

3 The Fundamental Theorem of Asset Pricing

In this section we will discuss what is sometimes called the fundamental theorem of asset pricing. Much of the material in this section is from Du±e (pp. 3-13). Du±e's proof of the central no-arbitrage result uses a nasty construction called the separating hyperplane theorem. Ingersoll (pp. 52-58) derives similar results with some theorems from linear programming. Varian (pp. 383-385) also uses linear programming results. The article by Dybvig and Ross from the New Palgrave is another useful reference.

Why is no-arbitrage an attractive principle to start with?

3.1 Notation

As assumed above, there are S possible states of the world, indexed by s . There are also N assets, indexed by i : The $N \times S$ matrix D represent the payoffs of the N securities in the S states. Security prices are given by the N -vector q , and portfolio weights are given by the N -vector μ . The portfolio μ costs $q \cdot \mu$ and it pays $D \cdot \mu$. A state-price vector is a strictly positive S -vector, \tilde{A} , with $q = D \tilde{A}$. A state price, \tilde{A}_j , is basically the marginal cost of a unit of payoff in state j .

3.2 The Theorem

An arbitrage is a portfolio that satisfies one of two criteria:

Arbitrage of first type. $q \cdot \mu = 0$; and $D \cdot \mu > 0$; where the $>$ sign means that at least one element of $D \cdot \mu$ is positive.

Arbitrage of second type. $q \cdot \mu < 0$; and $D \cdot \mu \geq 0$; where the $<$ sign means that the portfolio must have a negative price.

What do these arbitrage conditions mean?

The Fundamental Theorem of Asset Pricing. The following three conditions are equivalent:

- 2 The absence of arbitrage
- 2 The existence of a positive linear pricing rule (state prices)
- 2 The existence of an optimal portfolio for some agent who prefers more to less.

This is such an important result that we need to take the time to prove it. First, we will prove that the absence of arbitrage is equivalent to the existence of state prices. Before delving into the proof, however, we need to know the separating hyperplane theorem.

3.2.1 Separating Hyperplane Theorem

We need to use a theorem known as the separating hyperplane theorem to prove our result. We will not prove the theorem, but we will discuss the intuition behind it. The theorem is discussed in Du±e (pages 275-277) and in references cited by Du±e.

What is a hyperplane? Let's think first about a "regular" plane. The equation of a 3 dimensional plane is $F(x) = a + b_1x_1 + b_2x_2$: We can draw a regular plane:

The equation of an n -dimensional hyperplane is $F(x) = a + \sum_{i=1}^n b_i x_i$. It is difficult to draw hyperplanes with $n > 2$. Notice that the equation for a hyperplane is a linear functional."

Separating Hyperplane Theorem. Suppose that A and B are convex disjoint subsets of \mathbb{R}^n . There is some nonzero linear functional F such that $F(x) < F(y)$ for each x in A and y in B . Moreover, if x is in the interior of A or y is in the interior of B , then $F(x) < F(y)$.

So what does the theorem mean? Draw a picture:

We need a special version of the separating hyperplane theorem for our result. This version holds for cones, which are spaces that include all the points described by $\{x; \delta x > 0 \text{ for each } x \text{ in the set.}$ What does a cone look like? Some examples:

Linear Separation of Cones. Suppose M and K are closed convex cones in \mathbb{R}^n that intersect precisely at zero. If K is not a linear subspace,² then there is a nonzero linear functional F such that $F(x) < F(y)$ for each x in M and nonzero y in K .

3.2.2 Proof

Now that we have the separating hyperplane theorem, we can prove the first part of the fundamental theorem stated above. Let the sets

$$\begin{aligned} L &= \{x \in \mathbb{R}^n\}; \\ M &= \{(\sum_{i=1}^n \mu_i D_i) : \mu_i \geq 0, \sum_{i=1}^n \mu_i = 1\}; \\ K &= \{x \in \mathbb{R}^n : x \geq 0\} \end{aligned} \tag{30}$$

Note that both K and M are closed, convex subsets of L . They are also both cones. M is a linear space, but K is not linear since it does not contain any negative elements. No arbitrage means that K and M intersect precisely at zero. Why? Notice first that

²A nonempty set L of elements x, y, z, \dots is said to be a linear space (or vector space) if it satisfies the following three axioms:

1. Any two elements $x, y \in L$ uniquely determine a third element $x + y \in L$, called the sum of x and y , such that
 - ² $x + y = y + x$ (commutativity);
 - ² $(x + y) + z = x + (y + z)$ (associativity);
 - ² There exists an element $0 \in L$, called the zero element, with the property that $x + 0 = x$ for every $x \in L$;
 - ² For every $x \in L$ there exists an element $-x$, called the negative of x , with the property that $x + (-x) = 0$.
2. Any number α and any element $x \in L$ uniquely determine an element $\alpha x \in L$, called the product of α and x , such that $\alpha(-x) = (\alpha^-)x$ and $1x = x$.
3. The operations of addition and multiplication obey two distributive laws, $(\alpha + \beta)x = \alpha x + \beta x$, and $\alpha(x + y) = \alpha x + \alpha y$.

M must contain f_0 to avoid arbitrage. Now consider $Du_{\pm e}$'s pictures:

Suppose $K^T M = \{0\}$. The separating hyperplane theorem says that a linear functional exists, $F : L \rightarrow \mathbb{R}$ such that $F(z) < F(x)$ for all $z \in M$ and nonzero $x \in K$. Since M is a linear space,

$$\begin{aligned} F(z) &= 0; \quad \forall z \in M; \\ F(x) &> 0 \quad \forall x \in K \setminus \{0\}. \end{aligned} \tag{31}$$

Since M is a linear space, $F(z)$ must equal zero for all $z \in M$: Suppose, for example, that $F(z^a) < 0$ for some $z^a \in M$. Since the element $\lambda z^a \in M$ exists for any scalar λ , we could find $F(\lambda z^a)$, of any arbitrary value, contradicting the separating hyperplane theorem.

The fact that $F(x) > 0$ for all nonzero $x \in K$ implies that F is represented by some $\bar{q} > 0 \in \mathbb{R}^N$ and a strictly positive $\bar{A} \in \mathbb{R}^{S \times N}$ by $F(v; c) = \bar{q} \cdot v + \bar{A} \cdot c$ for any $(v; c) \in L$. This implies that $\bar{q} + \bar{A} \cdot (D^0 \mu) = 0 \quad \forall \mu \in \mathbb{R}^N$. Thus, the vector $\frac{\bar{A}}{\bar{q}}$ is a state price vector.

Conversely, if a state price vector exists, then there is clearly no arbitrage. Why not? Suppose that \hat{v} is an arbitrage opportunity. This means that the $(S + 1)$ -vector $\hat{v} = [\hat{v}_0; \hat{q} : D]$ is strictly nonnegative, with at least one element greater than zero.

Using our no-arbitrage result,

$$q = D\tilde{A}; \quad (32)$$

we know that,

$$0 = \sum_i q_i [D_i - \tilde{A}_i]; \quad (33)$$

This last statement is a contradiction since we assumed that q was an arbitrage and \tilde{A} is strictly positive.

3.3 The Existence of an Optimal Portfolio

Why do we care about the second part of the fundamental theorem? The second part of the theorem emphasizes why we are so concerned about arbitrage in the first place. We will show that if there are arbitrage opportunities, there is no optimal portfolio for agents to hold. When there are arbitrage opportunities, agents cannot maximize expected utility because they will always prefer to take unbounded positions in order to take advantage of the arbitrage opportunity. Thus, stating that the absence of arbitrage is equivalent to an optimum existing is just another way to say that arbitrage opportunities cannot exist in equilibrium.

We will not prove the second part of the theorem exactly as it is stated. Rather, we will prove a slightly less general but much more tractable version that appears in *Du±e*. This version requires assuming the existence of an agent with an increasing utility function over consumption, $U(c)$; and a random endowment, e . This agent's problem can be stated as:

$$\begin{aligned} & \text{Max } U(c) \\ & \text{Subject to: } c \cdot e + D^0 \mu; \quad q^0 \mu = 0; \quad c_s \geq 0 \text{ in all states;} \end{aligned} \quad (34)$$

where the utility function $U(c)$ is defined on the vector of consumptions in the S possible states. We also need to assume that at least one asset exists that has a positive payoff[®]

and a positive price. Now we are ready to state the simpler version that we will prove. **Existence of an Optimum in the Absence of Arbitrage.** If there is a solution to (34) then there is no arbitrage. If $U(c)$ is continuous and there is no arbitrage, then there is a solution to (34).

You should prove this result as an exercise.

3.4 Homework Problems

1. Suppose that there are two assets in a two state economy. Let the payoff matrix be:

$$\begin{array}{cc}
 & \begin{array}{cc} S_1 & S_2 \end{array} \\
 \begin{array}{c} P_1 \\ P_2 \end{array} & \begin{array}{cc} 0 & 1 \\ 1 & 2 \end{array}
 \end{array} \quad (35)$$

where S_1 and S_2 are the possible states of the world and P_1 and P_2 are possible payoffs. Let the probabilities of the two states be 0.2 for state 1 and 0.8 for state

2. Let the security prices be 0.6 and 1.0 for assets 1 and 2 respectively. What does the risk-free rate of return have to be to avoid arbitrage in this economy? Calculate state prices, the value of the pricing kernel in each state, and the risk-neutral probabilities that correspond to this example (assuming the no-arbitrage risk-free rate).

2. What is the difference between the law of one price and the absence of arbitrage? Think of an example that violates one but not the other.
3. Prove the second part of the fundamental theorem of asset pricing (stated in section 3.3). You may want to (but don't have to) use the results listed below in your proof:

The Theorem of the Maximum. Let the set $D \subset \mathbb{R}^N$ be closed and bounded (closed + bounded = compact), and let $f : D \rightarrow \mathbb{R}$ be continuous. Then f assumes its

maximum (and its minimum) values at some point x_0 (and x_1) of D .

Definition of Bounded. A set $D \subset \mathbb{R}^n$ is said to be bounded if there is a number, $Q > 0$, such that $\|x\| < Q \quad \forall x \in D$. A set is closed if it contains all its boundary points.

Contrapositive Argument. For any statements p and q , $(p \rightarrow q) \equiv (\text{not } q \rightarrow \text{not } p)$: