

7 Representative Agent Theorems

Most of the economic models that we will apply can be considered representative agent models. A representative agent model is a model in which all agents act in such a manner that their cumulative actions might as well be the actions of one agent maximizing its expected utility function. Economists construct representative agents in order to deal with the complicated issue of aggregation. It is relatively simple to model the behavior of one person given some preferences and constraints. It is more difficult to model the behavior of a group of people, or an entire economy. Aggregation is what economists call the summing up of individuals' behavior to derive the behavior of a market or an economy. Models that aggregate individuals without using a representative agent device are sometimes called heterogeneity models.

The representative agent only exists under certain circumstances, and those circumstances are the subject of this section. Most of the material in this section comes from Huang and Litzenberger (1987), chapter 5.

7.1 Complete Markets

The biggest assumption involved in creating a representative agent is that markets are complete. A complete market is a market in which there are at least as many assets with linearly independent payoffs as there are states. Returning to the notation used in section 3, we will think again of an $N \times S$ matrix D that represent the payoffs of N securities in S states. A complete market in this notation is characterized by the condition that the rank of D is S . This requires that $N \geq S$. It also means that we can assume that D is square. If there are more securities than states then at least $S \leq N$ of the securities must have payoffs that are linear combinations of other securities. As long as the law of one price holds, these redundant securities can be ignored.

Some of the implications of complete markets are discussed in section 2. For example, in complete markets worlds, we can create Arrow-Debreu securities and every agent

can use these securities to smooth consumption. When markets are not complete, they can sometimes be made complete by the use of derivatives. Derivatives generally have payoffs that are nonlinear functions of their underlying assets, so adding derivatives to the set of assets available can increase the rank of D .

7.2 Pareto Optimality

In order to see how to construct a representative agent, we are going to need a slightly more general asset pricing model than the CAPM. We will consider a consumption-based one period model. Throughout this section we will assume that there is one perishable consumption good that serves as our numeraire. All the uncertainty in our model is about which state will be revealed at the end of period one. Agents can consume at time zero and at time one, but for their consumptions to be feasible, it must be true that

$$\sum_{i=1}^I c_{i0} = C_0; \quad (109)$$

and

$$\sum_{i=1}^I c_{i!} = C_! \quad \forall ! = 1, \dots, S; \quad (110)$$

where there are I individuals indexed by i , C_0 denotes aggregate time zero consumption available, and $C_!$ means the total amount of consumption possible in state $!$. A set of state contingent consumption allocations is pareto optimal if it is feasible and if there do not exist other feasible allocations that can strictly increase the utility of one individual without decreasing the utilities of others.

There is a well know result that is associated with the second welfare theorem of economics (described in Varian, pp. 329-335) that states that corresponding to every Pareto optimal allocation, there exist a set of non-negative numbers, $f_{s,i} g_{i=1}^!$, such that the same allocation can be achieved by a social planner maximizing a linear combination of individuals' utility functions using $f_{s,i} g_{i=1}^!$ as weights. The social planner solves the

problem

$$\max_{\{c_{i0}, c_{i1}\}} \sum_{i=1}^I \lambda_i \sum_{l=1}^L \frac{1}{2} \lambda_{il} u_{il}(c_{i0}, c_{i1}) \quad (111)$$

subject to the constraints (109) and (110) listed above, where $\frac{1}{2} \lambda_{il}$ denotes individual i 's subjective probability of state l occurring and $u_{il}(c_{i0}, c_{i1})$ is individual i 's utility over c_{i0} and c_{i1} .

What is the intuition behind this result? We will show that λ_i can be interpreted as the reciprocal of the marginal utility of income of agent i . Agents that have relatively large incomes therefore have more weight in the maximization than agents with low incomes. Thus, how the problem turns out depends on initial endowments. Since we have assumed that utility is strictly increasing, the weights are strictly positive, $\lambda_i > 0; i = 1; 2; \dots; I$.

Forming the Lagrangian for the social planner, we obtain

$$\max_{\{c_{i0}, c_{i1}\}} L = \sum_{i=1}^I \lambda_i \sum_{l=1}^L \frac{1}{2} \lambda_{il} u_{il}(c_{i0}, c_{i1}) + \bar{A}_0 \sum_{i=1}^I c_{i0} + \sum_{l=1}^L \bar{A}_l \sum_{i=1}^I c_{i1} \quad (112)$$

Since the utility functions used here are strictly concave and the λ_i are strictly positive, the first order conditions are necessary and sufficient for maximization in this problem. Those conditions are:

$$\sum_{l=1}^L \lambda_{il} \frac{\partial u_{il}(c_{i0}, c_{i1})}{\partial c_{i0}} = \bar{A}_0; \quad i = 1; 2; \dots; I \quad (113)$$

$$\lambda_i \sum_{l=1}^L \lambda_{il} \frac{\partial u_{il}(c_{i0}, c_{i1})}{\partial c_{i1}} = \bar{A}_l; \quad l = 1; 2; \dots; L; \quad i = 1; 2; \dots; I \quad (114)$$

plus the constraints, (109) and (110). We can get rid of the weights, λ_i , by examining the ratio of these two conditions:

$$\frac{\sum_{l=1}^L \lambda_{il} \frac{\partial u_{il}(c_{i0}, c_{i1})}{\partial c_{i1}}}{\sum_{l=1}^L \lambda_{il} \frac{\partial u_{il}(c_{i0}, c_{i1})}{\partial c_{i0}}} = \frac{\bar{A}_l}{\bar{A}_0}; \quad l = 1; 2; \dots; L; \quad i = 1; 2; \dots; I \quad (115)$$

From this condition for maximization, it is clear that a feasible allocation of state contingent consumption is Pareto optimal if and only if, for each state, marginal rates of substitution between present consumption and future state contingent consumption are equal across individuals. In other words, a Pareto optimal outcome is one in which all individuals share risk perfectly. This does not mean, of course that each individual has equal consumption or equal utility in all states. Agents with a larger endowment will have higher consumption in all states than agents with a smaller endowment. It just means that everyone's relative unhappiness in a bad state is the same.

Pareto optimal allocations are always possible in competitive economies with complete securities markets. Suppose that markets are complete and that \tilde{A}_i is the price of the Arrow-Debreu security that provides one unit of consumption in state $!$. Then the individual's problem can be stated as:

$$\max_{C_{i0}; C_{i!}} \sum_{! \in \Omega} \frac{1}{2} \lambda_{i!} u_{i!}(C_{i0}; C_{i!}) \quad (116)$$

$$s.t: C_{i0} + \sum_{! \in \Omega} \tilde{A}_! C_{i!} = e_{i0} + \sum_{! \in \Omega} \tilde{A}_! e_{i!}; \quad (117)$$

where e_{i0} and $e_{i!}$ represent agent i 's endowment in period zero and state $!$. We assume that these endowments are such that wealth is strictly positive at time zero.

The Lagrangian for each agent is

$$\max_{C_{i0}; C_{i!}} L = \sum_{! \in \Omega} \frac{1}{2} \lambda_{i!} u_{i!}(C_{i0}; C_{i!}) + \mu_i (e_{i0} - C_{i0} - \sum_{! \in \Omega} \tilde{A}_! e_{i!} + \sum_{! \in \Omega} \tilde{A}_! C_{i!}); \quad (118)$$

which has first order conditions,

$$\sum_{! \in \Omega} \frac{1}{2} \lambda_{i!} \frac{\partial u_{i!}(C_{i0}; C_{i!})}{\partial C_{i0}} = \mu_i; \quad (119)$$

$$\frac{1}{2} \lambda_{i!} \frac{\partial u_{i!}(C_{i0}; C_{i!})}{\partial C_{i!}} = \mu_i \tilde{A}_!; \quad ! \in \Omega; \quad (120)$$

plus the budget constraint. Once again, we can get rid of μ_i by forming the ratio

$$\frac{\lambda_i \frac{\partial U_i(C_{i0}; C_{i1})}{\partial C_{i1}}}{\lambda_i \frac{\partial U_i(C_{i0}; C_{i1})}{\partial C_{i0}}} = \tilde{A}_i; \quad i = 1, 2, \dots \quad (121)$$

In a market equilibrium, the feasibility constraints, (109) and (110), are always satisfied. If we set $\lambda_0 = 1$, $\lambda_i = \tilde{A}_i$, and $\mu_i = \frac{1}{\lambda_i}$ then we can see that the conditions for the optimality of a single agent are equivalent to the conditions for a Pareto optimal allocation discussed above. Conversely, to achieve a Pareto optimal allocation from this competitive economy, the social planner assigns a weight of μ_i^{-1} to each individual.

This derivation reinforces what we already know about the weights, μ_i . The weight of agent i is the reciprocal of agent i 's "shadow price of the budget constraint." The shadow price of the budget constraint is set equal to the marginal utility of consumption in period zero here. This all makes intuitive economic sense - the social planner considers people with more at stake (people with a higher initial wealth) more important in the total maximization than people with a smaller market weight.

Where does this derivation break down if markets are not complete? We need state prices to evaluate the agent's endowment. But we have assumed that we have an equilibrium, so there must not be any arbitrage opportunities available in our economy. This means, of course, that a state price vector exists regardless of whether or not markets are complete. The problem is that the state price vector is not unique if markets are not complete. If state prices are not unique then we can sort of think of one representative agent existing for each set of state prices that we assume people use to value their claims. This construction, however, assumes that all agents use the same state prices to value their endowments. If different people use different state price vectors then the whole representative agent idea breaks down.

Duane has a nice, concise discussion of the representative agent in his first chapter. He says that it is not always necessary for markets to be complete for Pareto optimality, and thus for a representative agent to exist. Complete markets (and other assumptions)

are just sufficient for the representative agent, they are not necessary. However, Deaton also points out that "... it can be shown that, with incomplete markets and under natural assumptions on utility, for almost every endowment, the equilibrium allocation is not Pareto optimal." Thus, we should not expect to have a representative agent with incomplete markets.

7.3 Constructing the Representative Agent

Now we are ready to derive the representative agent result. We assume that markets are complete and that the economy is competitive. We also assume that individuals have homogeneous beliefs and time-additive, state-independent utility functions that are strictly concave, increasing, and differentiable. This means that the conditions for maximization by a single agent, (119) and (120), can be stated as:

$$\frac{\partial u_{i0}(c_{i0})}{\partial c_{i0}} = \mu_i; \quad (122)$$

and

$$\mu_i \frac{\partial u_{i1}(c_{i1})}{\partial c_{i1}} = \mu_i \tilde{A}_i; \quad (123)$$

where μ_i is the subjective probability of state ω that all agents agree on.

Let \tilde{A}_i represent state prices as defined in section 2 of the notes. Define u_0 and u_1 to be:

$$u_0(z) = \max_{\{z_i\}_{i=1}^I} \sum_{i=1}^I \mu_i u_{i0}(z_i) \quad (124)$$

$$\text{s.t: } \sum_{i=1}^I z_i = z;$$

and

$$u_1(z) = \max_{\{z_i\}_{i=1}^I} \sum_{i=1}^I \mu_i u_{i1}(z_i) \quad (125)$$

$$\text{s.t: } \sum_{i=1}^I z_i = z;$$

where $\mu_i = \frac{1}{\mu_i}$ is the social planner's weight for each individual. An immediate conse-

quence of these definitions is that

$$u_0^0(C_0) = \sum_{i=1}^I u_{i0}^0(C_{i0}) \frac{\partial C_{i0}}{\partial C_0} = \sum_{i=1}^I \mu_i \frac{\partial C_{i0}}{\partial C_0} \quad (126)$$

$$= \sum_{i=1}^I \frac{\partial C_{i0}}{\partial C_0} = 1; \quad (127)$$

and

$$u_1^0(C_1) = \sum_{i=1}^I u_{i1}^0(C_{i1}) \frac{\partial C_{i1}}{\partial C_1} = \sum_{i=1}^I \mu_i \frac{\tilde{A}_i}{\gamma_i} \frac{\partial C_{i1}}{\partial C_1} \quad (128)$$

$$= \frac{\tilde{A}_i}{\gamma_i} \sum_{i=1}^I \frac{\partial C_{i1}}{\partial C_1} = \frac{\tilde{A}_i}{\gamma_i}; \quad (129)$$

These results use the constraints (109) and (110) to conclude that

$$\sum_{i=1}^I \frac{\partial C_{i0}}{\partial C_0} = 1 \quad (130)$$

and

$$\sum_{i=1}^I \frac{\partial C_{i1}}{\partial C_1} = 1; \quad (131)$$

Now think of a representative agent that has endowments of C_0 and C_1 ; $I \geq 2$ - in periods one and two, respectively. Let the representative agent's subjective probabilities be γ_i , and let its utility for period zero and one consumption be $u_0(C_0)$ and $u_1(C_1)$ respectively. The state prices in this type of economy must be equal to \tilde{A}_i for the representative agent to exist.

In this economy, for the markets to clear, the representative agent must not want to trade away from its endowment. Thus, using time zero consumption as the numeraire, the state price for state i must be equal to the representative agent's marginal rate of substitution (MRS) between time zero consumption and time one, state i consumption. This MRS is:

$$\frac{\gamma_i u_1^0(C_1)}{u_0^0(C_0)}; \quad (132)$$

After substituting (127) and (129) into (132), we know that the representative agent's MRS is equal to

$$\frac{u_1^0(C_1)}{u_0^0(C_0)} = \frac{u_1 \tilde{A}_1}{u_1} = \tilde{A}_1 : \quad (133)$$

Thus, the representative agent exists in this case.

This representative agent, however, never wants to trade. Equilibriums in which this agent is satisfied will never involve trading. This has led many to look for models that say something about the quantity of trade going on.

7.4 Other Aggregation Results

The representative agent's utility function in the previous results depended on the initial endowments of individuals through the parameter ω_i . Therefore, the prices in the economy will vary with the initial endowments of agents. For some utility functions, the utility function of the representative agent does not depend on the endowments of the agents in the economy. Such utility functions have what is called the aggregation property. Two utility functions that satisfy the aggregation property are the power utility function,

$$u_i(z) = \frac{1}{1 + B} (A_i + Bz)^{1 + \frac{1}{B}}; \quad (134)$$

and the negative exponential utility function,

$$u_i(z) = A_i \exp\left(-\frac{z}{A_i}\right); \quad (135)$$

The power function includes log utility,

$$u_i(z) = \ln(A_i + z); \quad (136)$$

for the case when $B = 1$.

In the case of negative exponential utility, it can even be shown that if agents have

different time preference parameters and different subjective probabilities than the representative agent's time preference and probabilities are composites of the individuals' parameters. See Huang and Litzenberger for more discussion of these issues.

7.5 Thoughts on the Representative Agent

We have shown that under fairly general conditions a representative agent exists. We have not shown, however, that the representative agent is always an interesting construction. It may be that while a representative agent exists in most circumstances, a simpler way to characterize prices is possible. However, most of the economic models that are prominent in Finance use some sort of representative agent formulation.

A few models that don't rely on the representative agent argument have been suggested in recent years. Most of these models rely on some sort of simulation in order to determine prices and things because they are fairly complicated. Heterogeneity models or models with incomplete markets seem like a fairly fertile ground for future study. For examples of heterogeneity models, see Heaton and Lucas (1995),⁵ Aiyagari (1993),⁶ Telmer (1993),⁷ and Constantinides and Du±e (1996).⁸

7.6 Homework Problem

1. Show that u_0 and u_1 defined in (124) and (125) are both strictly concave and increasing (Huang and Litzenberger problem 5.1).
2. Provide some intuition for equation (115). In particular, without using any math,

⁵Heaton, J. and D. Lucas (1995) "The Importance of Investor Heterogeneity and Financial Market Imperfections for the Behavior of Asset Prices," Carnegie-Rochester Conference Series on Public Policy 42 1-32.

⁶Aiyagari, S. R. (1993) "Explaining Financial Market Facts: The Importance of Incomplete Markets and Transaction Costs," Federal Reserve Bank of Minneapolis Quarterly Review, Winter, 17-31.

⁷Telmer, C. (1993) "Asset-Pricing Puzzles and Incomplete Markets," Journal of Finance, XLVIII, 1803-1832.

⁸Constantinides, G., and D. Du±e (1996) "Asset Pricing with Heterogeneous Consumers," Journal of Political Economy, 219-240.

explain why agents will want to equalize (across states) marginal rates of substitution between current consumption and future consumption.

3. What kind of asset pricing model does equation (133) imply? To answer, write down the representative agent's pricing model in the form $E(R_m) = 1$. Your expression for m should involve the representative agent's utility function. Is your expression consistent with our previous interpretations of m ?