# 4 Option Valuation with Stochastic Volatility

The Black-Scholes option pricing model assumes that the volatility of the underlying security is either constant or a known function of time. We shall now discuss a model for pricing options when the volatility is also stochastic. This is a model by Heston published in the RFS in 1993. I should hand out a copy of the article in class. Before jumping into the model, we need to talk a little more about the concept of the market price of risk. These notes were basically created by a Ph.D. student named Anand Goel last year.

## 4.1 Market Price of Risk

We have seen that simple call and put options are examples of derivatives dependent on the price of some underlying asset such as a stock. When the underlying is a traded security, we can create a continuous time hedge to value the derivative. In this case, risk attitudes of investors do not matter and we can use risk-neutral valuation. However, when the price of a derivative security is dependent on a variable that is not traded, risk-neutral valuation may not be possible and risk attitudes may matter. The following discussion taken from Hull's book shows how this risk is priced by the market.

Suppose that $\theta$ is a variable that follows the stochastic process

$$\frac{d\theta}{\theta} = \mu dt + \sigma dz, \tag{105}$$

where $dz$ is a Weiner process, and suppose that there are two assets that both depend on $\theta$ that are traded in the market, $f$ and $g$. Using Ito's lemma, we can state that

$$\frac{df}{f} = (f_t + f_\theta \mu + \frac{1}{2} f_{\theta\theta}\sigma^2)dt + f_\theta \sigma dz \equiv \mu_f dt + \sigma_f dz, \tag{106}$$

and we can write down an analogous equation for $g$. If we form a portfolio with $\sigma_g g$ units of $f$ and $-\sigma_f f$ units of $g$ it will have a value equal to

$$\Pi = \sigma_g g f - \sigma_f f g, \tag{107}$$

which will be riskless with changes equal to

$$d\Pi = [\mu_f \sigma_g f g - \mu_g \sigma_f f g]dt \tag{108}$$

since this portfolio is riskless, it must have a return equal to the risk-free rate, so that

$$d\Pi = r\Pi dt, \tag{109}$$

which can be solved to yield

$$\frac{\mu_f - r}{\sigma_f} = \frac{\mu_g - r}{\sigma_g} = \lambda. \tag{110}$$

In fact, any risky asset with a payoff that depends only on $\theta$ will satisfy this equation. We call $\lambda$ the price of risk of $\theta$.

Now consider the problem when there are two "factors," or two sources of uncertainty in prices. Suppose $\theta_1$ and $\theta_2$ are two variables that follow stochastic processes

$$\frac{d\theta_i}{\theta_i} = m_i dt + s_i dz_i \tag{111}$$

for $i = 1, 2$, where $dz_i$ are Weiner processes with

$$dz_1 dz_2 = \rho_{12}. \tag{112}$$

The parameters $m_i$ and $s_i$ are drift rates and volatilities and may be functions of $\theta_i$ and time. We do *not* assume that $\theta_i$ are traded securities. We assume there are at least three traded securities whose prices depend on these variables. These could be options with payoffs that are different functions of these variables. Let $f_1, f_2$ and $f_3$ denote the prices of the three traded securities and $r$ be the risk-free rate. Using Ito's lemma,

$$df_j = \mu_j f_j dt + \sigma_{1j} f_j dz_1 + \sigma_{2j} f_j dz_2 \tag{113}$$

where

$$\begin{aligned} \mu_j f_j = & \tfrac{\partial f_j}{\partial t} + \tfrac{\partial f_j}{\partial \theta_1} m_1 \theta_1 + \tfrac{\partial f_j}{\partial \theta_2} m_2 \theta_2 + \tfrac{1}{2} s_1^2 \theta_1^2 \tfrac{\partial^2 f_j}{\partial \theta_1^2} + \\ & \tfrac{1}{2} s_2^2 \theta_2^2 \tfrac{\partial^2 f_j}{\partial \theta_2^2} + \rho_{12} s_1 s_2 \theta_1 \theta_2 \tfrac{\partial^2 f_j}{\partial \theta_1 \partial \theta_2}, \\ \sigma_{1j} f_j = & \tfrac{\partial f_j}{\partial \theta_1} s_1 \theta_1, \\ \sigma_{2j} f_j = & \tfrac{\partial f_j}{\partial \theta_2} s_2 \theta_2. \end{aligned} \tag{114}$$

We can use the 3 traded securities to create a portfolio that is instantaneuosly riskless. We need to choose weights so that stochastic components due to $\theta_1$ and $\theta_2$ are eliminated. Let $k_j$ be the amount of security $j$ in the portfolio $\Pi$ so that

$$\Pi = k_1 f_1 + k_2 f_2 + k_3 f_3, \tag{115}$$

$$k_1 \sigma_{11} f_1 + k_2 \sigma_{12} f_2 + k_3 \sigma_{13} f_3 = 0, \tag{116}$$

$$k_1 \sigma_{21} f_1 + k_2 \sigma_{22} f_2 + k_3 \sigma_{23} f_3 = 0. \tag{117}$$

The return on portfolio is given by

$$d\Pi = (k_1 \mu_1 f_1 + k_2 \mu_2 f_2 + k_3 \mu_3 f_3) dt. \tag{118}$$

The portfolio is instantaneously riskless so it must earn risk-free interest rate. Therefore, we have

$$k_1 \mu_1 f_1 + k_2 \mu_2 f_2 + k_3 \mu_3 f_3 = r(k_1 f_1 + k_2 f_2 + k_3 f_3)$$

or

$$k_1 f_1(\mu_1 - r) + k_2 f_2(\mu_2 - r) + k_3 f_3(\mu_3 - r) = 0 \tag{119}$$

Since all $k_j$'s are not zero, equations (116), (117) and (119) can not be linearly independent. We must therefore, have

$$f_j(\mu_j - r) = \lambda_1 \sigma_{1j} f_j + \lambda_2 \sigma_{2j} f_j$$

or

$$\mu_j - r = \lambda_1 \sigma_{1j} + \lambda_2 \sigma_{2j}$$

for some $\lambda_1$ and $\lambda_2$. The above equation holds for any traded asset whose price depends on $\theta_1$ and $\theta_2$. If $f$ is the price of a traded security with price dependent on $\theta_1$ and $\theta_2$ with

$$df = \mu f dt + \sigma_1 f dz_1 + \sigma_2 f dz_2 \tag{120}$$

then

$$\mu - r = \lambda_1 \sigma_1 + \lambda_2 \sigma_2 \tag{121}$$

The parameter $\lambda_i$ is called the market price of risk of $\theta_i$ and is in general a function of $\theta_i$ and $t$ but independent of security $f$. The variable $\sigma_i$ can be interpreted as the quantity of $\theta_i$-risk in $f$. Equation (121) says that the excess expected return on security $f$ to compensate for risk of $\theta_i$ equals the quantity of $\theta_i$-risk multiplied by the market price of $\theta_i$-risk.

## 4.2 Stochastic Volatility Model

We assume that there is an asset whose price $S$ follows the stochastic process

$$dS(t) = \mu S dt + \sqrt{v(t)} S dz_1, \tag{122}$$

where $z_1(t)$ is a Weiner process. The volatility is a Ornstein-Uhlenbeck process,

$$d\sqrt{v(t)} = -\beta\sqrt{v(t)}dt + \delta dz_2(t), \tag{123}$$

where $z_2(t)$ has a correlation $\rho$ with $z_1(t)$. We can use Ito's lemma to get the stochastic process followed by $v(t)$

$$\begin{aligned} dv(t) &= [\delta^2 - 2\beta v(t)]dt + 2\delta\sqrt{v(t)}dz_2(t) \\ &= \kappa[\theta - v(t)]dt + \sigma\sqrt{v(t)}dz_2(t), \end{aligned} \tag{124}$$

where $\kappa, \theta$, and $\sigma$ are constants. Let $C$ be the price of a European call option on asset S with strike price $K$ and maturing at time $T$. The price is a function of asset price $S$, volatility $v$ and time $t$. Using Ito's lemma,

$$\begin{aligned} dC = &\ \mu_C C dt + \sigma_{C1} C dz_1 + \sigma_{C2} C dz_2, \\ \mu_C C = &\ \tfrac{\partial C}{\partial t} + \mu S \tfrac{\partial C}{\partial S} + \kappa[\theta - v]\tfrac{\partial C}{\partial v} + \tfrac{1}{2} v S^2 \tfrac{\partial^2 C}{\partial S^2} \\ + &\ \rho\sigma v S \tfrac{\partial^2 C}{\partial S \partial v} + \tfrac{1}{2}\sigma^2 v \tfrac{\partial^2 C}{\partial v^2}, \\ \sigma_{C1} C = &\ \sqrt{v} S \tfrac{\partial C}{\partial S}, \\ \sigma_{C2} C = &\ \sigma\sqrt{v}\tfrac{\partial C}{\partial v}. \end{aligned} \tag{125}$$

The call option is a traded security whose price depends on S and v so we can use the result in equation (121) to get

$$\mu_C - r = \lambda_1 \sigma_{C1} + \lambda_2 \sigma_{C2} \tag{126}$$

Since, the underlying asset is also a traded security, its price process also follows equation (121).

$$\mu - r = \lambda_1 \sqrt{v} \tag{127}$$

Substituting from (125) and (127) into (126) and simplifying, we get

$$\begin{aligned} \tfrac{\partial C}{\partial t} + rS\tfrac{\partial C}{\partial S} + \{\kappa[\theta - v] - \lambda_2\sigma\sqrt{v}\}\tfrac{\partial C}{\partial v} + \\ \tfrac{1}{2}vS^2\tfrac{\partial^2 C}{\partial S^2} + \rho\sigma vS\tfrac{\partial^2 C}{\partial S\partial v} + \tfrac{1}{2}\sigma^2 v\tfrac{\partial^2 C}{\partial v^2} &= rC. \end{aligned} \tag{128}$$

We still do not know the functional form of $\lambda_2$. Heston motivates the form of risk premium $\lambda_2\sigma\sqrt{v}$ from the risk-premium obtained by Cox, Ingersoll and Ross (1985) in their general equilibrium model. They show that for a single state variable following the stochastic process in (124), the risk premium is proportional to the state variable. Therefore, we assume $\lambda_2\sigma\sqrt{v} = \lambda v$ for some constant $\lambda$. The constant can be estimated using the price of another option dependent on the volatility just as we estimate Black-Scholes implies volatility.

## 4.3 Solving the Option Price PDE

We guess the form of solution to be a generalization of Black-Scholes formula.

$$C(S, v, t) = SP_1(S, v, t) - Ke^{-(T-t)}P_2(S, v, t). \tag{129}$$

Transforming variable $S$ to $x = ln(S)$ and substituting for the risk premium in (128), we get the PDE

$$\begin{aligned} \tfrac{\partial C}{\partial t} + (r - \tfrac{1}{2}v)\tfrac{\partial C}{\partial x} + \{\kappa[\theta - v] - \lambda v\}\tfrac{\partial C}{\partial v} + \\ \tfrac{1}{2}v\tfrac{\partial^2 C}{\partial x^2} + \rho\sigma v\tfrac{\partial^2 C}{\partial x\partial v} + \tfrac{1}{2}\sigma^2 v\tfrac{\partial^2 C}{\partial v^2} &= rC. \end{aligned} \tag{130}$$

Substituting the proposed solution (129) into (130), $P_1$ and $P_2$ must satisfy the PDEs

$$\begin{aligned} \tfrac{\partial P_j}{\partial t} + (r + u_j v)\tfrac{\partial P_j}{\partial x} + (a - b_j v)\tfrac{\partial P_j}{\partial v} \\ + \tfrac{1}{2}v\tfrac{\partial^2 P_j}{\partial x^2} + \rho\sigma v\tfrac{\partial^2 P_j}{\partial x\partial v} + \tfrac{1}{2}\sigma^2 v\tfrac{\partial^2 P_j}{\partial v^2} &= 0. \end{aligned} \tag{131}$$

for $j = 1, 2$ where $u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, a = \kappa\theta, b_1 = \kappa + \lambda - \rho\sigma, b_2 = \kappa + \lambda$. $P_1$ and $P_2$ must satisfy the terminal conditions

$$P_j(x, v, T, ln(K)) = 1_{\{x \geq ln(K)\}} \tag{132}$$

## 4.4 Risk-Neutral Probabilities and Characteristic Functions

To solve (131) and (132), we consider an interpretation of $P_j$ as risk-neutral probabilities. Suppose $x(t)$ follows the stochastic process

$$\begin{aligned} dx(t) = &\quad (r + u_j v)dt + \sqrt{v(t)}dz_1(t), \\ dv = &\quad (a - b_j v)dt + \sigma\sqrt{v(t)}dz_2(t), \end{aligned} \tag{133}$$

for $j = 1$ or 2. Consider a function $g(x, v)$. Let $f_j$ be the conditional expectation of $g$ at terminal date defined as

$$f_j(x, v, t) = E[g(x(T), v(T)) \mid x(t) = x, v(t) = v]. \tag{134}$$

Ito's lemma shows that

$$\begin{aligned} df_j = \Big\{ &\tfrac{\partial f_j}{\partial t} + (r + u_j v)\tfrac{\partial f_j}{\partial x} + (a - b_j v)\tfrac{\partial f_j}{\partial v} + \tfrac{1}{2}v\tfrac{\partial^2 f_j}{\partial x^2} + \rho\sigma v\tfrac{\partial^2 f_j}{\partial x \partial v} \\ &+ \tfrac{1}{2}\sigma^2 v\tfrac{\partial^2 f_j}{\partial v^2}\Big\}dt + \sqrt{v}\tfrac{\partial f_j}{\partial x}dz_1 + \sigma\sqrt{v}\tfrac{\partial f_j}{\partial v}dz_2. \end{aligned} \tag{135}$$

The law of iterated expectations implies that $f_j$ must be a martingale. Equating the drift of $f$ to zero, we get the Fokker-Planck forward equation

$$\begin{aligned} &\tfrac{\partial f_j}{\partial t} + (r + u_j v)\tfrac{\partial f_j}{\partial x} + (a - b_j v)\tfrac{\partial f_j}{\partial v} + \tfrac{1}{2}v\tfrac{\partial^2 f_j}{\partial x^2} \\ &\qquad + \rho\sigma v\tfrac{\partial^2 f_j}{\partial x \partial v} + \tfrac{1}{2}\sigma^2 v\tfrac{\partial^2 f_j}{\partial v^2} \quad = 0 \end{aligned} \tag{136}$$

The technique used above to get the differential equation by equating drift of $f_j$ to zero is useful when we are interested in finding expected value of a function of an evolving stochastic process. The terminal condition for $f_j$ is

$$f_j(x, v, T) = g(x, v). \tag{137}$$

If $g(x, v) = 1_{\{x \geq ln(K)\}}$, then the equations (136) and (137) for $f$ are the same as the equations (131) and (132) for $P_j$. Thus, $P_j$ is the conditional probability that the option expires in the money when $x$ follows the stochastic process (133).

$$P_j(x, v, t; ln(K)) = Pr(x(T) \geq ln(K) \mid x(t) = x, v(t) = v). \tag{138}$$

We do not have a direct closed form solution for these probabilities. However, these probabilities are related to characteristics functions which have closed form solutions. If $g(x, v) = e^{\iota\phi x}$ ($\iota$ is a square root of $-1$), $g$ is called the characteristic function of $x$. In this case, we guess the solution for (136) and (137) to be of the form

$$f_j(x, v, t) = \exp(A_j(T - t) + B_j(T - t)v + \iota\phi x) \tag{139}$$

Substituting the above function into (136), we get two ordinary differential equations,

$$\begin{aligned} -\tfrac{\partial B_j}{\partial \tau} + \iota u_j \phi - b_j B_j - \tfrac{1}{2}\phi^2 + \iota\rho\sigma\phi B_j + \tfrac{1}{2}\sigma^2 B_j^2 &= 0, \\ -\tfrac{\partial A_j}{\partial \tau} + \iota r \phi + a B_j &= 0, \end{aligned} \tag{140}$$

subject to

$$A_j(0) = 0, B_j(0) = 0.$$

The solution to these differential equations (after dropping the subscripts) is

$$\begin{aligned} f(x,v,t;\phi) &= e^{A(T-t;\phi)+B(T-t;\phi)v+\iota\phi x}, \\ A(\tau;\phi) &= \iota r\phi\tau + \tfrac{a}{\sigma^2}\Big\{(b-\iota\rho\sigma\phi+d)\tau - 2ln\Big(\tfrac{1-ge^{d\tau}}{1-g}\Big)\Big\}, \\ B(\tau;\phi) &= \tfrac{b-\iota\rho\sigma\phi+d}{\sigma^2}\Big\{\tfrac{1-e^{d\tau}}{1-ge^{d\tau}}\Big\}, \\ g &= \tfrac{b-\iota\rho\sigma\phi+d}{b-\iota\rho\sigma\phi-d}, \\ d &= \sqrt{(\iota\rho\sigma\phi-b)^2-\sigma^2(\iota 2u\phi-\phi^2)}. \end{aligned} \tag{141}$$

We can thus evaluate the characteristic functions in closed form. However, we are interested in the risk-neutral probabilities $P_j$. These can be inverted from characteristic functions by performing the following integration.

$$P_j(x,v,t;ln(K)) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\Big[\frac{e^{-\iota\phi ln(K)}f_j(x,v,t;\phi)}{\iota\phi}\Big]d\phi. \tag{142}$$

To verify the above equation, consider the expression on the right.

$$\begin{aligned} \tfrac{1}{2}+ \quad & \tfrac{1}{\pi}\textstyle\int_0^\infty Re\Big[\tfrac{e^{-\iota\phi ln(K)}f_j(x,v,t;\phi)}{\iota\phi}\Big]d\phi \\ &= \tfrac{1}{2} + \tfrac{1}{\pi}\textstyle\int_0^\infty Re\Big[\tfrac{E[e^{\iota\phi\{x-ln(K)\}}|x(t)=x,v(t)=v]}{\iota\phi}\Big]d\phi \\ &= E\Big[\tfrac{1}{2} + \tfrac{1}{\pi}\textstyle\int_0^\infty Re\Big[\tfrac{e^{\iota\phi\{x-ln(K)\}}}{\iota\phi}\Big]d\phi \mid x(t)=x, v(t)=v\Big] \\ &= E\Big[\tfrac{1}{2} + \tfrac{1}{\pi}\textstyle\int_0^\infty \tfrac{sin(\phi\{x-ln(K)\})}{\phi}d\phi \mid x(t)=x, v(t)=v\Big] \\ &= E\Big[\tfrac{1}{2} + sgn(x-ln(K))\tfrac{1}{\pi}\textstyle\int_0^\infty \tfrac{sin(\phi)}{\phi}d\phi \mid x(t)=x, v(t)=v\Big] \\ &= E\Big[\tfrac{1}{2} + \tfrac{1}{2}sgn(x-ln(K)) \mid x(t)=x, v(t)=v\Big] \\ &= E[1_{\{x\geq ln(K)\}} \mid x(t)=x, v(t)=v]. \end{aligned} \tag{143}$$

where we have used the Dirichlet formula $\int_{-\infty}^{\infty}\frac{sin(v)}{v}dv = 1$ and $sgn$ function is defined as $sgn(x) = 1$ if $x > 0$, 0 if $x = 0$ and $-1$ if $x < 0$.

The solution for the price of the European call option is given by equations (129), (141) and (142). The option price depends on strike, maturity, asset price, volatility, risk-free rate and the parameters of the volatility process. It is independent of the drift of the asset price.