

2 Utility Maximization in Continuous-Time

Having discussed continuous-time mathematics, we are ready to derive some basic asset-pricing models. We will do this by examining some of the results of Bob Merton, whose entire career has been dedicated to continuous-time financial models. We will discuss two particular models of Merton's called the continuous-time CAPM (or CT CAPM) and the intertemporal CAPM (ICAPM). While the original papers that developed these models can be found in Merton's book, Ingersoll's chapter 13 is probably an easier source for this material.

2.1 Assumptions for the CT CAPM

We will derive the continuous-time CAPM with assumptions similar to those that we used to derive the one period CAPM. In particular, we will make an assumption that is very similar to the assumption that all returns are jointly normally distributed. First, though, we assume an economy with n risky assets, one risk free asset and a representative investor. Next we define:

- $P_i(t)$ = price of firm i at t ,
- $W(t)$ = wealth,
- $C(t)$ = consumption,
- $B[W(T), T]$ = bequest function,
- $U[C, t]$ = utility function,
- w_i = fraction of wealth invested in asset i ,
- r_f = return on the risk free asset (asset 0)
- E_t = expectation conditional on information available at t .

Our big assumption is that the stochastic process generating returns consists of a geometric Brownian motion process,

$$r_i(t + dt) = dP_i/P_i = \alpha_i dt + \sigma_i dz_i, \quad (26)$$

with properties

$$\begin{aligned} E[r_i] &= \alpha_i dt, \\ E[r_i^2] &= \text{Var}[r_i] = \sigma_i^2 dt, \\ \text{Cov}[r_i, r_j] &= \sigma_{ij} dt. \end{aligned} \quad (27)$$

We will also assume (for simplicity) that α_i , σ_i , and σ_{ij} are all constants. Note that this assumption is particularly valid for continuously-compounded or logarithmic returns; log returns can vary from positive to negative infinity. Assuming that log returns follow Brownian motion is the continuous-time equivalent of assuming that all returns are jointly normally distributed.

2.2 Deriving the CT CAPM

The problem now is to maximize the sum of future discounted expected utility. Since we are in continuous time, we integrate expected utility rather than sum it. The problem is

$$\max E_0 \left[\int_0^T U[C(t), t] dt + B[W(T), T] \right], \quad (28)$$

subject to the wealth accumulation equations,

$$\begin{aligned} W(t + dt) &= [W(t) - C(t)dt] \sum_{i=0}^n w_i(t) [1 + r_i(t + dt)], \\ \sum_{i=0}^n w_i(t) &= 1 \quad \forall t \\ W(0) &= W_0. \end{aligned} \quad (29)$$

We will solve this problem with something like continuous-time dynamic programming, but first we will have to convert it into something more familiar.

To use our dynamic programming results, we will write the continuous-time problem as the sum of a bunch of discrete-time problems. First, we write the problem as a sum of small continuous-time problems,

$$\begin{aligned} & \max E_0 \left[\int_0^T U[C(t), t] dt + B[W(T), T] \right] \\ & = \max E_0 \left[\sum_{t=0}^{T-dt} \int_t^{t+dt} U[C(s), s] ds + B[W(T), T] \right]. \end{aligned} \quad (30)$$

The solution to each of the small integrals in (30) is a function of time, t , so each integral can be expanded with an exact Taylor series,

$$\begin{aligned} & \int_t^{t+dt} U[C(s), s] ds \\ & = U[C(t), t] dt + \frac{1}{2} U_t[C(t^*), t^*] dt^2 \\ & = U[C(t), t] dt, \end{aligned} \quad (31)$$

where t^* is between t and dt . Since there is essentially no space between t and dt , t^* and t are the same value. Thus, we don't need to worry about the partial derivative of $C(t)$ in the Taylor series. Intuitively, for small enough time increments cumulative utility is just equal to total utility.

We also alter the budget constraint of the problem. In particular, we write it in the differential form,

$$dW = -C(t)dt + [W(t) - C(t)dt] \sum_{i=0}^n w_i(t)r_i(t + dt). \quad (32)$$

Exploiting the Brownian motion process that we have assumed for returns and using continuous-time math rules to drop several terms,

$$dW = \left[W(t) \sum_{i=0}^n w_i \alpha_i - C(t) \right] dt + W(t) \sum_{i=0}^n w_i \sigma_i dz_i. \quad (33)$$

Now the change in wealth process has the properties

$$E_t[dW] = -C(t)dt + W(t) \sum_{i=0}^n w_i(t)\alpha_i dt, \quad (34)$$

$$\text{Var}_t[dW] = [W(t) - C(t)dt]^2 \text{Var} \left[\sum_{i=0}^n w_i(t)r_i(t + dt) \right] = W(t)^2 \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij} dt \quad (35)$$

We will follow Merton's convention and use the symbol J to mean the value function for our problem,

$$J(W, t) = \max E_t \left[\int_t^T U[C(s), s] ds + B[W(T), T] \right] \text{ s.t. (33)}. \quad (36)$$

Using our dynamic programming results and our discrete-time representation of cumulative utility, we can express the value function as a Bellman equation,

$$J(W, t) = \max E_t \left[\int_t^{t+dt} U[C(s), s] ds + J[W(t + dt), t + dt] \right] \text{ s.t. (33)}. \quad (37)$$

Using Ito's lemma to expand $J[W(t + dt), t + dt]$,

$$\begin{aligned} dJ &= J[W(t + dt), t + dt] - J[W(t), t] \\ &= J_t dt + J_W dW + \frac{1}{2} J_{WW} dW^2 + \frac{1}{2} J_{tt} dt^2 + J_{Wt} dt dW \\ &= J_t dt + J_W dW + \frac{1}{2} J_{WW} dW^2 \end{aligned} \quad (38)$$

Taking the expected value of $J[W(t + dt), t + dt]$ yields,

$$E_t J[W(t + dt), t + dt] = J[W(t), t] + J_t dt + J_W E[dW] + \frac{1}{2} J_{WW} \text{Var}[dW]. \quad (39)$$

Now we are ready to put all these pieces together and come up with a kinder and gentler objective function. Substituting (31), (34), (35), and (39) into (37) and

dropping the dt terms gives

$$0 = \max \left[U[C, t] + J_t + J_W \left(W \sum_{i=0}^n w_i \alpha_i - C \right) + \frac{W^2}{2} J_{WW} \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij} \right] \quad (40)$$

The choice variables in this maximization problem are consumption and the portfolio weights, and we can use ordinary calculus to derive first order conditions. Before proceeding, we eliminate the constraint that $\sum_{i=0}^n w_i(t) = 1$ for all values of t by expressing the expected portfolio return as

$$\sum_{i=0}^n w_i(t) \alpha_i = \sum_{i=1}^n w_i(t) [\alpha_i - r_f] + r_f. \quad (41)$$

The envelope condition for this problem is

$$U_C(C^*, t) = J_W(W, t), \quad (42)$$

and the first order conditions with respect to portfolio weights (after dividing both sides by W) are

$$0 = J_W(W, t)(\alpha_i - r_f) + J_{WW}(W, t)W \sum_{j=1}^n w_j^* \sigma_{ij}, \quad i = 1, 2, \dots, n. \quad (43)$$

Since the representative investor must hold the market portfolio, the w_j^* must equal the relative weights of the market in equilibrium. Therefore, the last term in (43), $\sum_{j=1}^n w_j^* \sigma_{ij}$, is equal to the covariance of asset i 's return with the return on the market portfolio, $\text{Cov}(r_{it+1}, r_{mt+1})$ in our previous discrete-time notation. Equation (43) can be written

$$\alpha_i - r_f = - \left(\frac{\sigma_m^2 J_{WW}(W, t) W}{J_W(W, t)} \right) \beta_{im}, \quad (44)$$

which looks a lot like the ordinary CAPM. Since there is nothing individual-specific in the risk premium of (44), everyone must hold the market portfolio and the market

portfolio must be mean-variance efficient. This leads directly to the continuous-time CAPM statement that is the analog to the traditional statements of the CAPM,

$$\alpha_i = r_f + \beta_{im}[\alpha_m - r_f]. \quad (45)$$

2.3 The Intertemporal CAPM

The intertemporal CAPM was developed by Merton a few years after the CT CAPM. It has much of the same flavor as the CT CAPM, but it is more interesting because it allows for more than one factor to be priced in returns. The ICAPM is the original multifactor model; Merton published his paper on the ICAPM in *Econometrica* in 1973, several years before the APT.

To derive the ICAPM we will make all the assumptions that we made to get the CT CAPM. We will also assume that there is a state variable besides aggregate wealth that people care about, and we will call it X . We can think of X as containing information about how the parameters that we assumed to be constant in the previous section ($\alpha_i, \sigma_i, \sigma_{ij}$) change through time. Merton originally motivated this state variable as a way to capture changes in the investment opportunity set through time. We assume that X follows Brownian motion,

$$dX = \mu dt + s dZ, \quad (46)$$

and we allow X to be correlated with returns,

$$\text{Cov}[dX, r_i] = \sigma_{iX} dt. \quad (47)$$

Nothing else has changed in our problem, so we can proceed as we did before, except

that now our value function is

$$J = J[W(t), X(t), t], \quad (48)$$

so our Bellman equation is

$$J(W, X, t) = \max E_t \left[\int_t^{t+dt} U[C(s), s] ds + J[W(t+dt), X(t+dt), t+dt] \right], \quad (49)$$

subject to equation (33). Now our expansion of $J[W(t+dt), X(t+dt), t+dt]$ is

$$\begin{aligned} dJ &= J[W(t+dt), X(t+dt), t+dt] - J[W(t), X(t), t] \\ &= J_t dt + J_W dW + J_X dX + \frac{1}{2} J_{XX} dX^2 + \frac{1}{2} J_{WW} dW^2 + J_{WX} dX dW, \end{aligned} \quad (50)$$

with conditional expected value

$$\begin{aligned} E_t J[W(t+dt), X(t+dt), t+dt] &= J[W(t), X(t), t] + J_t dt + J_W E[dW] \\ &+ J_X E[dX] + \frac{1}{2} J_{XX} \text{Var}[dX] + \frac{1}{2} J_{WW} \text{Var}[dW] + J_{WX} \text{Cov}[dX, dW]. \end{aligned} \quad (51)$$

Combining again (34), (35), (41), (46), (47), (51) and (49), and dropping the dt terms, we get the objective function

$$\begin{aligned} 0 = \max & \left[U[C, t] + J_t + J_W W \left(\sum_{i=1}^n w_i (\alpha_i - r_f) + r_f \right) - J_W C + \right. \\ & \left. \frac{W^2}{2} J_{WW} \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij} + J_X \mu + \frac{1}{2} J_{XX} s^2 + J_{WX} W \sum_{i=1}^n w_i \sigma_{iX} \right] \end{aligned} \quad (52)$$

Now the optimal weights satisfy the first order condition

$$0 = J_W (\alpha_i - r_f) + J_{WW} W \sum_{j=1}^n w_j^* \sigma_{ij} + J_{WX} \sigma_{iX}, \quad i = 1, 2, \dots, n. \quad (53)$$

which can be written in matrix form as

$$\mathbf{0} = J_W (\alpha - r_f \mathbf{1}) + J_{WW} W \Sigma \mathbf{w}^* + J_{WX} \sigma \quad (54)$$

where α is the vector of expected returns, Σ is the covariance matrix of returns and σ is the vector of covariances of asset returns with the state variable X . This equation has solution

$$\mathbf{w}^* = \frac{-J_W}{J_{WW} W} \Sigma^{-1} (\alpha - r_f \mathbf{1}) - \frac{J_{WX}}{J_{WW} W} \Sigma^{-1} \sigma, \quad (55)$$

so people buy shares in two mutual funds,

$$\mathbf{w}^* = D\mathbf{t} + H\mathbf{h}, \quad \mathbf{t} \propto \Sigma^{-1} (\alpha - r_f \mathbf{1}), \quad \mathbf{h} \propto \Sigma^{-1} \sigma \quad (56)$$

The portfolio called \mathbf{t} is the tangency portfolio that we have seen before. Its weights are proportional to what we said any mean-variance efficient portfolio weights ought to equal way back in the CAPM chapter. They are also proportional to the optimal weights that can be derived for the CT CAPM.

The portfolio called \mathbf{h} is a portfolio that people use to hedge movements in the state variable, X . It is proportional to the portfolio that has maximum correlation with X . To see this, note that the squared correlation coefficient of any portfolio's return with the state variable is

$$\rho^2 = \frac{(\mathbf{w}' \sigma)^2}{s^2 \mathbf{w}' \Sigma \mathbf{w}}. \quad (57)$$

The first order condition for this squared correlation is

$$\mathbf{0} = \frac{\partial \rho^2}{\partial \mathbf{w}} = \frac{(\mathbf{w}' \Sigma \mathbf{w})(\mathbf{w}' \sigma) \sigma - (\mathbf{w}' \sigma)^2 \Sigma \mathbf{w}}{s^2 (\mathbf{w}' \Sigma \mathbf{w})^2} \quad (58)$$

which implies that

$$\mathbf{w} \propto \Sigma^{-1} \sigma. \quad (59)$$

Thus, the portfolio holdings implied by the ICAPM have intuitive meaning.

What does the ICAPM say about expected returns? We can rearrange (54), the first order condition for portfolio weights as

$$(\alpha - r_f \mathbf{1}) = \frac{-J_{WW}}{J_W} \Sigma \mathbf{w}^* W + \frac{-J_{WX}}{J_W} \sigma. \quad (60)$$

Since \mathbf{w}^* is the vector of weights of the representative agent, they must be market portfolio weights. This means that $\Sigma \mathbf{w}^*$ represents a vector that is proportional to market betas. Furthermore, using the definition of portfolio \mathbf{h} , we can find the covariance of each asset with portfolio \mathbf{h} ,

$$(\sigma_{ih}) = \Sigma \mathbf{h} \propto \Sigma \Sigma^{-1} \sigma = \sigma, \quad (61)$$

so the beta of each asset on the state variable X is proportional to that asset's beta on the portfolio \mathbf{h} . Thus, expected returns are given by

$$\alpha_i = r_f + \beta_{im} \gamma_1 + \beta_{ih} \gamma_2. \quad (62)$$

If we assume that we can pick a state variable that is orthogonal to the market return, then since both the market portfolio and portfolio \mathbf{h} must fit (62),

$$\alpha_i = r_f + \beta_{im}(\alpha_m - r_f) + \beta_{ih}(\alpha_h - r_f). \quad (63)$$

This is the expected returns statement of the ICAPM.

One valuable lesson that we learn from the ICAPM is that continuous-time models are extremely useful. Think back to the derivations that we just went through. We dropped lots of terms along the way by invoking the properties of continuous-time processes. Without continuous-time math, we would have much more complicated mathematical problems to solve and it is not clear that we could get a decent model.

The proper reason to use continuous-time methods in any paper is that they simplify the analysis. Improper reasons include “its been done in discrete time so I thought it would be cool,” trying to impress people, no good reason, etcetera.

The ICAPM can be generalized in a fairly simple manner to account for more than one state. In fact, you will get to show that the model can be generalized for a homework assignment.

2.4 Homework Problems

1. Derive the CT CAPM without assuming that a risk-free asset exists. You will want to impose a constraint on the maximization problem that we were able to substitute out. Imposing the constraint will produce an “extra” lagrange multiplier term.
2. Suppose that there are two state variables that people care about other than aggregate wealth (X and Z). Let these two state variables be correlated. What does the ICAPM say about expected returns now?