9 Consumption-Based Models

In this section we will derive a typical consumption-based asset pricing model in discrete time with dynamic programming. These models have been developed both by macroeconomists and ¯nancial economists. Macroeconomists want to understand the behavior of consumption under uncertainty while ¯nancial economists want to understand asset returns. Consumption and asset returns turn out to be intricately related to eachother. The model that we will examine is similar to that of Lucas.[10] The consumption-based model has motivated lots of empirical research, including the development of GMM.[11]

9.1 Assumptions and Notation

To begin, we make several assumptions:

1. There is a representative agent.
2. Markets are perfect - no rationing constraints, transactions costs, etcetera.
3. There exists a (real) riskless asset.
4. Individuals can freely borrow and lend at the risk-free rate.
5. labor income is given exogenously and it is diversi¯able.
6. The representative investor has additively separable Von Neumann-Morgenstern utility.
7. The representative investor lives until period T.

We also need to describe the notation to be used:

[10] Lucas, R. (1978), \Asset Prices in an Exchange Economy," Econometrica, 46, 1429-45.

[11] Hansen, L. and K. Singleton (1982), \Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models," Econometrica, 50, 1269-1288.

- $u(c_t)$ is the agent's (undiscounted) utility of consumption at period t.
- β is the representative agent's discount factor for future utility.
- E_t is the expectations operator conditional on all information available at t.
- A_t is the agents total wealth in period t.
- y_t is the agent's exogenous labor income in period t.
- r_{it+1} is the return on asset i from t to t + 1. It is not known until t + 1.
- $r_{0t+1} = r_{ft}$ is the risk-free return from t to t + 1. It is known at t.
- w_{it} is the fraction of the agent's wealth invested in asset i from t to t + 1. It is chosen in period t.

9.2 Derivation

Under these assumptions, the representative investor solves the problem,

$$\max_{c_t; w_{it}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t); \tag{160}$$

$$s.t.\ \ A_{t+1} = (A_t + y_t - c_t) \left[\sum_{i=0} (1 + r_{it+1}) w_{it} \right]; \tag{161}$$

$$\sum_{i=0} w_{it} = 1. \tag{162}$$

The term in brackets in equation (161) is just the return on the agent's portfolio. The budget constraint just says that wealth next period is equal to the wealth that we start with this period minus what we consume plus what we earn times our portfolio return. The additive utility assumption expressed in (160) is significant. Assuming additive separability ties together the concepts of risk aversion and intertemporal substitution. A big point in the work of Epstien and Zin (cited earlier) is that we may want to be able to identify risk aversion and intertemporal substitution separately. Using the fact that $r_{0t+1} = r_{ft}$, we can combine (161) and (162) to obtain

$$A_{t+1} = (A_t + y_t - c_t)\left[1 + r_{ft} + \sum_{i=1}^{n}(r_{it+1} - r_{ft})w_{it}\right]. \tag{163}$$

Now the summation term is the excess return on our portfolio of risky assets. We can express the value function in period t as

$$V_t(A_t) = \max_{c_t, w_{it}} E_t \sum_{s=t}^{T} \beta^{s-t} u(c_s); \quad \text{s.t. } (163); \tag{164}$$

or as the bellman equation,

$$V_t(A_t) = \max_{c_t, w_{it}} [u(c_t) + \beta E_t V_{t+1}(A_{t+1})]; \quad \text{s.t. } (163). \tag{165}$$

We can substitute (163) into (165) to get the final objective function:

$$V_t(A_t) = \max_{c_t, w_{it}} \left\{ u(c_t) + \beta E_t V_{t+1}\left((A_t + y_t - c_t)\left[1 + r_{ft} + \sum_{i=1}^{n}(r_{it+1} - r_{ft})w_{it}\right]\right)\right\}. \tag{166}$$

Given our objective function, we can derive first order conditions for the representative investor. The first order condition for consumption is,

$$\text{FOC for } c_t: \quad u'(c_t) = \beta E_t \left(V'_{t+1}(A_{t+1}) \left[1 + r_{ft} + \textstyle\sum_{i=1}^{n}(r_{it+1} - r_{ft})w_{it}\right]\right); \quad t = 1, 2, 3, \ldots, T. \tag{167}$$

This condition characterizes the optimal consumption-savings path for our agent. It says that she should always set the marginal utility of consumption today equal to the discounted expected return of her portfolio times the marginal value of wealth next period. This demonstrates another type of consumption smoothing behavior predicted by economic models. In intertemporal models, agents smooth their consumptions across time. It would probably not be optimal, for example, for our agent to consume 90% of her wealth in period 0 and leave the remaining 10% for the rest of her lifespan. The first order condition for portfolio weights look like,

$$\text{FOC for } w_{it}: \quad E_t[V'_{t+1}(A_{t+1})(r_{it+1} - r_{ft})] = 0; \quad i = 1, 2, 3, \ldots, N; \quad t = 1, 2, 3, \ldots, T. \tag{168}$$

Besides these first order conditons, we can also use the envelope condition,

$$V'_t(A_t) = \beta E_t \left(V'_{t+1}(A_{t+1}) \left[1 + r_{ft} + \sum_{i=1} (r_{it+1} - r_{ft}) w_{it} \right] \right). \tag{169}$$

Combining (167) and (169), we obtain

$$V'_t(A_t) = u'(c_t), \tag{170}$$

so the FOC for the portfolio weights, (168), can be written as

$$E_t[u'(c_{t+1})(r_{it+1} - r_{ft})] = 0; \quad i = 1, 2, 3, \ldots, N; \quad t = 1, 2, 3, \ldots, T. \tag{171}$$

We immediately recognize that (171) is of the form

$$E_t[M_{t+1} r^e_{it+1}] = 0, \tag{172}$$

and we are done. The consumption-based model basically implies that the pricing kernel, M_{t+1}, is equal to the marginal utility of consumption in the next period, $u'(c_{t+1})$.

How do we interpret this model's results? Where are the demand and supply curves? The representative agent in this model is actually setting prices and choosing consumption simultaneously to make markets clear. We could probably derive demand and supply curves for this model, but their forms would be messy. Like most representative agent models, this model predicts no trading.

This asset pricing model provides some intuition about what a model with heterogeneous agents and incomplete markets might look like. The first order conditions of the representative agent are the same conditions that any ordinary consumer would solve in the same circumstances. Thus, we can think of consumers choosing their consumptions to meet the conditions above. If we aggregate consumers in some way without a representative agent, it is not clear what sort of equilibrium we will get.

Models similar to this have been developed with the first order conditons of firms rather than consumers. Production-based asset pricing models basically solve a similar problem in which firms are maximizing the present value of future cash flows with the constraints that output is produced by a production function and capital in period t is equal to depreciated capital from period $t - 1$ plus investment. These models basically stipulate that expected stock returns should be set equal to the expected returns on investment.

9.3 Homework Problems

1. Derive the consumption based model without assuming the existence of a risk free asset. You will have to add a constraint that we relaxed in class. You may want to give that constraint a lagrange multiplier and proceed.

2. Suppose that the representative agent has quadratic utility. What sort of asset pricing model results (this is a famous model - you may not know the name yet). Now suppose that the representative agent's consumption is a linear function of his return. What sort of asset pricing model results?

3. Suppose that ¯ = $\frac{1}{1+r_{ft}}$ and suppose that you have no labor income. Using all the ¯rst order conditions we derived, derive an expression that relates consumption today to future consumption only (you should get portfolio returns to cancel out somehow). What does this expression mean in economic terms?