpha)) dG^B(v).$$

Note that for $\alpha = \frac{1}{2}$, the price $p^N(\frac{1}{2})$ equates the expected surplus of buyers and sellers.

The price $p^N(\alpha)$ depends on the distribution of bargaining power, and the price is

²⁹Note, however, that within every period almost no trade takes place. Therefore, almost no new traders enter the market and the surplus with respect to the inflow converges to one (see the comment at the end of this section).

³⁰The main differences are that DeFraja and Sakovics (2001) include an entry stage and have discounting instead of an exit rate. In addition, they use a noisy search technology, i.e., they assume that a buyer is matched with a random number of sellers. None of these differences affects the main conclusions.

³¹Using the techniques by Gale (1987), see for example the teaching notes by Wright, <http://www.ssc.upenn.edu/~rwright/courses/rw.pdf>.

strictly increasing in α . Thus, generically, the limiting outcome fails to be Walrasian:³² Only for a single point α^* does $p^N(\alpha^*)$ equal p^w , while for all $\alpha \in (0, 1) \setminus \alpha^*$, $p^N(\alpha) \neq p^w$ and hence $(\bar{Q}^S, \bar{Q}^B) \notin Q^W$. This is not necessarily bad: for every $p^N(\alpha)$, the expected limiting payoff is *above* S^* . Simple algebra reveals that for all $p^N(\alpha) \neq p^w$

$$S^* < \int_{p^N(\alpha)}^1 (v - p^N(\alpha)) dG^B(v) + \int_0^{p^N(\alpha)} (p^N(\alpha) - c) dG^S(c).$$

To illustrate the failure we look at the extreme case with $\alpha \rightarrow 0$, i.e., when buyers enjoy all the bargaining power. In this case, the condition requires that the price must become zero, $\lim_{\alpha \rightarrow 0} p^N(\alpha) = 0$. (We take the limit of outcomes with respect to $\delta \rightarrow 0$ first and with $\alpha \rightarrow 0$ afterwards.) This corresponds to our introductory example, with $\varepsilon = p^N(\alpha)$ being small. Thus, expected equilibrium payoffs among buyers become approximately $\int_0^1 v dG^B(v)$ while expected payoffs to sellers become zero. Hence, $\lim_{\alpha \rightarrow 0} \lim_{\delta_k \rightarrow 0} S(A_k^C(\alpha)) = \int_0^1 v dG^B(v) - 0$. This is strictly larger than S^* - how can this happen?

Note that in the limiting outcome, almost all buyers trade with certainty, while almost no seller trades. Thus, the mass buyers who trade converges to $\int_0^1 1 dG^B(v) = 1$, while the mass of sellers who trade converges to $\int_0^1 0 dG^S(c) = 0$. Therefore, for some δ_k and α small enough, the mass balance condition (11) is violated.

Why is it possible with cloning that all buyers can trade? For any $\alpha \in (0, 1)$, in any period, the probability that a buyer is matched with a seller who accepts to trade at the price $p^N(\alpha)$ is strictly positive, since $G^S(p^N(\alpha)) > 0$. So with $\delta_k \rightarrow 0$, it becomes certain that a buyer will be matched with a seller who agrees to trade. In the original model, without cloning, this is not true: If all trade occurs at a price $p^N(\alpha)$ close to zero, sellers with costs $c \leq p^N(\alpha)$ would become scarce, and the share of such sellers in the pool would become zero.

Since our specification of the matching and bargaining protocol in the above example is standard, the peculiar results are only due to the cloning assumption. Nevertheless, De Fraja and Sakovics (2001) explicitly introduce cloning as a *technical* assumption.³³ They do not claim that cloning is meant to reflect underlying economic conditions. But because this assumption has such a strong implication for the results, one might try to use means other than cloning to solve possible technical problems.

³²DeFraja and Sakovics interpret this and similar results as indicating the importance of "local market conditions" for limiting outcomes, reflected here in the distribution of bargaining power.

³³DeFraja and Sakovics have infinitely lived traders and assume an entry stage (see Section 6.2 of our paper). They write that they assume cloning to ensure the stationarity of the mass of traders who decide *not* to enter the pool (see p. 846). They do not explicitly state any problem that would arise otherwise. Actually, in Gale (1987, section 6) and Satterthwaite and Shneyerov (2006), the mass of non-entering types is "infinite" without causing problems. (Basically, this mass just plays no role in any of the above papers and is not even explicitly defined.)

Also, one might take the inflows as fundamental objects as argued by Gale in 1987. Let us therefore consider the surplus with respect to the entering traders. For this, let $A^{IN} = [V^S, V^B, G^{SC}, G^{BC}, Q^S, Q^B]$ denote an outcome where the c.d.f.s $G^{SC}(\cdot)$ and $G^{BC}(\cdot)$, refer to the endogeneous inflows of clones. The expected surplus of the entering traders is given by

$$S^C(A^{IN}) = \int_0^1 V^S(c) dG^{SC}(c) + \int_0^1 V^B(v) dG^{BC}(v).$$

Thus, maximization of the surplus requires not only maximization with respect to the expected payoff of each type of the clones, i.e., with respect to V^S and V^B , but also with respect to their endogeneous distributions G^{SC} and G^{BC} . In particular, an outcome which satisfies the condition of Lemma 3, i.e., pairwise efficiency for all types, does not need to be efficient. In fact, the limiting equilibrium outcome associated with small α is pairwise efficient as we know from (33) but it is nevertheless quite inefficient when α is close to the zero: Then, the share of sellers who actually trade (i.e., those with costs below α) is almost zero. But then almost no buyer can trade in a given period. Thus, the inflow of clones who replace those buyers who trade, must be almost zero. So with $A_k^{IN}(\alpha)$ being the equilibrium outcome for given α and k , the limiting surplus is zero for $k \rightarrow \infty$ and $\alpha \rightarrow 0$:

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \lim_{k \rightarrow \infty} S^C(A_k^{IN}(\alpha)) &= \lim_{\alpha \rightarrow 0} \lim_{k \rightarrow \infty} \int_0^1 \underbrace{V_k^S(c)}_{\rightarrow 0} \underbrace{dG_k^{SC}(c)}_{\rightarrow 0} + \int_0^1 \underbrace{V_k^B(v)}_{\rightarrow 1} \underbrace{dG_k^{BC}(v)}_{\rightarrow 0} \\ &= 0. \end{aligned}$$

This finding is our analogue to the almost trivial observation that prices are market clearing with respect to the endogeneous inflow.

8 Conclusion

We have introduced a new approach to the analysis of decentralized markets with vanishing frictions. By directly characterizing sequences of trading outcomes independently of the fine details of the trading institution, we have shown which conditions imply convergence to efficiency across different models. We then have validated this approach by showing that sequences of equilibrium outcomes for models in the literature satisfy these conditions.

Several open questions remain. First, we assume that p^w is known ex ante. In many markets, however, traders are uncertain about the supply and demand, and p^w is a random variable. Can we expect decentralized markets to converge to efficiency even if traders have to learn the state of the market? Second, when analyzing trading with an entry stage we had to exclude by assumption sequences which are trivial and in which no trader ever enters. Are there conditions on the economic fundamentals that ensure that every sequence is non-trivial? Finally, what can we say about more general preferences

and production technologies, in particular, what can we say about the interaction of markets for several different goods, like, for example, the markets for labor and capital?

A Appendix

A.1 Proof of Lemma 1

Let

$$S_M(M^T) \equiv \max_{Q \in \hat{Q}} S_Q(\cdot) \\ \text{st. } M^T = \int_0^1 Q^S(c) dG^S(c) = \int_0^1 Q^B(v) dG^B(v), \quad (34)$$

and note that

$$\max_{Q \in \hat{Q}} S_Q(\cdot) = \max_{M^T \in [0,1]} S_M(\cdot).$$

Let $p^S(M^T)$ be such that $G^S(p^S) = M^T$ and $p^B(M^T)$ be such that $1 - G^B(p^B) = M^T$. Then clearly

$$Q(M) \equiv \arg \max_{Q \in \hat{Q}} S_Q(\cdot) \text{ st. (34)} \\ = \left\{ Q \in \hat{Q} \mid \int_0^1 \left| Q^S(c) - 1_{c \leq p^S(M)}(c) \right| dc + \int_0^1 \left| Q^B(v) - 1_{v \geq p^B(M)}(v) \right| dv = 0 \right\}$$

and thus

$$S_M(M^T) = \int_{p^B(M^T)}^1 v dG^B(v) - \int_0^{p^S(M^T)} c dG^S(v).$$

Note that $S_M(\cdot)$ is continuously differentiable in M^T and the second derivative of $S_M(M^T)$ is $-\left(\frac{1}{dG^B(p^B(M^T))} + \frac{1}{dG^S(p^S(M^T))}\right)$; so the surplus is strictly concave in M^T . Therefore, a necessary and sufficient condition for $M^* \in \arg \max S_M(\cdot)$ is that the first derivative is zero:

$$p^B(M^*) - p^S(M^*) = 0, \quad (35)$$

which implies that the cutoffs must be the market clearing price p^w : By definition of M^T , $M^T = G^S(p^S(M^T)) = 1 - G^B(p^B(M^T))$. This is true at $p^B(M^*) = p^S(M^*)$ only for $p^B(M^*) = p^S(M^*) = p^w$. Thus:

$$Q^W = Q(G^S(p^w)) = \arg \max_{Q \in \hat{Q}} S_Q(\cdot). \quad \blacksquare$$

A.2 Proof of Lemma 2

The "if" part follows directly from continuity of $S_Q(\cdot)$ and from Lemma 1. For the "only if" part, recall that we say $\lim_{k \rightarrow \infty} Q_k = Q^W$ if $d(Q_k, Q') \rightarrow 0$ for all $Q' \in Q^W$.

By Helly's selection theorem, every sequence of monotone functions has a convergent subsequence. Take such a subsequence and let \bar{Q} denote its limit. Lebesgue's bounded convergence theorem implies that $S_Q(\bar{Q}) = S^*$. Therefore $\bar{Q} \in Q^W$ from Lemma 1. Hence, every convergent subsequence has its limit in Q^W , and thus the sequence itself converges to Q^W (see Lemma 7 in the Appendix) ■

A.3 Proof of Lemma 3

Suppose the limiting statement does not hold. According to the Bolzano-Weierstrass theorem, this implies that there is some $\varepsilon > 0$ and some subsequence indexed by k' , such that $S(A_{k'})$ converges and $\lim_{k' \rightarrow \infty} S(A_{k'}) \leq S^* - \varepsilon$. Take some subsubsequence indexed by k'' such that $V_{k''}^S, V_{k''}^B$ converge pointwise. Such a subsubsequence exists by $(V_{k''}^S, V_{k''}^B) \in \Sigma_+ \times \Sigma_-$ and Helly's selection theorem. Let \bar{p} be defined as before: $\bar{p} \equiv \inf_{c \leq p^w} (\bar{V}^S(c) + c)$. Along the subsubsequence, $\lim_{k'' \rightarrow \infty} V_{k''}^S(c) \geq \bar{V}^S(c) \geq \bar{p} - c$ for all $c \leq p^w$, and similarly, $\lim_{k'' \rightarrow \infty} \inf V_{k''}^B(v) \geq v - \bar{p}$ for all $v \geq p^w$, by the condition of the lemma. Hence

$$\lim_{k'' \rightarrow \infty} \inf S(A_{k''}) \geq \int_{p^w}^1 (v - \bar{p}) dG^B(v) + \int_0^{p^w} (\bar{p} - c) dG^S(c) = S^*,$$

where the last equality follows the observation in the first part of the proof. This contradicts the starting hypothesis $\lim_{k' \rightarrow \infty} S(A_{k'}) \leq S^* - \varepsilon$. ■

A.4 Proof of Convergence of Sequences

The sets of monotone functions Σ_+ and Σ_- are sequentially compact and satisfy the conditions of the following lemma (see footnote 3.1). The proof of the first part is standard and the second part is a straightforward extension:

Lemma 7 *Let (X, τ) be a sequentially compact topological space. Suppose there is some sequence $\{x_n\} = x_1, x_2, \dots$ in X and some $y \in X$ such that every convergent subsequence converges to y . Then*

$$x_n \rightarrow y.$$

Similarly, suppose there is some subset $Y \subset X$ such that every convergent subsequence of $\{x_n\}$ converges to some $y \in Y$ (possibly y is different for each subsequence). Then for every neighborhood $G \supset Y$, there is some $N(G)$, st. $x_n \in G$ for all $n \geq N(G)$, and we say $x_n \rightarrow Y$.

Proof: We prove the first part by contradiction: Suppose not, then by the definition of convergence, there is some neighborhood G of y that does not contain all elements of the sequence from some index onwards, i.e., for every N there is some $n'(N) \geq N$ such that $x_{n'(N)} \notin G$. This allows the construction of a subsequence $\{x_{n'}\}$ such that $x_{n'} \notin G$ for all n' . By X being sequentially compact, there is some convergent subsubsequence of $\{x_{n'}\}$. By the hypothesis of the lemma, this subsubsequence converges to y . This contradicts $x_{n'} \notin G$ for all n' . The second statement follows similarly: Suppose not, then there

would be some neighborhood $G \supset Y$ and some subsequence $\{x_{n'}\}$ such that $x_{n'} \notin G$ for all n' . Again, we can find a convergent subsubsequence by X being sequentially compact; this implies a contradiction to the definition of $\{x_{n'}\}$ \blacksquare

A.5 Proof of $q^{PS} \rightarrow 1$

We want to show that $\lim_{k \rightarrow \infty} Q_k^S(c) = 1$ implies $\lim_{k \rightarrow \infty} q^{PS}(p_k^S(\cdot, c), c | \sigma_F, \delta_k) = 1$. Note that $V_k^S(\cdot)$ is decreasing and $V_k^B(\cdot)$ is increasing. This implies that reservation prices $r_k^S(\cdot)$ and $r_k^B(\cdot)$ are monotone. Furthermore, if $r_k^B(v) = p_k^S(c, v)$ then $r_k^S(c) = p_k^B(v, c)$ because $p_k^S(c, v) = r_k^B(v)$ if and only if the continuation payoff is below the reservation price, i.e., if and only if

$$\begin{aligned} (1 - \delta_k) U^S(p_k^S, r_k^S, c | \sigma_{kF}) &\leq r_k^B(v) - c \\ &= v - (1 - \delta) U^B(p^B, r^B, v | \sigma_F) - c \end{aligned}$$

and hence $v - r_k^S(c) \geq (1 - \delta) U^B(p^B, r^B, v | \sigma_F)$. Therefore, the probability to trade is independent of whether one is a proposer or a responder. So $D^S(p_k^S, r_k^S, c | \sigma_{Fk}, \delta_k) = P^S(p_k^S, r_k^S, c | \sigma_{Fk}, \delta_k)$. This implies that if $\lim_{k \rightarrow \infty} Q_k^S(c) = 1$, then $\lim_{k \rightarrow \infty} q^{PS}(p_k^S(\cdot, c), c | \sigma_F, \delta_k) = 1$: With $P_k^S(p_k^S) \equiv P^S(p_k^S, r_k^S, c | \sigma_{Fk}, \delta_k)$

$$\lim_{k \rightarrow \infty} Q_k^S(c) = \lim_{k \rightarrow \infty} \frac{P_k^S(p_k^S)}{1 - (1 - \delta_k)(1 - P_k^S(p_k^S))} = 1$$

implies $\lim_{k \rightarrow \infty} \delta_k [P_k^S(p_k^S)]^{-1} = 0$ and therefore $\lim_{k \rightarrow \infty} \delta_k [\alpha P_k^S(p_k^S)]^{-1} = 0$. Thus $q^{PS}(p_k^S(\cdot, c), c | \sigma_F, \delta_k) \rightarrow 1$ by

$$q^{PS}(p_k^S(\cdot, c), c | \sigma_F, \delta_k) = \frac{\alpha P_k^S(p_k^S)}{1 - (1 - \delta_k)(1 - \alpha P_k^S(p_k^S))}. \quad \blacksquare$$

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