

5 No-Arbitrage in Continuous Time

This section of the notes develops some no-arbitrage concepts in continuous time. As you might suspect, the continuous-time no-arbitrage results are very similar to the discrete-time results that you may have seen before. The continuous-time results are fairly high tech. We will not prove most of the results - we just review them so that you are aware of their existence. The reference material that I will distribute about this is a chapter from Duffie's book. It is a little bit difficult to read, in part because it refers to other sections of the book. I will try to make Duffie as clear as possible to anyone interested.

5.1 No-Arbitrage Results in Discrete Time

There are a number of classic arbitrage results in discrete time that are similar to the results we will discuss in continuous time. It is worthwhile to briefly review these results before discussing new results. The most important result is often called the fundamental theorem of asset pricing:

The Fundamental Theorem of Asset Pricing. *The following three conditions are equivalent:*

- *The absence of arbitrage*
- *The existence of a positive linear pricing rule (state prices)*
- *The existence of an optimal portfolio for some agent who prefers more to less.*

You can find a proof of the fundamental theorem in a number of places, including Duffie's book and a set of notes used in my BA 855 class and available on my website.

What does the theorem mean by the existence of a positive linear pricing rule? There are at least four interesting manifestations of the pricing rule. One is known as the pricing kernel (or stochastic discount factor) and is usually denoted m . The pricing

kernel is a random variable, and it satisfies the property

$$E_{t-1}(R_{i,t}m_t) = 1 \quad \forall i, t, \quad (144)$$

where $R_{i,t}$ is the return on asset i between period $t-1$ and t , and E_{t-1} is the conditional expectation operator, conditional on all information available at time $t-1$. There is a great deal of discussion about the stochastic discount factor in BA 855. Again, if you have not had BA 855 and the pricing kernel interests you, you may want to download the notes about pricing kernels from my website. The appropriate set of notes is called “General properties of asset pricing models.”

The pricing kernel, a manifestation of the positive linear pricing rule, is equivalent to other representations. One familiar representation is a factor model,

$$E_{t-1}r_{i,t} = r_f + \beta_{i,m}\gamma_{t-1}, \quad (145)$$

where $\beta_{i,m}$ is the beta of the return of asset i in period t regressed on the stochastic discount factor, m_t , conditional on all information available at time $t-1$. This factor model should look familiar - it is very similar to the CAPM. In fact, it is trivial to show that the CAPM and the APT are special cases of equation (145). The factor model listed above is a conditional factor model, as is the stochastic discount factor model. Unconditional versions of these models may look form familiar:

$$E[R_i m] = 1 \iff E(r_i) = r_f + \beta_{i,m}\gamma \quad (146)$$

Another, potentially less familiar representation of the positive linear pricing rule is the risk-neutral probability. If there are S possible future states of the world with subjective probability of occurrence of π_s and associated stochastic discount value m_s ,

then risk-neutral probabilities $\hat{\pi}_s$ take the form,

$$\begin{aligned}\hat{\pi}_s &= [m_s \pi_s] / \psi_0, \\ \psi_0 &\equiv \sum_s m_s \pi_s, \\ q_i &= \frac{1}{1+r_f} \sum_s \hat{\pi}_s d_{i,s},\end{aligned}\tag{147}$$

where q_i is the price of asset i , and $d_{i,s}$ is the payoff (dividend) of asset i in future state of the world s . With risk-neutral probabilities pricing is very simple. The price of any asset is its discounted expected payoff when calculated with the risk-neutral probabilities.

The final way that we will express our central asset pricing result of $E[Rm] = 1$ is with a set of strictly positive state prices, an S -vector ψ , that solve

$$\mathbf{q} = \mathbf{D}\psi,\tag{148}$$

where \mathbf{q} represents an N -vector of security prices and \mathbf{D} is the N by S matrix of state-contingent payoffs. State prices can be related to the stochastic discount factor with the relation

$$\psi_s = m_s \pi_s$$

While the state price form of the pricing kernel is useful for intuition and discussions of market completeness, we will not use it very much here.

5.2 Equivalent Martingale Measure

We want to describe our set of risk-neutral probabilities with a much fancier name – our probabilities are actually an equivalent martingale measure. Understanding this new name requires some discussion of each of its three terms. First, we call our probabilities a measure because any set of probabilities can be thought of as a measure.

Second, any two measures that assign positive probability to the same set of possible

outcomes are called equivalent. Since only states that have positive probability receive non-zero state prices, the risk-neutral measure is equivalent to the subjective measure of the representative agent, which is equal to the objective probability measure by rational expectations. If we call our subjective probability measure P and our risk-neutral measure Q , then the equivalence of P and Q is the same as

$$P(A) > 0 \iff Q(A) > 0, \quad (149)$$

where A is a possible realization of the random variable over which P and Q are defined.

Third, a measure is a martingale measure with respect to a particular process if it converts the process to a martingale. Remember that a martingale with respect to the information set \mathcal{F}_{t-1} is a stochastic process with the properties,

$$\begin{aligned} E[|X_t|] &< \infty \quad \forall t \\ E[X_t | \mathcal{F}_{t-1}] &= x_{t-1}. \end{aligned} \quad (150)$$

Now think about how we have defined our new probability measure. Our risk-neutral measure has the property that

$$E^Q[S_{t+1}]e^{-r} = S_t, \quad (151)$$

so it converts the process for the discounted stock price into a martingale. Since our risk-neutral probability measure is equivalent to the subjective probability measure and since it converts discounted stock price processes to martingales we call it an equivalent martingale measure.

Associated with every change of measure is a mapping of how to go from one measure to another measure. This mapping is called a *Radon-Nikodym* derivative, and it is commonly written as $\frac{\partial Q}{\partial P}$. You can think of the Radon-Nikodym derivative as the ratio of the probability densities that describe probability under measure Q and P . A

Radon-Nikodym derivative is actually a strictly positive random variable. Heuristically, it is a random variable because you don't know at which point you will have to evaluate the ratio of probability densities for a draw from the space of possible outcomes until the draw has been made. Thinking of the Radon-Nikodym derivative as the ratio of two densities makes it clear why it is important to change from one probability measure to another measure that is equivalent. If some states have zero probability under measure P and positive probabilities under Q then the Radon-Nikodym derivative will not be defined over all possible states.

In a continuous-time setting, the Radon-Nikodym derivative becomes a stochastic process. Again, it is stochastic because the point at which the ratio of densities will be evaluated at each point in time is not known until that time is reached. The Radon-Nikodym process has the properties,

$$E^Q[S_T] = E^P \left[\frac{\partial Q}{\partial P} S_T \right], \quad (152)$$

$$E^Q[S_t | \mathcal{F}_\omega] = \frac{E^P \left[\frac{\partial Q}{\partial P} S_t | \mathcal{F}_\omega \right]}{E^P \left(\frac{\partial Q}{\partial P} | \mathcal{F}_\omega \right)}, \quad \omega \leq t \leq T. \quad (153)$$

The Radon-Nikodym process is closely related to the stochastic discount factor that we discussed in Section 5.1. Using (151) and (152), the stochastic discount factor that satisfies the price version of $E[Rm] = 1$,

$$S_t = E_t[S_{t+1}m_{t+1}], \quad (154)$$

is just a Radon-Nikodym derivative multiplied by a discount factor (or divided by either R_f or e^{rt}). If we want to be fancy and high tech, we call the stochastic discount factor a Radon-Nikodym derivative.

5.3 No-Arbitrage Results

So what do we use all of this stuff for? We need it to define the absence of arbitrage in continuous time. Our notion of arbitrage in the discrete-state, one period world of earlier results was a portfolio with zero or negative cost and a payoff that is greater than or equal to zero (at least one state greater than zero for a zero-cost portfolio). Now our notion of arbitrage expands a little bit. Rather than talk about portfolio returns, we think of returns to dynamic strategies that are dictated by a process $\theta(t) \in \mathfrak{R}^N$. The process $\theta(t)$ must be *adapted*, or measurable at time t . This is required to make the strategy well defined. A strategy is self-financing if

$$\theta(t) \cdot S(t) = \theta(0) \cdot S(0) + \int_0^t \theta(\tau) dS(\tau) \quad t \leq T, \quad (155)$$

where the integral here is a stochastic integral and $S(t)$ is the N -dimensional process of asset prices. The idea here is that a self-financing strategy requires you to put up some money at the beginning of your strategy but it does not require any additions of capital. A self-financing strategy is an arbitrage if $\theta(0) \cdot S(0) < 0$ and $\theta(T) \cdot S(T) \geq 0$ or $\theta(0) \cdot S(0) \leq 0$ and $\theta(T) \cdot S(T) > 0$.

The fun thing about defining no-arbitrage as we have above is that this definition allows for much more complex strategies than the simple portfolios that we allowed in our one-period results. In particular, there are a set of strategies called “doubling strategies” that will produce arbitrage profits if strategies are unlimited. To avoid doubling strategies, we impose some constraints on strategy processes. To impose the constraints, we define some subsets of the set of all adapted processes, $\mathcal{L}(S)$:

$$\mathcal{L}^1 = \left\{ \theta \in \mathcal{L} : \int_0^T |\theta(t)| dt < \infty \text{ a.s.} \right\}, \quad (156)$$

$$\mathcal{L}^2 = \left\{ \theta \in \mathcal{L} : \int_0^T \theta(t)^2 dt < \infty \text{ a.s.} \right\}, \quad (157)$$

$$\mathcal{H}^1 = \left\{ \theta \in \mathcal{L}^1 : E \left(\int_0^T |\theta(t)| dt \right) < \infty \right\}, \quad (158)$$

$$\mathcal{H}^2 = \left\{ \theta \in \mathcal{L}^2 : E \left(\int_0^T \theta(t)^2 dt \right) < \infty \right\}, \quad (159)$$

We will actually just use one of these sets. We list them all because Duffie refers to several of these sets in the readings. It is also nice to see how the sets compare in restrictiveness. With these subsets defined, we can discuss several different versions of a no-arbitrage pricing rule (like we did in Section 5.1), but we will just concentrate on one:

No-Arbitrage and Equivalent Martingale Measure. *If the price process S (meaning the multidimensional vector of all prices) admits an equivalent martingale measure, then there is no arbitrage in $\mathcal{H}^2(S)$.*

We will not prove this theorem. Its proof is somewhat similar to the proof of the fundamental theorem.

One final result that is often useful is known as **Girsanov's theorem**. Girsanov's theorem basically says that you can take a Brownian motion process with any sort of drift and, under some technical conditions, find a measure that converts your process to another Brownian process with a different drift. Changing measure usually amounts to changing drift. We will use a special case of Girsanov's theorem:

Diffusion Invariance Principle. *Let $X \in \mathfrak{R}^d$ be an Ito process with $dX = \mu_t dt + \sigma_t dW$. If X is a martingale with respect to an equivalent probability measure Q then there is a standard Brownian motion \hat{W} in \mathfrak{R}^d under Q such that $dX = \sigma_t d\hat{W}$, $t \in [0, T]$.*

We will apply the concepts that we have just discussed with an example below.

5.4 Example: The Black-Scholes Model

To illustrate the usefulness of the risk-neutral pricing techniques, we will derive the Black-Scholes model one more time. We begin by again assuming that the stock price

process follows geometric Brownian motion,

$$\frac{dS}{S} = \alpha dt + \sigma dW \quad (160)$$

This time, rather than construct a continuous-time hedge, we apply the risk-neutral results directly. Using Girsanov's theorem, we can convert the process for the stock's price to an equivalent Brownian motion process with drift equal to the risk-free rate,

$$\frac{dS}{S} = r dt + \sigma d\hat{W}, \quad (161)$$

where \hat{W} is a standard Wiener process. Now we know that under the risk-neutral measure we calculate the call option's value by taking the expectation of the option's terminal value,

$$\begin{aligned} C &= e^{-rt} E^Q \max[(S - k), 0] \\ &= e^{-rt} (E^Q[S|S > k] - k) Q[S > k], \end{aligned} \quad (162)$$

where $Q[S > k]$ is the probability under Q that $S > k$.

To calculate the terms above, we will need some distributional information for S . Since the stock price follows geometric Brownian motion, we know that stock prices are lognormally distributed under the risk-neutral measure. In particular,

$$E^Q[S_t|S_0] = S_0 e^{rt}, \quad (163)$$

$$E^Q[\ln S_t|S_0] = \ln(S_0) + rt - \sigma^2 t/2 \quad (164)$$

$$\text{Var}^Q[\ln S_t|S_0] = \sigma^2 t. \quad (165)$$

We also know some of the properties of the lognormal density. In particular, it is a standard result (see Ingersoll's mathematical introduction) that if a random variable,

Z , is lognormally distributed with

$$\begin{aligned} E[Z] &= e^{\mu+\sigma^2/2}, \\ \text{Var}[\ln Z] &= \sigma^2, \end{aligned} \tag{166}$$

then

$$E[Z|Z > a]\Pr[Z > a] = \int_a^\infty z f(z) dz = e^{\mu+\sigma^2/2} \mathcal{N}\left(\frac{-\ln a + \mu}{\sigma} + \sigma\right), \tag{167}$$

where $\mathcal{N}(\cdot)$ again indicates the cumulative normal distribution. We also know that $\ln[S_t]$ is normally distributed.

We want to combine these results to calculate (162), specifically the quantities, $E^Q[(S - k)|S > k]$ and $Q[S > k]$. Using our distributional results,

$$C = e^{-rt} \left[S_0 e^{rt} \mathcal{N}\left(\frac{\ln(S_0) + rt - \sigma^2 t/2 - \ln(k)}{\sigma\sqrt{t}} + \sigma\sqrt{t}\right) - kQ(S > k) \right]. \tag{168}$$

We can see immediately that the cumulative normal distribution in (168) is evaluated at the quantity that we previously defined as d_1 . Using the fact that $\ln S_t$ is normally distributed, and that $1 - F(X) = F(-X)$ when F is the normal CDF, the quantity $Q(S > k)$ can be written as

$$Q(S > k) = \mathcal{N}\left(\frac{\ln(S_0) + rt - \sigma^2 t/2 - \ln(k)}{\sigma\sqrt{t}}\right) = \mathcal{N}(d_2). \tag{169}$$

Thus, we have again the Black-Scholes formula,

$$C = S_0 \mathcal{N}(d_1) - e^{-rt} k \mathcal{N}(d_2). \tag{170}$$

This concludes our derivation of the Black-Scholes model using risk-neutral pricing arguments. This derivation is in some ways more satisfying than the continuous-hedge argument that we gave before. We derived the model without having to wave hands

about the heat transfer equation. Moreover, this argument should be valid even if it is difficult to do a dynamic hedge. This demonstrates how useful risk-neutral pricing can be.

This risk-neutral derivation also sheds some light on the interpretation of the terms $\mathcal{N}(d_1)$ and $\mathcal{N}(d_2)$. We already said that $\mathcal{N}(d_1)$ is equal to the option's delta, or the derivative of the option price with respect to the stock price. Now we know that $\mathcal{N}(d_2)$ represents the probability, under the risk-neutral measure, that the call option finishes in the money.

5.5 Homework Problems

1. Find an expression for the Radon-Nikodym derivative in the simple world that we discussed in Section 5.1 of the notes (this should not be hard).
2. Would you expect the risk-neutral probability that an equity call option is in the money to be greater or less than the corresponding objective probability? Why? What about the probability that an equity put is in the money? What does this imply about expected option returns?
3. Show mathematically that your answer to first part of the previous question is true.