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Abstract: We use simulation-based techniques to compare and contrast two methods for solving state-dependent pricing models: discretization, which solves and simulates the model on a grid; and collocation, which relies on Chebyshev polynomials. While both methods produce qualitatively similar results, statistically significant quantitative differences do arise. We present evidence favoring discretization over collocation in this context, given a lack of robustness in the latter.

Keywords: state-dependent prices, discretization, collocation, value function iteration

JEL classification: E31, E37, C63, C68

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I Introduction

Computational advances have revolutionized economics, allowing for the solution and simulation of previously intractable problems. Macroeconomists developing micro-founded models have especially benefited from these developments. In particular, large databases underlying computation of price indexes have produced new evidence on pricing behavior of firms, which in turn has spurred interest in developing models to match and explain these facts. These models have generally relied on state-dependent pricing (SDP) mechanisms, since SDP models allow for a broader range of outcomes than their time-dependent counterparts, but they lack the explicit closed-form solutions of the latter due to their nonlinearities and nondifferentiabilities. Thus they are a prime candidate for solving via computational methods.

This paper compares and contrasts two methods of solving SDP models using simulation techniques. The first method considered is *discretization*: we convert the problem's state variables to a grid, perform value function iteration, construct a policy matrix, and simulate the model by constraining actions and outcomes to the discretized states. The second method considered is *collocation*: we use Chebyshev polynomials to approximate the solution to the value function and use this solution to simulate arbitrary state realizations. We consider performance across the methods through a variety of indicators—such as macro (business cycle) moments, micro (pricing) moments, impulse responses, and computational aspects (processing time, memory, and numerical precision)—and for alternative parameterizations of the structural model.

Our findings suggest that the discretization and collocation solution methods generally provide results that are qualitatively similar. However, the results tend to exhibit statistically significant quantitative differences. This latter point is important for economists using these models to calibrate or estimate structural parameters.

In light of this discrepancy, we view the evidence as favoring discretization over collocation for SDP models using simulation techniques. This conclusion is based on several facts. First, moments produced using discretization converge without requiring extremely large numbers of grid points or computational time. We do not find the same result for collocation. Second, discretization can require considerably less time than collocation—a relevant fact for practical implementation. Third, discretization appears to be a more reliable solution method than collocation in terms of robustness to alternative parameterizations and the addition of grid points/nodes. This conclusion contrasts with that presented by Hatchondo et al. (2008) for a model of sovereign default, but is in line with the warning in Aruoba et al. (2006) that nondifferentiabilities (as are present in SDP models) may prove problematic for collocation methods.

We also contribute to the growing literature that seeks to match empirical micro pricing evidence with SDP models. In particular, the model incorporates firm-specific factor markets for labor as one source of real rigidity in the midst of idiosyncratic productivity shocks. Under such an assumption, however, we find that the menu costs would need to average 2.1% of revenues and productivity shocks would need a standard deviation of 22.5%—numbers that would seem to be implausibly large. The finding that firm-specific labor markets are difficult to reconcile within SDP models is consistent with similar findings for diminishing returns to labor in Golosov and Lucas (2007) and for kinked demand curves in Klenow and Willis (2006), supporting Nakamura and Steinsson's (2007) conjecture that, in order to be consistent with micro pricing evidence, real rigidity must emanate from other sources.

The outline of the paper is as follows. Section II develops the model and Section III discusses the discretization and collocation methods. Section IV calibrates the model. Section V presents results from solving and simulating the model using discretization and collocation, and Section VI discusses their relevance. Section VII concludes. An Appendix (Section VIII) contains additional robustness exercises, extensions, and explanations.

II The Model

The model is relatively standard in the New Keynesian tradition (cf. Woodford 2003), featuring a representative household and monopolistically competitive firms. Firms face explicit "menu" costs they must pay to change their prices, generating state-dependent pricing decisions. The ability to optimize over the timing of price changes is a feature unique to SDP models, in contrast to the more common assumption in the sticky-price literature of Calvo-style price setting in which the timing of price changes is random.

II.1 Households and Firms

A representative household maximizes utility given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{\delta}{1+\phi} \int_0^1 l_{jt}^{1+\phi} dj \right], \tag{1}$$

where β is the subjective discount factor, σ is the inverse intertemporal elasticity of substitution, ϕ is the inverse labor supply elasticity, and δ is a parameter determining the relative weighting of labor to consumption in the utility function. The amount of labor supplied to firm j is l_{jt}. For conciseness, we henceforth omit time subscripts and use $'$ and $_{-1}$ to denote the next and the previous period's values, respectively.

The consumption composite is $C = [\int_0^1 c_j^{(\theta-1)/\theta} dj]^{\theta/(\theta-1)}$, with elasticity of substitution

$\theta > 1$ and c_j consumption of the jth good. The price level $P = [\int_0^1 p_j^{1-\theta} dj]^{1/(1-\theta)}$ is the minimum cost of purchasing one unit of the consumption aggregate, with p_j the price of good j. Given this framework, consumers maximize consumption for a given level of expenditure. Total consumption expenditures are $\int_0^1 p_j c_j dj = PC$, and demand for good j is $c_j = C\,(p_j/P)^{-\theta}$.

The household faces the budget constraint $C \leq \int_0^1 \left(\frac{w_j}{P} l_j + \Pi_j\right) dj$, where w_j and Π_j are the nominal wage and real profits, respectively, earned from firm j.[1] Thus labor supply is a function of the real wage and consumption,

$$\frac{w_j}{P} = \delta l_j^{\phi} C^{\sigma}. \tag{2}$$

We also impose a binding cash-in-advance constraint, $PC = M$, with M the money supply.

Firms are monopolistic competitors producing differentiated goods. Market clearing requires that aggregate output Y equal aggregate consumption C, and that consumption and production of each good be equal ($y_j = c_j$) for all j. Thus demand for firm j's product is

$$y_j = Y \left(\frac{p_j}{P}\right)^{-\theta}. \tag{3}$$

Firms satisfy all demand at their price p_j via the production function $y_j = a_j \ell_j^{\alpha}$, with ℓ_j the amount of labor used to produce good j, returns to scale in labor α, and productivity for firm j given by a_j. Firm-specific productivity is stochastic and follows

$$\ln a_j = \rho_a \ln a_{j,-1} + \xi_j, \tag{4}$$

with $\rho_a \in (0,1)$ and $\xi_j \sim$ i.i.d. $\mathrm{N}\big(0, \sigma_\xi^2\big)$.

The labor supplied to firm j from the household's problem, l_j, may differ from the

[1] Rotemberg (1987) develops and discusses a similar model along these lines.

amount used in production, ℓ_j, since it is costly for a firm to adjust its price. Specifically, a firm that sets $p_j \neq p_{j,-1}$ expends a fixed amount of labor, Φ, on this change—i.e., there is a menu cost to adjusting prices. The relationship between labor supplied to firm j and labor demand is $l_j = \ell_j + I_j\Phi$, where the indicator I_j equals one if a price change occurs and zero otherwise.

When making pricing decisions, firms realize they face an upward-sloping labor supply curve given by (2) and demand for their product given by (3). Real profits for firm j are

$$\Pi\left(\frac{p_j}{P}, a_j, Y, I_j\right) = Y\left(\frac{p_j}{P}\right)^{1-\theta} - \delta Y^{\sigma}\left[\left(\frac{Y}{a_j}\right)^{1/\alpha}\left(\frac{p_j}{P}\right)^{-\theta/\alpha} + I_j\Phi\right]^{1+\phi}. \tag{5}$$

Contemporaneous profits if $I_j = 0$ and the firm keeps the previous period's price are denoted Π^K, and Π^C denotes profits if $I_j = 1$ and it changes its price to $\widetilde{p}_j$. With Z denoting a vector of aggregate variables, the value to the firm of keeping its old price is

$$V^K\left(\frac{p_{j,-1}}{P}, a_j, Z\right) = \Pi^K\left(\frac{p_{j,-1}}{P}, a_j, Y\right) + \beta EV\left(\frac{p_{j,-1}}{P'}, a'_j, Z'\right). \tag{6}$$

The value to the firm of changing its price is

$$V^C\left(\frac{p_{j,-1}}{P}, a_j, Z\right) = \max_{\widetilde{p}_j} \Pi^C\left(\frac{\widetilde{p}_j}{P}, a_j, Y\right) + \beta EV\left(\frac{\widetilde{p}_j}{P'}, a'_j, Z'\right), \tag{7}$$

with the firm using Φ units of labor in the price change captured by Π^C. The firm optimizes over these choices such that

$$V = \max\left\{V^K, V^C\right\}. \tag{8}$$

II.2 Monetary Policy and Expectations

The typical convention in SDP models is for monetary policy to take the form of a rule for the level of money or money growth. We consider the AR(1) process

$$\Delta m = \mu(1-\rho) + \rho\Delta m_{-1} + \varepsilon, \tag{9}$$

where $\rho \in (0, 1)$, $\Delta m = \ln(M/M_{-1})$, and $\varepsilon \sim$ i.i.d. $\mathrm{N}(0, \sigma_\varepsilon^2)$.

Firms form expectations over productivity and money growth via (4) and (9), respectively. To form expectations over other aggregate variables, we assume firms employ a forecasting rule. The use of a forecasting rule is related to Krusell and Smith (1998) and was introduced into SDP models by Willis (2002). While simplifying computation of expectations, the forecasting rule is consistent with the real-world idea that information is costly to acquire and process.[2] Such a stylized fact renders the full-information, rational expectations equilibrium—which requires every firm to know all the state variables of all other firms in the economy—infeasible to implement.[3]

We posit that agents use the forecasting rule

$$\ln Y = b_0 + b_1 \ln Y_{-1} + b_2 \Delta m. \tag{10}$$

The rule has several notable features. First and foremost, it is parsimonious, thereby keeping the number of state variables to a minimum. Second, it has economic significance: b_1 measures the persistence of real output movements, and b_2 measures the response of output to nominal shocks.

To ensure that expectations are on average consistent with outcomes, we guess a starting value for the vector of coefficients from the forecasting rule, denoted B_0, and simulate the model using the implied forecasting rule. One then estimates (10) with the realizations of the aggregate variables, producing a set of estimates $\hat{B}_0$. For some critical value τ^c, if

[2] Among others, see the lines of research by Mankiw and Reis (2002), Sims (2003), and Reis (2006). Zbaracki et al. (2004) offer empirical evidence of information costs in a case study of a large industrial manufacturer.

[3] As an alternative, one could impose distributional restrictions on the state space to make the problem tractable, as in Dotsey et al. (1999), for instance. However, such restrictions would need to be consistent with the behaviors of real-world price setters.

$\left|\hat{B}_0 - B_0\right| < \tau^c$ then agents' expectations in the model are consistent with outcomes and vice versa. If not, we choose new coefficients B_1 and iterate to convergence.

III Methods for Solving and Simulating the Model

State-dependent pricing models have the virtue of enabling firms to optimize over the timing of their price changes. But this benefit comes with a cost. In particular, SDP models can be cumbersome to work with, given the number of state variables involved, and the discrete nature of the problem—firms either keep their old price or pay a fixed cost and change it as they see fit—creates nonlinearities and nondifferentiabilities. Computational methods are therefore especially important for researchers in this field. This section discusses two alternative methods of solving and simulating these types of models.

III.1 Discretization

The first method we employ is discretization. We discretize the relevant state variables to solve the firms' problem via value function iteration and then constrain actions when simulating the model so as to remain on the grid. Papers using variants of this approach in the SDP literature include Willis (2002), Klenow and Willis (2006, 2007), Knotek (2006), and Nakamura and Steinsson (2007, 2008).

We discretize the firms' real prices p_j/P, the idiosyncratic productivity states a_j, and the aggregate variables (denoted by Z): money growth Δm, output Y, and inflation π. The real price grid contains 349 points in increments of 0.15%, fine enough to capture interesting pricing behaviors in the model, including potential monetary neutrality. Following Tauchen (1986), productivity is converted into its discrete Markov representation spanning two standard deviations around the unconditional process mean. Similarly, the aggregate variables are discretized after combining the money growth rule (9) and the forecasting rule (10) into a VAR and converting it to its Markov representation, as in Terry and Knotek (2008). These Markov representations are used to form expectations during value function iteration. The discretized states for money growth and output imply a discretization of inflation via the cash-in-advance constraint.

Value function iteration proceeds in three steps. First, we initialize V via a method that guesses a common value V_0 for all elements in the value function and then iteratively assesses whether the guess should have been higher (or lower), moving in the direction indicated. Second, from V_0 we use an accelerating algorithm, starting with a small number of grid points for the aggregate variables (but the full number for p_j/P and a_j). After iterating to convergence, we add aggregate grid points and linearly interpolate. The third step iterates to convergence on the full version of V. With the final value of V, we construct a policy matrix that returns the price firm j sets (and, thus, whether it changes its price or not) as a function of $p_{j,-1}/P$, a_j, and the aggregate variables Z.

To simulate the model, random values of a_j and Δm are generated using their Markov approximations. We solve for the other aggregate variables on the grid through a guess-and-verify procedure, such that firms' real prices $p_{j,-1}/P$, productivity a_j, and elements of Z are consistent with pricing outcomes suggested by the policy matrix and with each other. While a_j and Z are always elements of the grid, real prices are "nudged" to the nearest grid point since there is no guarantee that $p_{j,-1}/\left(P_{-1}e^{\pi}\right)$ is a point on the real-price grid.[4]

[4] This constrasts with other approaches—e.g., Aruoba et al. (2006)—which use linear interpolation when simulating the model.

III.2 Collocation

The second method we employ is a projection technique known as collocation. Collocation allows us to convert the Bellman problem into a system of equations involving linear combinations of orthogonal Chebyshev polynomials that can be dealt with using standard nonlinear solution techniques for finite-dimensional problems.[5] Klenow and Kryvtsov (2008) employ Chebyshev collocation in SDP models, and Hatchondo et al. (2008) apply the method to a model of sovereign default.

To solve the Bellman problem, for a finite N let $\bar{\psi}_i$ be a Chebyshev basis and χ_i be coefficients, where $i = 1, ..., N$. The appropriate basis of Chebyshev polynomials is the tensor product of the corresponding univariate Chebyshev polynomial bases. Since the state S is four dimensional in $p_{j,-1}/P$, a_j, Δm, and Y, we also specify N collocation state nodes on a four-dimensional interval $I = [\underline{\iota}_1, \bar{\iota}_1] \times ... \times [\underline{\iota}_4, \bar{\iota}_4]$. The exact nodes $S_1, ..., S_N$ are formed by taking the Cartesian product of the four sets of univariate Chebyshev interpolation nodes, which are in turn determined by the selection of the endpoints in the interval I.[6] With this framework, we can proceed in one of two ways.

The first way is to approximate the value function itself, V, with Chebyshev polynomials: $V = \sum_{i=1}^{N} \chi_i^V \bar{\psi}_i(S)$. At each Chebyshev node S_k, $k = 1, ..., N$, we then require that this approximation satisfy the Bellman equation (8), which can be written as

$$\sum_{i=1}^{N} \chi_i^V \bar{\psi}_i(S_k) = \max \left\{ \begin{array}{c} \Pi^K(S_k) + \beta E \sum_{i=1}^{N} \chi_i^V \bar{\psi}_i(S'), \\ \max_{\tilde{p}_j} \; \Pi^C(S(k, \tilde{p}_j)) + \beta E \sum_{i=1}^{N} \chi_i^V \bar{\psi}_i(S'(\tilde{p}_j)) \end{array} \right\}. \tag{11}$$

[5] Judd (1998) provides some standard definitions describing the polynomial basis that we use; see also Miranda and Fackler (2002).

[6] The curse of dimensionality is present here since we must have that $N = N_1 * N_2 * N_3 * N_4$, where N_r is the size of the univariate Chebyshev basis $\psi_1^r, ..., \psi_{N_r}^r$ containing polynomials with maximum degree $N_r - 1$. Also note that each four-dimensional polynomial $\bar{\psi}_i = \psi^1_{i_1(i)} * \psi^2_{i_2(i)} * \psi^3_{i_3(i)} * \psi^4_{i_4(i)}$ is by definition of the tensor product equal to the product of four of these univariate basis polynomials.

This produces a system of N nonlinear equations depending on the N coefficients χ^V.

The second way is to separately approximate the firm's value to keeping its price and the value to changing its price as $V^K = \sum_{i=1}^N \chi_i^K \bar{\psi}_i(S)$ and $V^C = \sum_{i=1}^N \chi_i^C \bar{\psi}_i(S)$. At each Chebyshev node S_k, $k = 1, ..., N$, we require that the approximations satisfy equations (6) and (7), respectively:

$$\sum_{i=1}^N \chi_i^K \bar{\psi}_i(S_k) = \Pi^K(S_k) + \beta E \max \left\{ \sum_{i=1}^N \chi_i^K \bar{\psi}_i(S'), \sum_{i=1}^N \chi_i^C \bar{\psi}_i(S') \right\}, \tag{12}$$

$$\sum_{i=1}^N \chi_i^C \bar{\psi}_i(S_k) = \max_{\tilde{p}_j} \Pi^C(S\left(k, \tilde{p}_j\right)) + \beta E \max \left\{ \begin{array}{c} \sum_{i=1}^N \chi_i^K \bar{\psi}_i(S'\left(\tilde{p}_j\right)), \\ \sum_{i=1}^N \chi_i^C \bar{\psi}_i(S'\left(\tilde{p}_j\right)) \end{array} \right\}. \tag{13}$$

Note that since χ_i^K and χ_i^C depend on themselves and each other, this produces a single system of $2N$ nonlinear equations depending on the $2N$ coefficients $\left[\chi^K, \chi^C\right]'$.

Expectations in (11), (12), and (13) must be taken over ξ_j and ε to find S'. The implied integration is carried out numerically using Gauss-Hermite quadrature. The above systems can be expressed compactly as $\Psi\chi = F\left(\chi\right)$, or

$$\Psi\chi - F(\chi) = 0. \tag{14}$$

When approximating V, Ψ is $N \times N$ such that $\Psi_{k,i} = \bar{\psi}_i(S_k)$, χ is $N \times 1$ with i-th component χ_i^V, and $F(\chi)$ is $N \times 1$ with k-th component given by the quadrature-based evaluation of the right-hand side of (11). When approximating V^K and V^C, Ψ is $2N \times 2N$ block diagonal, $\chi = \left[\chi^K, \chi^C\right]'$, and $F(\chi)$ is a $2N \times 1$ vector with the stacked quadrature-based evaluations of the right-hand sides of (12) and (13), respectively. This replaces the solution of the integral Bellman equation with a finite-dimensional root-finding problem, depending solely on the coefficient vector χ. A variety of methods can solve for χ; we use Newton's method but discuss this and other options below.[7]

[7] In solving (14) for any χ, we also must perform the firm's underlying optimization over $\tilde{p}_j$ at each node

Given a coefficient solution χ, a firm's policy decision can in principle be accurately obtained via evaluation of the expressions on the right-hand sides of (11), or (12) and (13), replacing the node S_k with arbitrary state S to exploit the continuous nature of collocation. Unfortunately, repeated evaluation of these expressions is extremely time-consuming. To avoid this, note that with the coefficient solution χ^V one can obtain at each collocation node S_k the value to the firm of keeping its price $V^K\left(S_k;\chi^V\right)$, the value of changing its price $V^C\left(S_k;\chi^V\right)$, and the optimal price when changing $\tilde{p}\left(S_k;\chi^V\right)$. One can then interpolate these functions on the set of collocation nodes, producing the polynomial interpolants $\overline{V^K}(S)$, $\overline{V^C}(S)$, and $\overline{\tilde{p}}\,(S)$ for arbitrary S. For the case in which we approximate the value to keeping and the value to changing price directly, the coefficient solutions χ^K and χ^C allow one to evaluate $V^K = \sum_{i=1}^N \chi_i^K \bar{\psi}_i(S)$ and $V^C = \sum_{i=1}^N \chi_i^C \bar{\psi}_i(S)$ for arbitrary S, thereby only requiring one interpolant $\overline{\tilde{p}}\,(S)$.

To simulate the model, values for a_j and Δm are generated based on (4) and (9), respectively, and firms enter the period with nominal prices $p_{j,-1}$.[8] We then solve for equilibrium output (and by extension the price level and inflation) using a guess-and-verify procedure, wherein the guess completes current period state information for each firm and we evaluate whether firms would wish to keep their price or change it. In the case of the latter, we evaluate $\overline{\tilde{p}}$ to obtain the firm's new price. We then verify whether the guess is consistent with the aggregate outcomes, iterating to convergence as necessary.[9]

S_k. Because of the explicit form of the profit function and the Chebyshev basis polynomials, we analytically compute first and second derivatives with respect to the policy variable and apply a modified version of Newton's method to solve this optimization problem. We discuss this in more detail below.

[8]The values of a_j and Δm are constrained to their collocation/interpolation intervals in order to avoid pathological behavior outside of the intervals.

[9]Reiter (2006) presents an alternative solution method that combines elements of projection and perturbation methods.

IV Model Calibration

The structural parameters of the model are calibrated using values in line with the literature. Table 1 provides a detailed list. We track the pricing decisions of $n = 5,000$ firms and aggregate their decisions accordingly. Adding more firms did not materially affect the conclusions presented below. The parameters of the exogenous money growth process (9) were estimated for nominal GDP growth for the U.S. between 1984Q1 and 2007Q4. This $T = 96$ quarter period is also the length of a simulation. Forecasting rule coefficients in (10) were found by averaging over $S = 25$ simulations.

The model uses a quarterly frequency, hence $\beta = 0.99$. Utility over consumption takes a natural-log form ($\sigma = 1$). The persistence of productivity shocks is $\rho_a = 0.35$, translating the coefficient from Nakamura and Steinsson (2007) from a monthly into a quarterly frequency. The parameter δ is calibrated so the flexible-price rate of output is 1.

As a baseline case, we consider an economy characterized by real rigidity (or strategic complementarity in pricing decisions), with an elasticity of substitution $\theta = 5$, returns to scale in labor $\alpha = 2/3$, and inverse labor supply elasticity $\phi = 0.5$, generating a reduced-form real rigidity parameter around 0.31.[10] We explored variations on these parameters as well. Given the other calibrations of the model, the menu cost Φ (expressed in terms of labor) and the size of the idiosyncratic productivity shocks σ_ξ are calibrated to match evidence on the average duration between price changes and the average absolute size of price changes in Klenow and Kryvtsov (2008) for regular prices.

Finally, solving and simulating the model requires selecting the number of grid points

[10] Algebraically, the amount of real rigidity can be measured by $(\sigma - \kappa) / (1 - \theta\kappa)$, with $\kappa = 1 - (1 + \phi) / \alpha$.

for the discretization technique, and for the collocation technique the number of Chebyshev nodes and the number of Gauss-Hermite quadrature nodes. For discretization, results below range for grids from 100 thousand to 25 million points. For collocation, we present results ranging from 250 to 1600 Chebyshev nodes with 9 to 12 quadrature nodes. The Appendix contains more details.

V Comparisons between Solution Methods

The model's richness provides myriad opportunities for analyzing the discretization and collocation methods. We compare and contrast the solution techniques along a large number of dimensions, including value function characteristics, direct simulations, micro (pricing) moments, and macro (business cycle) moments. The Appendix provides further comparisons.

V.1 Evaluating and Comparing Value Functions

The different techniques' approaches to solving the Bellman problem suggest that examination of the value functions themselves is warranted. For collocation approximating the value function V directly, we assess the accuracy of the solution via comparison of the left- and right-hand sides of (11) after solving (14) for the coefficients χ^V. By definition, the Bellman equation is satisfied exactly at the collocation nodes; away from those nodes, the two sides will not perfectly coincide. When approximating the functions V^K and V^C instead of V, the appropriate comparison is, for any state value, the maximum of the left-hand sides of (12) and (13) versus the maximum of the right-hand sides of (12) and (13) after solving for χ^K and χ^C.

Figure 1(a) plots a cross-section of the left- and right-hand sides of the direct approximation of V at one collocation node, allowing real price to vary. The purely polynomial left-hand side tracks the right-hand side well for much of the time, with the Chebyshev approximation exhibiting traditional equioscillatory behavior. At nondifferentiable points—where the pricing decision switches between changing and keeping—the fit predictably deteriorates. Figure 1(b) plots the comparable cross-section of the left- and right-hand sides for V implied when approximating V^K and V^C directly. Because a single polynomial is no longer used to approximate a kinked function, the left-hand side plotted in panel (b) more precisely captures the nondifferentiable switching points.[11] Differences between the plotted left- and right-hand sides represent errors in the firms' problem, since firms' values, as represented by the right-hand side, are based in part on next period expectations involving the left-hand side. The discretization technique does not require similar assessment, since value function iteration produces discretized values for V that are internally consistent to a prescribed tolerance.[12]

Figure 2 compares the value functions across the solution techniques, varying real price and with all other states at steady-state values. The methods yield similar results. Qualitatively, the hump-shaped regions of firm inaction and the flat portions in which firms reoptimize nominal price cover virtually identical areas of the real-price space. Quantitatively, the results differ at most by 2.5%.

[11] The collocation equation (14) is solved to within a maximum absolute tolerance of 1E−8 in the standard norm. When approximating V with 448 collocation nodes, this yields a global maximum absolute percentage difference between the left- and right-hand sides of 0.1%. When approximating V^K and V^C directly, the global maximum absolute percentage difference between the implied left- and right-hand sides of V is 0.05%. The Appendix contains a detailed analysis of the precision of the collocation method given larger numbers of nodes.

[12] The maximum absolute percentage difference between value function iteration loops was set to 1E−6; varying this tolerance did not affect the analysis.

V.2 Comparing Simulated Economies

The ideal way to compare model simulations across solution techniques is to subject the methods to the same exogenous processes and examine the responses of endogenous variables. This is slightly problematic in the current context, since discretization requires approximation through Markov processes and remaining "on the grid" whereas collocation presumes normal shocks and can handle arbitrary states. To avoid this inconsistency, we use the Markov approximations to (4) and (9) to draw random series for idiosyncratic productivity and money growth, respectively, and simulate the responses of the discretized and collocation economies to these same series.

Figure 3 plots the inflation and output gap series generated by the solution techniques for a representative 96-quarter period. The inflation series move in virtually identical ways and are highly correlated with each other; the output gaps are similar but exhibit more differences to the naked eye, since small differences in inflation translate into relatively larger differences in output gaps. In general, adding more discretization grid points or Chebyshev nodes produces similar results. By contrast, if one were to use only a few hundred thousand discretization grid points, or very few Chebyshev nodes, the dynamics would differ quantitatively *and* qualitatively between the solution methods.

V.3 Computing Time and Resources

The more interesting comparisons between the models, however, embrace the fact that the discretization and collocation solution techniques generate differing exogenous processes because discretization uses Markov approximations for (4) and (9). The number of discretization grid points affects not only the Markov approximations for the exogenous processes but also the endogenous responses as well. We consider how these issues affect the mechanics of solving and simulating the model, as well as the ensuing dynamics, in the following sub-sections.

Crucial considerations for economists doing computational work are the time and memory required to solve and simulate—and, potentially, estimate—a model. Along these lines, we find a tradeoff between memory usage and computational time across solution techniques. Figure 4 plots the maximum memory required to solve and simulate the model using discretization and collocation for different numbers of discretization points and collocation nodes. For a reasonably fine grid with 7.5 million points, discretization requires more memory than collocation. Figure 5 plots the total amount of time required to solve and simulate the model across techniques. With 7.5 million grid points, discretization requires a fraction of the time of collocation.[13]

The reasons for the time-memory tradeoff are simple. Discretization stores the value and policy functions at every discrete point, absorbing a large amount of memory. Once this information is stored, model simulation is very fast because firms' actions are constrained to the grid and these are read directly from the policy matrix. The continuous nature of collocation avoids the need for large amounts of memory usage, but it requires polynomial evaluations to determine firms' actions at the continuously varying states. The time required for these evaluations makes collocation much slower than the grid technique, especially as the size of the matrix calculations involved becomes increasingly large.[14]

[13] A dual-processor/dual-core 3.0 GHz CPU with 8.0 GB of RAM was used for the computations. Each method required approximately $2,000$ lines of MATLAB code, with faster development time for the discretization method. Parallelization would reduce processor time for both discretization and collocation, more so for the latter. However, as not all economists may be able to take advantage of parallelization, we do not report times for it.

[14] In general, collocation via Newton's method requires less time to solve for the value function than

Note that there are missing observations associated with collocation in the figures. These are cases in which the solution technique failed to solve or simulate the model correctly for a given combination of Chebyshev nodes. We discuss this below in more detail.[15]

V.4 Macro Moments

Given series for money growth, inflation, and the output gap, one can calculate any number of business cycle statistics. For the sake of space, we summarize macro moments generated by the model along several dimensions. Others, such as standard deviations, autocorrelations, cross-correlations, and other regressions, are available upon request.

A useful and parsimonious description of the macro properties of the model comes from the coefficients from (10), b_1 and b_2, since these coefficients measure the persistence of real output movements and the response of output to nominal money shocks, respectively. Figure 6 plots the average regression value for b_1 and Figure 7 plots the average for b_2 across $s = 100$ simulations to mitigate the impact of differing shocks. In each case, the estimated discretization coefficients in panel (a) stabilize to a narrow band once the grid becomes "fine enough."

Panel (b) shows the same average coefficients from collocation, along with the estimates from the discretization case with 7.5 million grid points for comparison. Qualitatively, the average discretization and collocation coefficients all have the same (positive) sign. However, there are clear quantitative differences between the discretization coefficients and those from collocation. Moreover, there is little evidence to suggest that the collocation coefficients

discretization with value function iteration, though this discrepancy disappears with a large number of collocation nodes and quadrature nodes. In turn, this implies that value function iteration combined with interpolation (to allow firms to move off the grid) during simulation would be the worst of both worlds.

[15] The vertical dotted lines in the collocation figures highlight where changes were made to the number of nodes other than real price. The Appendix contains the combinations used.

converge as one adds more Chebyshev nodes, especially for the money growth coefficient.[16] This suggests that a quantitative analysis of the properties of this state-dependent pricing model may indeed be sensitive to the solution technique.

Impulse response analysis comes to a similar conclusion. Figure 8 plots generalized impulse responses (see Koop et al. 1996) for inflation and the output gap from a one standard deviation (1.6% annualized) shock to money growth in time period 0 for the discretization and collocation cases. Qualitatively, both discretization and collocation produce similar responses, though there are modest quantitative differences.

V.5 Micro Moments

SDP models have risen in popularity as economists have sought to construct models to match salient micro pricing facts. Thus the micro moments emanating from the models—and, implicitly, the structural parameters needed to produce those moments—are of quantitative importance. Figures 9 and 10 display two common statistics in this literature: the average duration between price changes, and the average (absolute) size of price changes. These moments are primarily determined by the size of the menu cost Φ and the standard deviation of productivity shocks σ_ξ, which are calibrated as in Table 1.

As the figures show, the micro pricing moments converge quite quickly as more discretization grid points are added; similar convergence is not apparent for feasible numbers of collocation nodes. While the moments generated by all the methods are broadly similar, statistically significant quantitative differences do arise. By implication, estimation of struc-

[16] These conclusions hold in statistically significant ways, though we omit the two standard error bands from the collocation cases to simplify the figures. For both the discretization and the collocation cases, a solution techniques' moments will converge to the "truth" implied by that technique as the number of grid points/nodes goes to infinity. Given the time demands of collocation (Figure 5), we were unable to find such convergence for feasible numbers of collocation nodes.

tural parameters Φ and σ_{ξ} via a moment-matching exercise would thus be affected by the choice of solution technique.

V.6 Robustness

Alternative parameterizations of the model are clearly possible. In one, we consider the case of strategic neutrality among price setters. The results generally echo those above: the solution techniques yield qualitatively similar results along the dimensions considered, but quantitative differences remain.

In another parameterization, we increased the amount of real rigidity in the model by increasing the elasticity of substitution to $\theta = 11$ and the inverse labor supply elasticity to $\phi = 1$, generating a reduced-form real rigidity parameter around 0.13—a substantial amount of strategic complementarity but within the range of plausible values suggested by Woodford (2003). Despite using the same procedures to solve and simulate the model as outlined above, the discretization and collocation techniques now generate differences quantitatively *and* qualitatively. This finding is well represented by Figure 11, which plots generalized impulse responses for the two solution methods.

The reasons for this discrepancy primarily reside with the collocation method, and in particular with the polynomial interpolants and approximants used to simulate the model.[17] These polynomials make model simulation practical, since the alternative—explicit evaluation of the right-hand sides of (11), (12), and (13)—takes more than 20 times longer to perform for every evaluation. But in this case with substantial real rigidity, the interpolants and approximants become imprecise and generate spurious results. This sensitivity to pa-

[17]Recall from Section III that we use the polynomial interpolants $\overline{V^K}(S)$ and $\overline{V^C}(S)$ for collocation for V, and the approximants $V^K = \sum \chi_i^K \bar{\psi}_i(S)$ and $V^C = \sum \chi_i^C \bar{\psi}_i(S)$ for collocation for V^K and V^C.

rameter calibration is an important drawback to the collocation method. The Appendix contains more details.

Collocation also suffers from a lack of robustness in the solution techniques used for (14). We solve for χ using a fairly standard application of Newton's method; we also considered quasi-Newton and function iteration methods, but none of these proved superior or even feasible in terms of accuracy and computing time. Using Newton's method comes with the substantial cost of local convergence, however, and with problems characterized by high-dimensionality from a large number of collocation nodes it can fail to produce a coefficient solution—as clearly evidenced by the missing observations in the above figures. These issues are exacerbated by the presence of discrete choices implying nondifferentiabilities at certain points of the state space, and they are also present in the underlying firm optimization for $\tilde{p}_j$ in (11) and (13), which is carried out using a univariate version of Newton's method augmented with an initial grid search over the real price space to obtain more accurate starting points for finding $\tilde{p}_j$. For the larger-scale problem (14), we also investigated a version of our technique which adaptively increases the number of real price nodes using previous solutions as subsets of new starting points, but this also fails to produce solutions for certain numbers of nodes and calibrations.[18] Alternative solution techniques, perhaps including derivative-free methods other than function iteration, might robustly produce coefficient solutions for a larger set of nodes and calibrations, but they would require increased computational time that would only add to the already lengthy time required to solve and simulate using collocation.[19]

[18]The techniques we consider are commonly used for solving problems of this type; see Aruoba et al. (2006) or Miranda and Fackler (2002).

[19]See Midrigan (2008) for an application of a simplex-based solution method to an SDP model.

VI Analysis

The common theme to emerge from the above findings is that the discretization and collocation methods generally provide qualitatively similar results, but the quantitative results differ in statistically significant ways. This latter point is important for economists using these models to calibrate or estimate structural parameters. Given this fact, which solution technique is preferable?

In our view, the weight of the evidence ranks discretization ahead of collocation. This conclusion is based on several facts. First, moments in the discretization technique appear to converge (Figures 6, 7, 9, and 10) without requiring extraordinarily large numbers of grid points and/or computational time. We do not find the same convergence in the collocation moments for feasible numbers of nodes. Second, even with a fairly fine grid (e.g., 7.5 million grid points), the time savings from discretization are staggering (Figure 5) and favor this as the more practical choice for economists doing estimation or repeated simulations. Third, the relative simplicity of discretization and its robustness to alternative numbers of nodes and calibrations make it a more reliable solution method than collocation. Aside from the issues documented in Section V.6 with the polynomial interpolants for the case of substantial real rigidity, the nonlinear root-finding methods used to solve (14) or for $\tilde{p}_j$ can suffer from a lack of robustness and fail to solve the model.

Our findings favoring discretization for SDP models can be compared with the results of two related papers. Aruoba et al. (2006) compare a broad range of computational methods for solving a smooth neoclassical growth model with a lower-dimensional state space and report results generally favoring collocation. We validate their warning that these results are likely not generalizable to problems with nondifferentiabilities such as ours. Hatchondo et al. (2008) compare discretization and collocation in a model of sovereign default and conclude that both qualitative and quantitative differences exist due to inaccuracies associated with discretization. Our problem differs from theirs in two important ways. First, their state-space is lower-dimensional, thus avoiding some of the constraints that we encounter using collocation. Second, we analyze moments that stabilize with discretization grids of around 7.5 million points. Even allowing for the larger dimension of our states, this implies much denser coverage than the 30 thousand discretization grid points Hatchondo et al. (2008) are constrained to due to memory limitations.

Finally, two economic points are worth noting. The first of these concerns the forecasting rule (10). In theory, the combination of real rigidity—or strategic complementarity, which makes firms' decisions dependent on the actions of others—and state-dependent pricing produces conditions for multiple equilibria, and these conditions may be worsened by the use of a forecasting rule, since the latter could act as a sunspot to coordinate firms' actions. Fortunately, this does not appear to be the case. The idiosyncratic shocks in the model—needed to match the size of price changes in empirical data—help to decouple firms' desired actions (Caballero and Engel 1993). At the same time, the forecasting rule does not appear to be powerful enough to dominate the intrinsic dynamics of the model. While we lack analytical proof, multiple simulations begun from distinct starting points for the forecasting rule consistently converge to the same final forecasting rule coefficients.

The second point concerns real rigidity, menu costs, and the size of the idiosyncratic productivity shocks. In the model, firms face specific factor markets for labor when $\phi > 0$—which, as Woodford (2003) notes, can be a powerful source of real rigidity while at the same time adding realism to a business cycle model, since factor prices cannot instantaneously adjust across all firms within an economy. In general, the greater is the real rigidity in the economy, the larger must be menu costs *and* productivity shocks in order to make the model consistent with the empirical data on the frequency and size of price changes. This implies that in the baseline calibration, menu costs must average 2.1% of revenues and productivity shocks require a standard deviation of 22.5% to be consistent with micro pricing evidence. These numbers that would appear to be implausibly large by most accounts: for instance, the industrial manufacturer in Zbaracki et al. (2004) paid literal menu costs of 0.04% of revenues, and Levy et al. (1997) document that supermarkets' menu costs amount to 0.7% of revenues.[20] That firm-specific labor markets are difficult to reconcile within SDP models is consistent with similar findings for diminishing returns to labor ($\alpha < 1$) in Golosov and Lucas (2007) and for kinked demand curves in Klenow and Willis (2006), supporting Nakamura and Steinsson's (2007) conjecture that real rigidity must emanate from other sources.

VII Conclusion

This paper solves and simulates a New Keynesian model with state-dependent sticky prices using two alternative methods: discretization and collocation. We compare and contrast the solution techniques along a variety of dimensions, including macro (business cycle) moments, micro (pricing) moments, impulse responses, and computational aspects. In general, we find that the models yield qualitatively similar results that can differ from each other

[20] Dotsey and King (2005) highlight a similar finding with regard to the size of menu costs (around 5.5% of revenues to generate durations around 4 quarters). However, their SDP model differs significantly: idiosyncratic shocks come in the form of randomized menu costs, rather than idiosyncratic productivity and a constant menu cost as in this paper. In addition, they focus only on the duration between price changes rather than their size as well.

quantitatively in statistically significant ways.

However, we also document some important shortcomings that can arise with the collocation method—such as imprecision in polynomial interpolants, the inability of locally convergent root-finding solution methods to handle large numbers of nodes and alternative calibrations, and the notable failure of moments to quickly converge as more nodes are added to the problem—which do not affect the discretization technique. We illustrate one example in which these shortcomings can produce spurious results under collocation, causing qualitative and quantitative discrepancies between the methods. Partly on this basis, we view the evidence as supporting discretization over collocation for state-dependent pricing models using simulation techniques.

An open question at this point is the extent to which these results hold for state-dependent pricing models that replace forecasting rules with simulation-free or linearization techniques—as well as the similarities and differences between these alternative ways of closing the model. We pursue this issue further in ongoing research.

VIII Appendix

The figures in the text present results from discretization grids varying in size from 100 thousand to 25 million points and results from collocation with 252 to 1625 nodes. Table A1 shows the exact number of points and nodes for each state variable for each combination.

To relax the assumption of strategic complementarity across price setters, we re-calibrate the model for strategic neutrality. Doing so requires common factor markets ($\phi = 0$, hence the utility function is linear in labor) and constant returns to labor ($\alpha = 1$). The elasticity of substitution is calibrated as in Golosov and Lucas (2007), $\theta = 7$. Matching data on the micro pricing moments requires $\Phi = 0.02075$ and $\sigma_\xi = 0.07025$. Along virtually all dimensions, the discretization and collocation methods produce results in line with those for the baseline real rigidity case. Direct simulation comparisons remain similar; collocation requires less memory but more time to solve and simulate the model; and average coefficients from (10) and micro pricing moments are qualitatively similar but quantitatively different in statistically significant ways. As with the case of additional real rigidity in Section V.6, we present a generalized impulse response in Figure A1 as a convenient summary of the above.

Section V.6 presents an alternative calibration with substantial real rigidity that delivers quantitatively *and* qualitatively different results across methods. As plotted in Figure A2, the baseline calibration approximations for V^K yield accurate firm decisions because polynomials do not distort the relationship between V^K and V^C, as seen in panels (a) and (b). When there is substantial real rigidity, the variation in V^K is large, as indicated by the different vertical axes in panels (c) and (d). Errors in the approximations absorb this change, and firms sometimes choose to keep their prices unchanged when the polynomial representing V^K inaccurately rises above V^C. These incorrect decisions lead directly to more firm-level price rigidity and the qualitative differences evident in Figure 11.

In principle, larger numbers of collocation nodes N can solve problems with the collocation method by providing more precise approximations. Figure A3 presents results under the baseline calibration. Varying N, we measure both the global maximum absolute percentage difference between the left- and right-hand sides of the Bellman equation and computational time. Values above the plotted line indicate that relative gains to the precision of the value function V are greater than the increase in required time. The reduced benchmark for this exercise is a case with 320 nodes.[21] When performing collocation on V directly, the tradeoff is less favorable due to the inability of a smooth polynomial to replicate the function's sharp points where the firm switches between keeping and changing its price. When performing collocation on V^K and V^C, the use of two polynomials for V makes capturing these kinks trivial, and it is less costly to reduce error. Note, however, that even with more real price nodes we never obtain relative increases of precision of more than 30.

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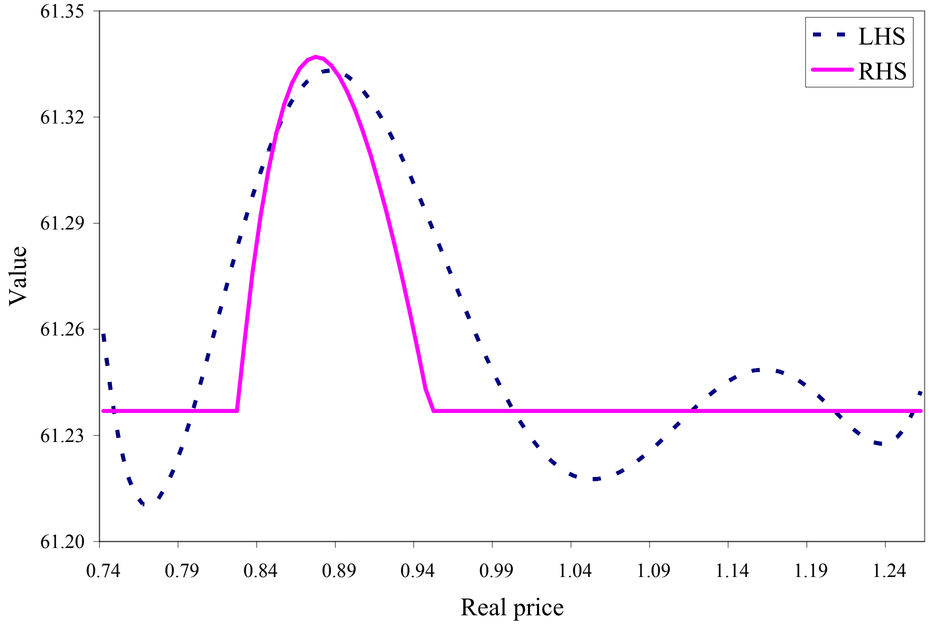
Table 1: Calibrations

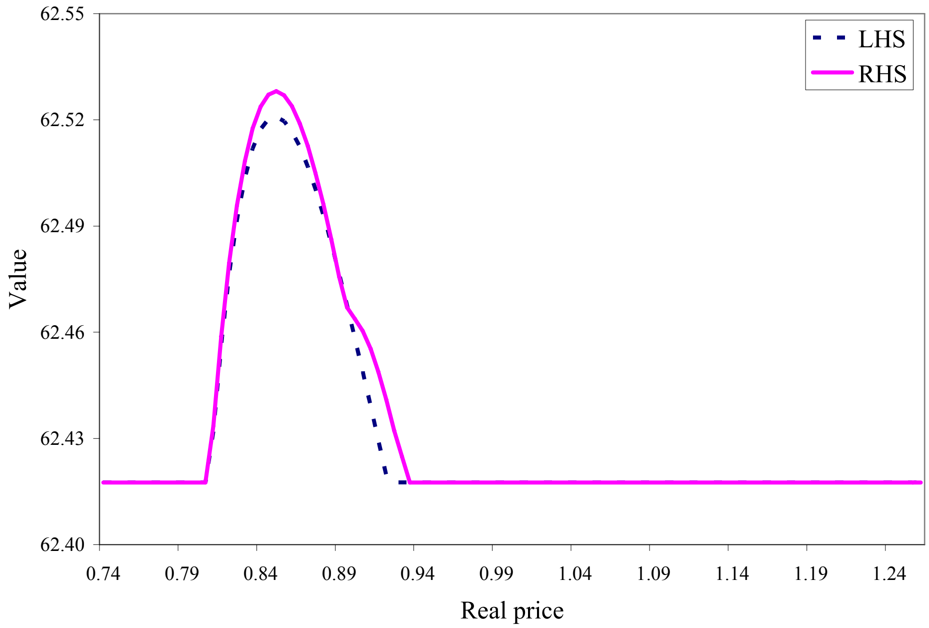
β	0.99	discount rate
σ	1	inverse intertemporal elasticity of substitution
ϕ	0.5	inverse labor supply elasticity
θ	5	elasticity of substitution
α	2/3	returns to labor
ρ_a	0.35	persistence of productivity
σ_ξ	0.225	standard deviation of productivity shocks
Φ	0.156	menu cost, in terms of labor
μ	0.006	steady state money growth
ρ	0.37	persistence of money growth
σ_ε	0.0048	standard deviation of money growth shocks
τ^c	0.005	tolerance for the forecasting rule
n	5000	number of firms
T	96	simulation length (quarters)
S	25	simulations for computing forecasting coefficients
s	100	simulations for moment computations

Figure 1: Collocation Precision

(a): Collocation using V

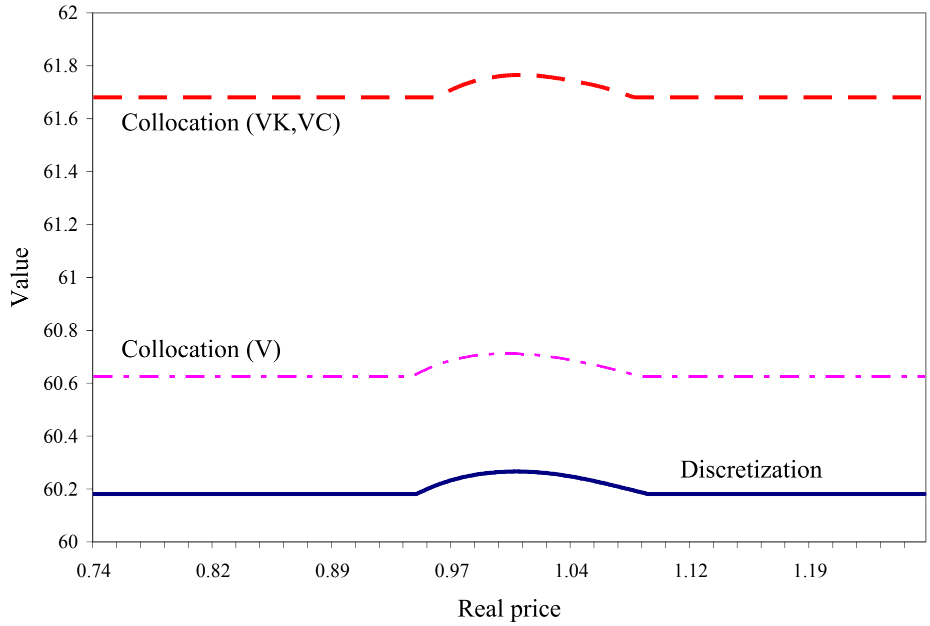


(b): Collocation using V^K, V^C



Notes: Collocation results use 448 Chebyshev nodes. In panel (a), LHS is the left-hand side of equation (11), and RHS is the right-hand side. In panel (b), LHS is the maximum of the left-hand sides of equations (12) and (13), and RHS is the maximum of the right-hand sides.

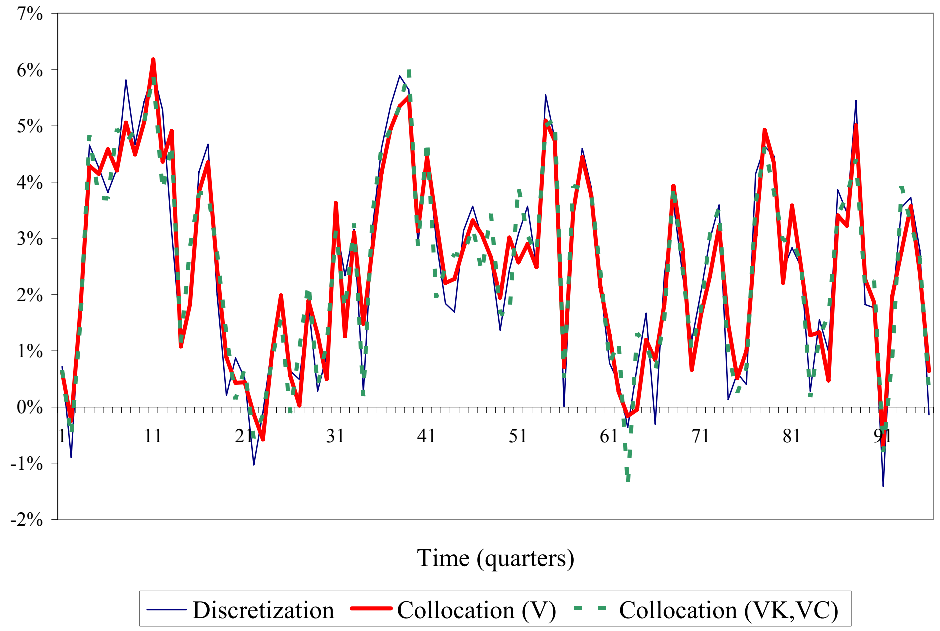
Figure 2: Comparing Value Functions



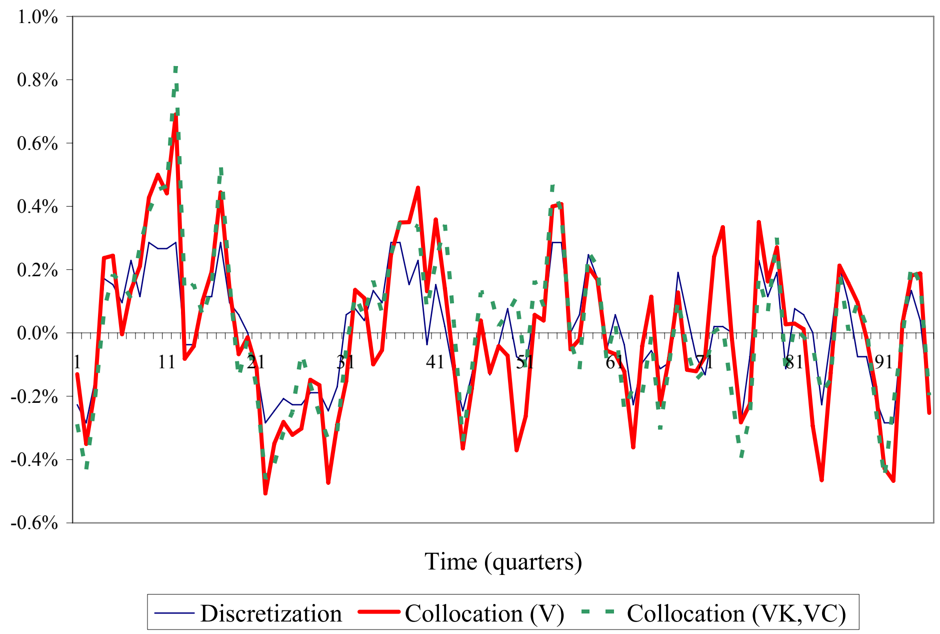
Notes: Discretization uses 7.5 million grid points, while collocation cases use 448 Chebyshev nodes. Productivity, money growth, and output are all constant at their steady state values.

Figure 3: Simulation with Identical Shocks

(a): Inflation (annualized)



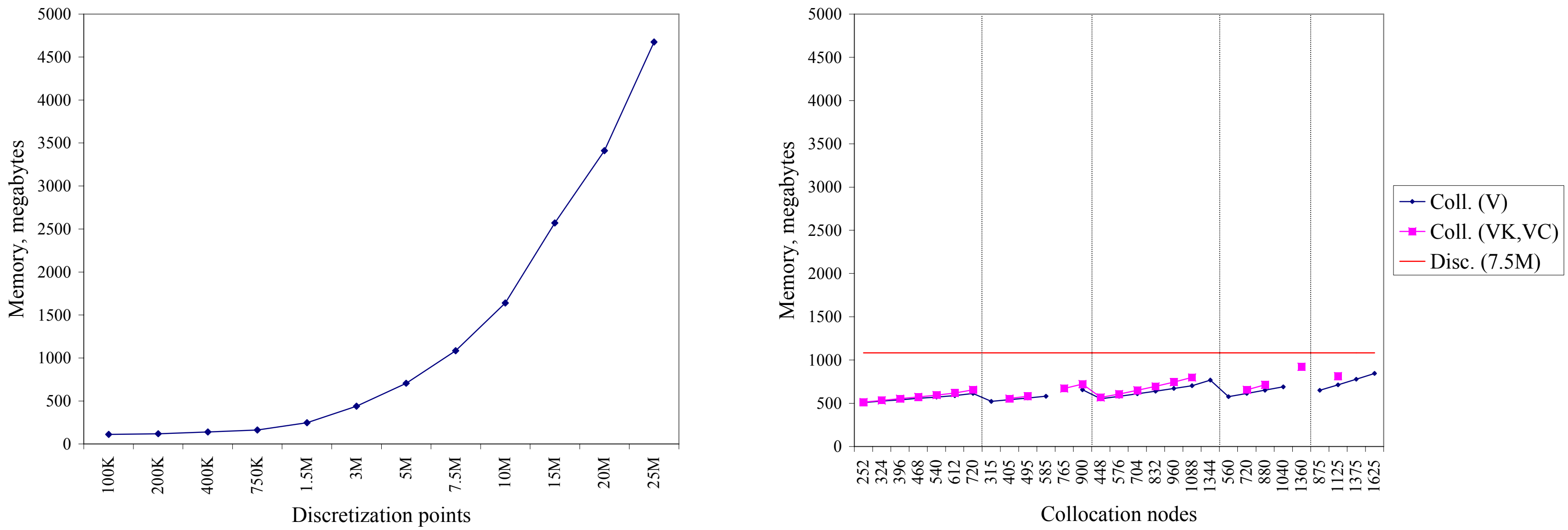
(b): Output gaps



Notes: Plotted are series for inflation and the output gap for one 96-quarter simulation with the same exogenous processes for money growth and productivity across the methods. Discretization uses 7.5 million grid points, while collocation cases use 448 Chebyshev nodes.

Figure 4: Maximum Memory Usage

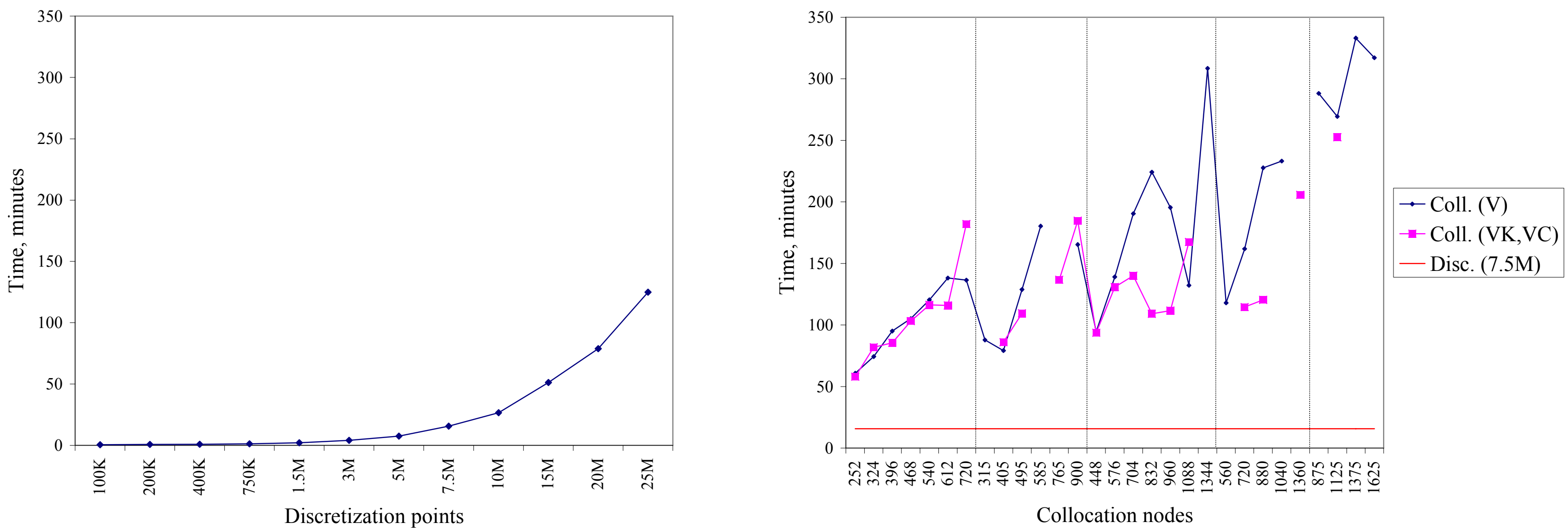
(a): Discretization (b): Collocation



Notes: In panel (a), K=thousand and M=million. In panel (b), coll. denotes collocation on either the value function (V) or for the value to keeping (V^K) and the value to changing (V^C). Disc. (7.5M) denotes discretization with 7.5 million grid points.

Figure 5: Time to Solve and Simulate Models

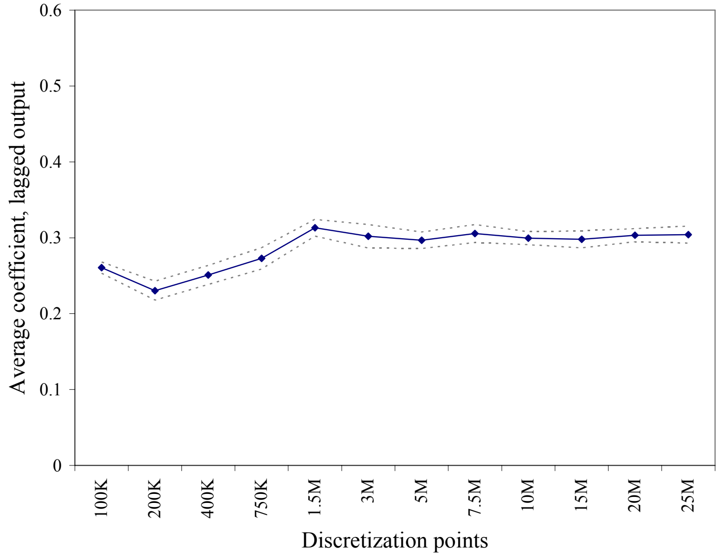
(a): Discretization (b): Collocation



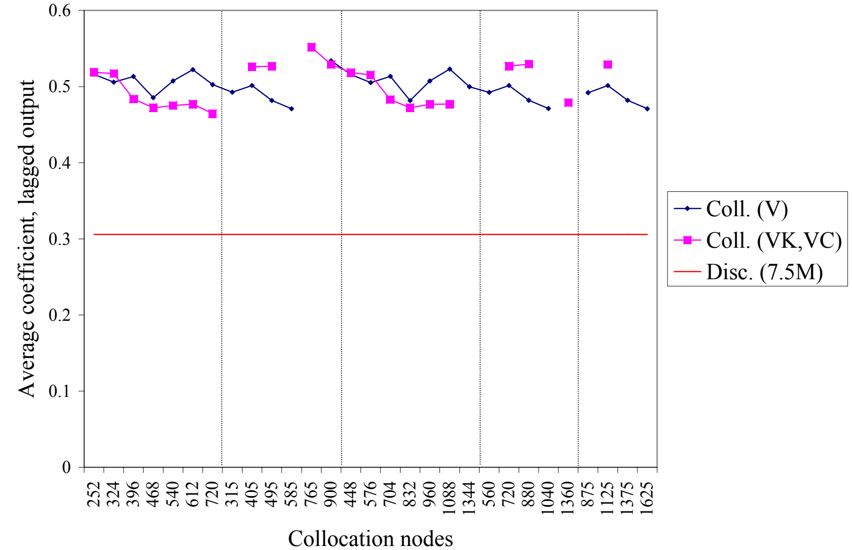
Notes: In panel (a), K=thousand and M=million. In panel (b), coll. denotes collocation on either the value function (V) or for the value to keeping (V^K) and the value to changing (V^C). Disc. (7.5M) denotes discretization with 7.5 million grid points.

Figure 6: Regression Coefficient on Lagged Output

(a): Discretization

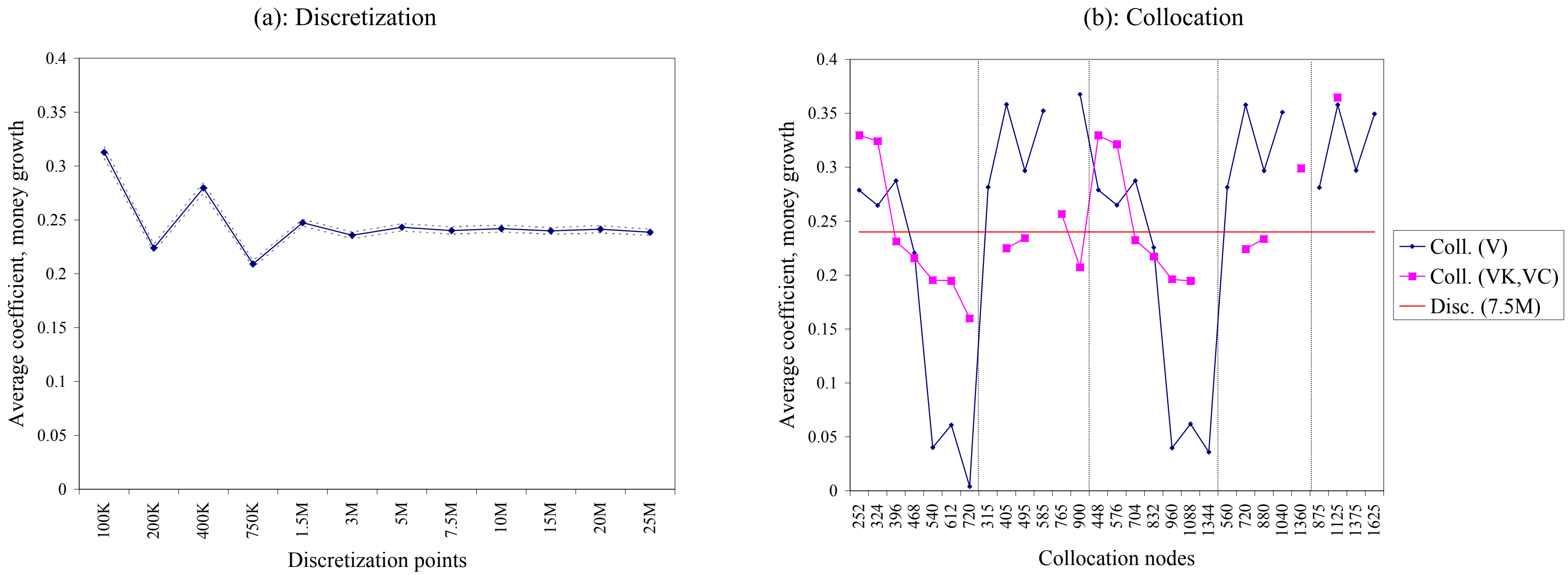


(b): Collocation



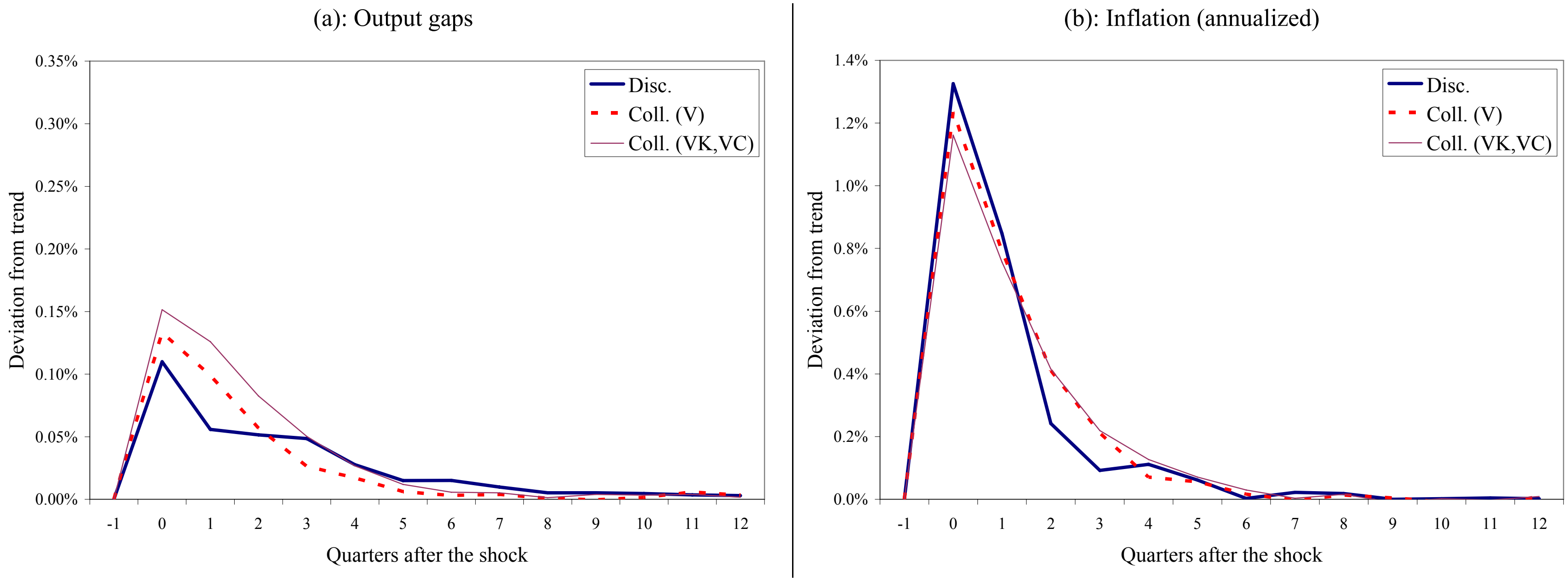
Notes: Plotted are the average b_1 coefficients across 100 simulations from regressions of $\ln Y = b_0 + b_1 \ln Y_{-1} + b_2 \Delta m$ on 96 quarters of simulated data. In panel (a), the dashed lines are two standard error bands, K=thousand and M=million. In panel (b), coll. denotes collocation on either the value function (V) or for the value to keeping (V^K) and the value to changing (V^C). Disc. (7.5M) denotes discretization with 7.5 million grid points.

Figure 7: Regression Coefficient on Money Growth



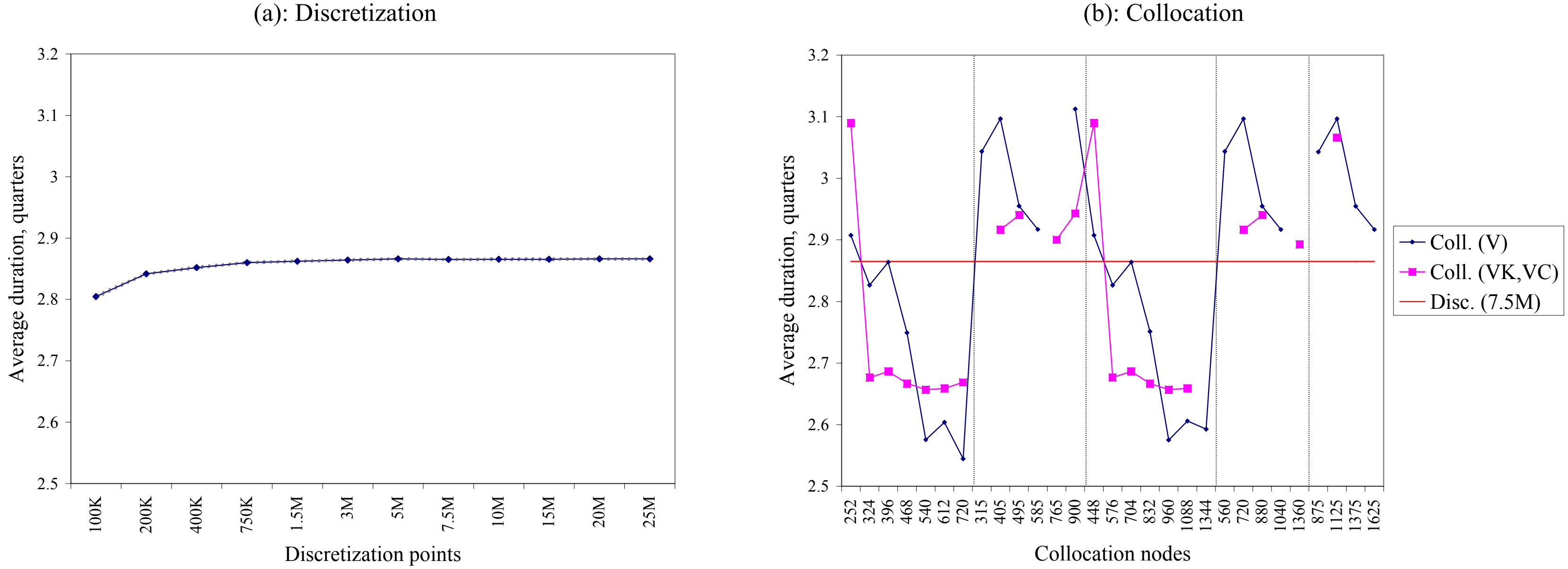
Notes: Plotted are the average b_2 coefficients across 100 simulations from regressions of $\ln Y = b_0 + b_1 \ln Y_{-1} + b_2 \Delta m$ on 96 quarters of simulated data. In panel (a), the dashed lines are two standard error bands, K=thousand and M=million. In panel (b), coll. denotes collocation on either the value function (V) or for the value to keeping (V^K) and the value to changing (V^C). Disc. (7.5M) denotes discretization with 7.5 million grid points.

Figure 8: Generalized Impulse Responses



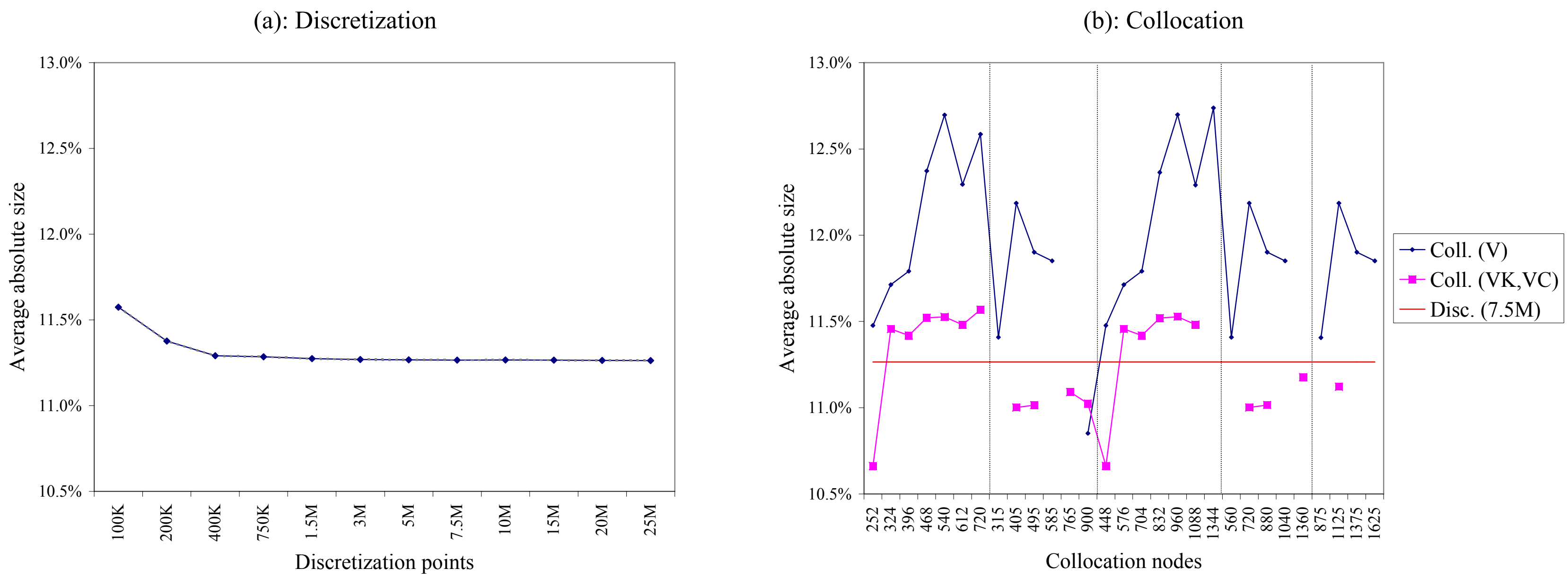
Notes: Responses of the output gap and inflation to a one standard deviation shock to money growth at time 0. Disc. denotes discretization with 7.5 million grid points. Coll. denotes collocation on either the value function (V) or for the value to keeping (V^K) and the value to changing (V^C) with 448 Chebyshev nodes.

Figure 9: Average Duration between Price Changes



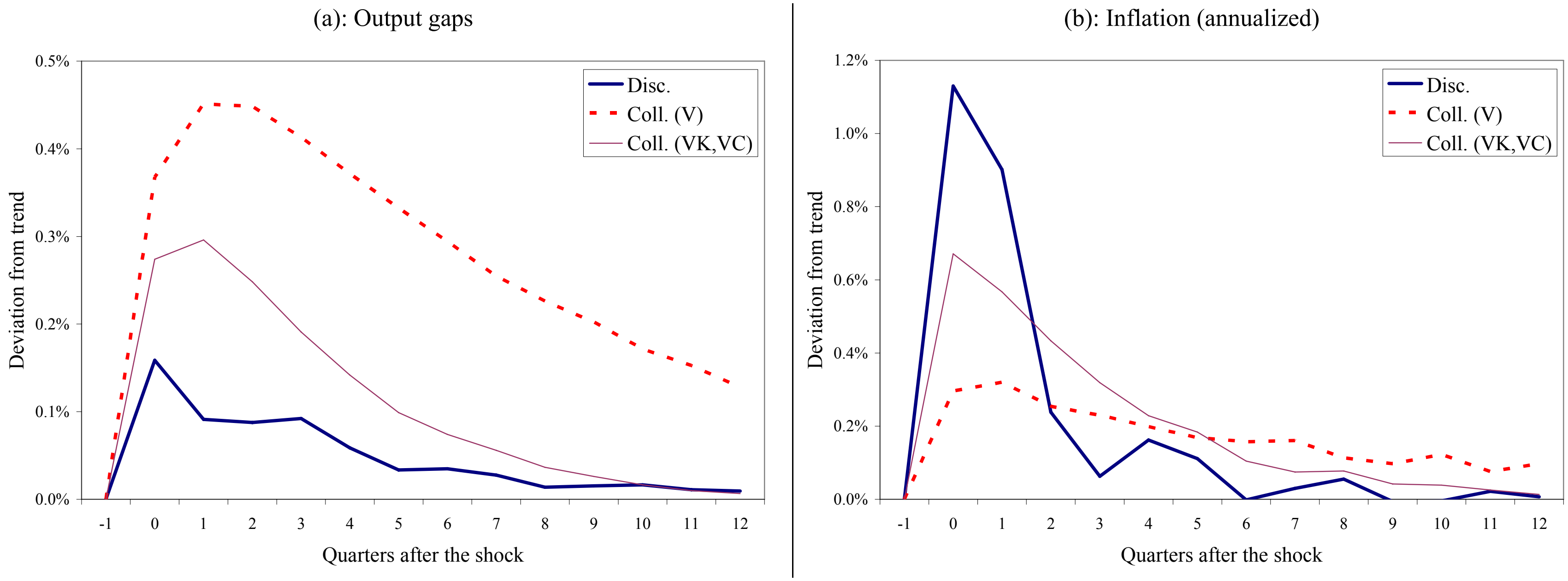
Notes: Plotted are the average duration between price changes across 100 sets of 96 quarters of simulated data. In panel (a), the dashed lines are two standard error bands (which nearly coincide with the observations), K=thousand and M=million. In panel (b), coll. denotes collocation on either the value function (V) or for the value to keeping (V^K) and the value to changing (V^C). Disc. (7.5M) denotes discretization with 7.5 million grid points.

Figure 10: Average Size of Price Changes



Notes: Plotted are the average absolute size of price changes across 100 sets of 96 quarters of simulated data. In panel (a), the dashed lines are two standard error bands (which nearly coincide with the observations), K=thousand and M=million. In panel (b), coll. denotes collocation on either the value function (V) or for the value to keeping (V^K) and the value to changing (V^C). Disc. (7.5M) denotes discretization with 7.5 million grid points.

Figure 11: Generalized Impulse Responses, Substantial Real Rigidity



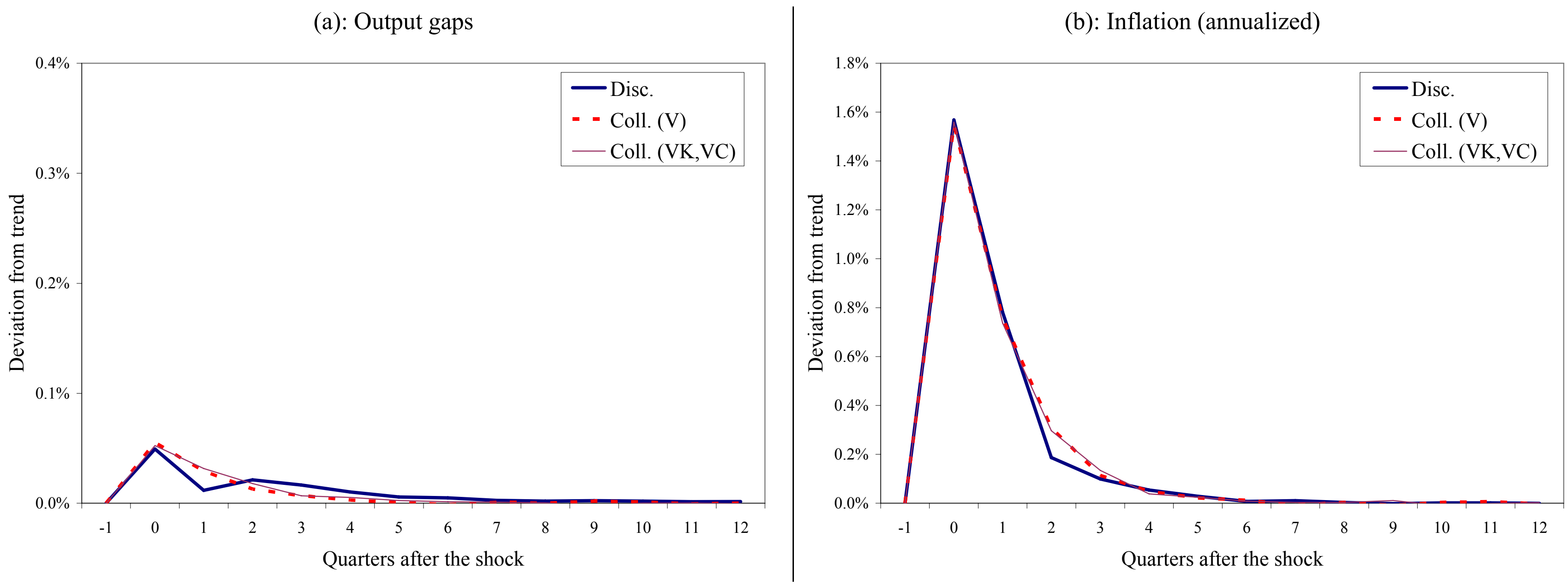
Notes: Responses of the output gap and inflation to a one standard deviation shock to money growth at time 0. Disc. denotes discretization with 7.5 million grid points. Coll. denotes collocation on either the value function (V) with 448 Chebyshev nodes or for the value to keeping (V^K) and the value to changing (V^C) with 900 Chebyshev nodes.

Table A1: Discretization Grid Points and Collocation Nodes

Discretization					Collocation				
Total number of grid points	Real price (p_i/P)	Productivity (a_i)	Money growth (Δm)	Output (Y)	Total number of nodes	Real price (p_i/P)	Productivity (a_i)	Money growth (Δm)	Output (Y)
100K	349	9	5	7	252	7	4	3	3
200K	349	13	5	9	324	9	4	3	3
400K	349	15	7	11	396	11	4	3	3
750K	349	23	7	13	468	13	4	3	3
1.5M	349	29	9	17	540	15	4	3	3
3M	349	35	11	23	612	17	4	3	3
5M	349	41	13	27	720	20	4	3	3
7.5M	349	45	15	31	315	7	5*	3	3
10M	349	51	17	35	405	9	5*	3	3
15M	349	55	19	41	495	11	5*	3	3
20M	349	63	21	43	585	13	5*	3	3
25M	349	67	23	47	765	17	5*	3	3
					900	20	5*	3	3
					448	7	4	4	4
					576	9	4	4	4
					704	11	4	4	4
					832	13	4	4	4
					960	15	4	4	4
					1088	17	4	4	4
					1344	21	4	4	4
					560	7	5*	4	4
					720	9	5*	4	4
					880	11	5*	4	4
					1040	13	5*	4	4
					1360	17	5*	4	4
					875	7	5	5	5
					1125	9	5	5	5
					1375	11	5	5	5
					1625	13	5	5	5

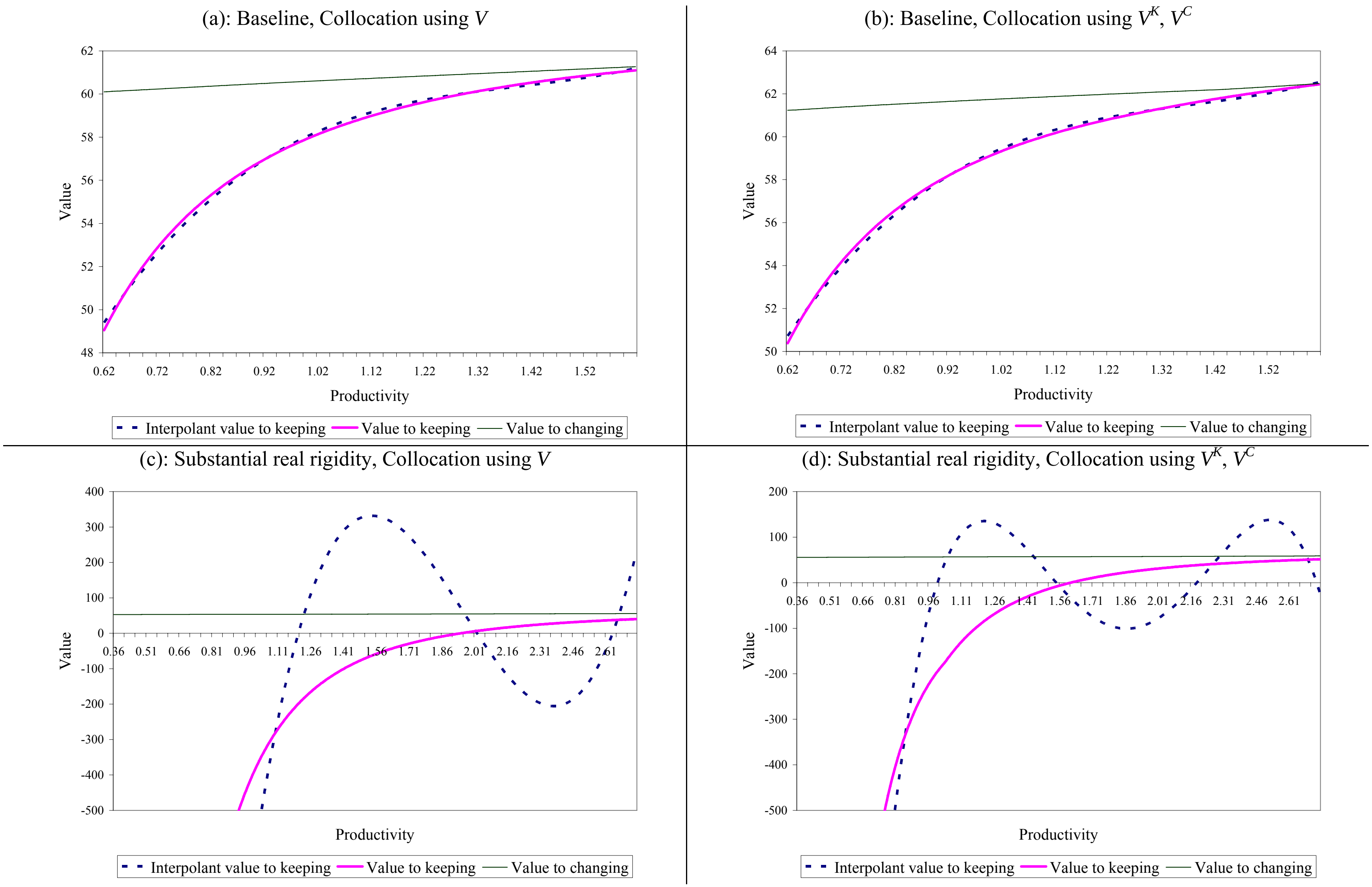
Notes: The total number of grid points is an approximation; K=thousand, M=million. For collocation, there are 3 quadrature nodes for money growth for all cases. For collocation approximating V^K and V^C, * denotes the use of 4 quadrature nodes; all other cases use 3 quadrature nodes for productivity.

Figure A1: Generalized Impulse Responses, Strategic Neutrality



Notes: Responses of the output gap and inflation to a one standard deviation shock to money growth at time 0. Disc. denotes discretization with 7.5 million grid points. Coll. denotes collocation on either the value function (*V*) or for the value to keeping (V^K) and the value to changing (V^C) with 448 Chebyshev nodes.

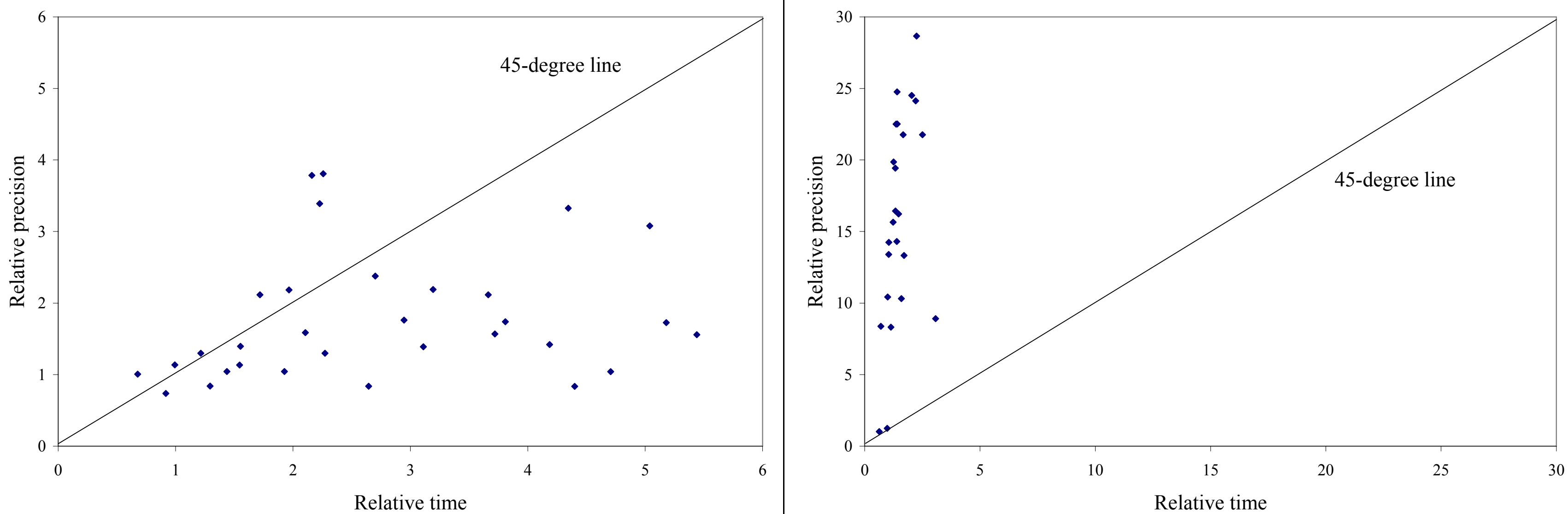
Figure A2: Interpolant versus Actual Values



Notes: For collocation using V, interpolant is $\overline{V^K}$. For collocation using V^K and V^C, interpolant is the left-hand side of (12).

Figure A3: Accuracy of Approximation with Additional Collocation Nodes

(a): Collocation using V (b): Collocation using V^K, V^C



Notes: Relative precision represents the ratio of the global maximum absolute percentage difference between the left- and right-hand sides of the Bellman equation under a benchmark number of collocation nodes to the same metric with a different number of nodes. Relative time represents the ratio of computing time required with an alternative number of nodes to computing time in the benchmark.