6 The CAPM

In this section we begin to look at models that have a little more economic content than pure arbitrage models. The ¯rst model we explore is the Capital Asset Pricing Model (CAPM). The CAPM is of mostly historical interest - it is used by practitioners and it is generally taught to MBA students, but very few active researchers still believe that it describes expected returns well. The CAPM is developed under fairly strong assumptions, meaning that it should only hold in speci¯c situations. Understanding the CAPM does help to develop intuition for more complicated asset-pricing models.

We will derive the CAPM with a risk-free asset. There is a version of the CAPM that does not use a risk-free asset, but it is probably not worth our time in class to derive all the possible CAPM relations.

6.1 Assumptions and Notation

Assumptions:

1. A Representative Agent Exists

   common time horizon

   homogeneous beliefs

2. Mean-Variance Analysis is Optimal

   normally distributed returns

   quadratic utility

3. Perfect Markets

   perfect competition

   unlimitted short sales and margin positions

   no transactions costs

assets are perfectly divisible

4. One Period World

5. There are n Risky Assets, 1 Risk-Free Asset, No Other Assets.

Notation:

μ^e = n-vector of expected excess returns on risky assets ($\mu - r_f$)

μ^e_p = the (scalar) expected excess portfolio return desired

r_f = the risk-free rate

w = n-vector of portfolio weights (risky assets)

w_0 = the weight on the risk-free asset

Σ = $n \times n$ variance-covariance matrix of returns on risky assets

1 = n-vector of ones

6.2 Mean-Variance Math

The mean-variance optimization problem can be stated as:

$$\text{minimize} \quad \frac{1}{2}\sigma_p^2 = \frac{1}{2}w'\Sigma w \tag{87}$$

$$\text{subject to:} \quad w'\mu^e = \mu^e_p. \tag{88}$$

What does this problem mean?

Note that Σ is positive definite (as are all well-specified variance-covariance matrices), so the objective function is convex and a first order condition will be necessary and sufficient for optimzation. Forming a lagrangian function,

$$L = \frac{w'\Sigma w}{2} - \lambda(w'\mu^e - \mu^e_p) \tag{89}$$

The first order conditions are:

$$\frac{\partial L}{\partial w} = \Sigma w - \lambda \mu^e = 0; \tag{90}$$

$$\frac{\partial L}{\partial \lambda} = w' \mu^e - \mu^e_p = 0; \tag{91}$$

By multiplying through by Σ^{-1}, (90) can be written as

$$w = \lambda \Sigma^{-1} \mu^e. \tag{92}$$

The value of the lagrangian multiplier, λ, can be found by substituting the expressions for w into (91),

$$\lambda = \frac{\mu^e_p}{\mu^{e'} \Sigma^{-1} \mu^e}. \tag{93}$$

If we define:

$$\begin{aligned} A &= 1' \Sigma^{-1} 1; \\ B &= 1' \Sigma^{-1} \mu = \mu' \Sigma 1; \\ C &= \mu' \Sigma^{-1} \mu; \end{aligned} \tag{94}$$

then we can express,

$$\lambda = \frac{\mu^e_p}{(\mu' - r_f 1') \Sigma^{-1} (\mu - r_f 1)} = \frac{\mu^e_p}{[C - 2 r_f B + r_f^2 A]}. \tag{95}$$

Now we can solve for the optimal weights by substituting the value of λ into (92).

$$w = \frac{\mu^e_p}{[C - 2 r_f B + r_f^2 A]} \Sigma^{-1} \mu^e; \quad w_0 = 1 - \sum_{i=1}^{N} w_i. \tag{96}$$

Any set of portfolio weights that satisfy (96) are mean-variance efficient. Substituting

(92) into the definiton for σ_p^2 yields

$$
\begin{aligned}
\sigma_p^2 &= w'\Sigma(\lambda\Sigma^{-1}\mu^e) \\
&= \lambda w'\mu^e \qquad (97) \\
&= \lambda\mu_p^e
\end{aligned}
$$

$$
\sigma_p^2 = \frac{\mu_p^{e2}}{[C - 2r_f B + r_f^2 A]}; \qquad (98)
$$

which is the equation of a parabola. In mean-standard deviation space, this becomes

$$
\sigma_p = \frac{|\mu_p - r_f|}{\sqrt{C - 2r_f B + r_f^2 A}}. \qquad (99)
$$

To plot this, we would draw two rays extending from r_f to the right. The rays would have slopes of positive and negative $\sqrt{C - 2r_f B + r_f^2 A}$. If we were to solve the same problem without a risk-free asset, the corresponding plot would be a hyperbola. These rays and this hyperbola are the shapes commonly depicted in the familiar "minimum-variance set" pictures.

We can identify the "tangency" portfolio by finding the minimum variance portfolio for which $\sum_{i=1}^{N} w_i = 1$. Equation (96) says that the optimal weights should be a constant multiplied by $\Sigma^{-1}\mu^e$. If we sum up the elements of $\Sigma^{-1}\mu^e$, they come to $B - r_f A$, so the tangency portfolio has weights,

$$w_t = \frac{\Sigma^{-1}\mu^e}{B - r_f A}. \tag{100}$$

The tangency portfolio has a mean return equal to

$$\mu_t = \frac{C - r_f B}{B - r_f A}, \tag{101}$$

and a variance equal to

$$\sigma_t^2 = \frac{C - 2r_f B + r_f^2 A}{(B - r_f A)^2}. \tag{102}$$

We can construct any portfolio on the minimum variance frontier with the tangency portfolio and the risk-free asset. Actually, we can use any two portfolios to create all other portfolios on the frontier, but it is intuitive to think of using the tangency portfolio and the risk-free asset.

6.3 Equilibrium Conditions

We have shown that mean-variance optimization leads the investor to choose from a set of minimum variance portfolios that are described by equation (96). Now we need to make some equilibrium arguments to finish our derivation of the CAPM.

Given our assumptions, we can state that investors will only want to hold minimum variance portfolios. Since all minimum variance portfolios can be generated with two minimum variance portfolios, the holdings of all investors can be generated by combining the weights of just two portfolios. This is an example of a mutual fund theorem. There are several mutual fund theorems in Ingersoll's chapter 6 if you are interested in them. Since all investors want to hold just two funds, we need to identify two minimum variance funds to let them hold. The easy fund to identify is just the risk-free asset. The other fund can be identified by noting that the risk-free asset is in zero net supply. This means that the aggregate value for w_0 is zero. Thus, the aggregate value for $w'1$ is one. The other minimum variance portfolio we will use is the tangency portfolio defined above. Since the tangency portfolio contains all risky assets in proportion to their market weights, it is usually called the market portfolio.

We can define a vector of covariances with the market portfolio as

$$\sigma_{im} = \Sigma w_m = \frac{\mu^e}{(B - r_f A)}. \tag{103}$$

By using the relation

$$\sigma_m^2 = w_m' \sigma_{im} = w_m' \Sigma w_m = \frac{\mu_m^e}{(B - r_f A)}; \quad \text{or} \quad (B - r_f A) = \frac{\mu_m^e}{\sigma_m^2}; \tag{104}$$

we can derive the CAPM relation

$$\mu^e = \beta_{im} \mu_m^e; \quad \beta_{im} = \frac{\sigma_{im}}{\sigma_m^2}. \tag{105}$$

This relation can, of course, also be written in a more familiar form,

$$E(r_i) = r_f + \beta_{im}[E(r_m) - r_f]; \tag{106}$$

which is the equation most often associated with the CAPM.

Notice, however, that any mean-variance efficient portfolio's weights can be expressed as

$$w_\mu = c\Sigma^{-1}\mu^e. \tag{107}$$

Thus, the covariance of each asset with this mean-variance efficient portfolio is

$$\sigma_{i\mu} = \Sigma w_\mu = \frac{\mu^e}{(B - r_f A)} \tag{108}$$

By now, this eqaution looks a lot like equation (103). It should not be surprising that we can derive an equation just like (106) for any mean-variance efficient portfolio. The observation that for any mean-variance efficient portfolio there is a linear relation between expected returns and covariances with the portfolio is a tautology. Furthermore, there will always be a mean-variance efficient portfolio that can be identified after the fact. The only real significance in the CAPM is the idea that the market portfolio must be mean-variance efficient. Put another way, the only important implication of the CAPM is that the pricing kernel should be a linear function of the market return and no other factors. Since the "true" market portfolio return is never observed, tests of the CAPM are quite difficult to perform. This idea is known as Roll's critique because Richard Roll was the first person to point this out.

The CAPM makes some very strong assumptions in order to conclude that

1. All investors hold just the market portfolio and the risk-free asset.

2. The market portfolio is mean-variance efficient.

3. Expected returns are proportional to market betas.

Other asset-pricing models are similar in spirit to the CAPM, so it is worthwhile knowing how it works.

We have already seen how the CAPM fits into the $1 = E(RM)$ framework above, so we won't return to that problem.

6.4 Homework Problems

1. Suppose that all of the assumptions underlying both the APT and the CAPM hold. Suppose, in particular, that returns are generated by a k-factor model and that equation (106) describes expected returns. Suppose also (for simplicity) that there are no error terms in the factor models that describe the generation of returns. What can you say about the risk premia (the ¸s) in the APT?

2. (Huang and Litzenberger, problem 3.1) In a CAPM world, let there be two securities with (random) returns r_i and r_j. Suppose that these securities have identical expected rates of return and identical variances. The correlation coe±cient between r_i and r_j is ½. Show that an equally-weighted portfolio of assets i and j achieves the minimum possible variance regardless of the value of ½.

3. (Huang and Litzenberger, problem 3.3) Let p be a mean-variance e±cient portfolio and let q be any portfolio having the same expected return. Show that $COV(r_p; r_q) = VAR(r_p)$ and, as a consequence, the correlation coe±cient of r_p and r_q lies in $(0; 1]$.