

# 1 Introduction

Finance has become an academic field over the past forty years. Before the 1950s, finance was mostly a descriptive field. In the 1950s, a few economists began asking (and answering) some fundamental finance questions. Since the 1950s, finance has grown to become either the largest subfield of economics or a field of its own. Finance has become large enough to have its own subfields.

It is useful to understand how the different pieces of finance fit together. This lecture note describes the major subfields of finance and explains (briefly) how they relate to each other.

## 1.1 Market Efficiency

Probably the first conjecture of finance to be systematically studied was the idea of market efficiency. Market efficiency is the notion that market prices are set conditional on all available information. In the old market efficiency studies, people assumed that to earn returns higher than the market return on average, or to “beat the market,” was impossible. As asset pricing theory has developed, people have come to believe that “while you can make excess returns, you cannot make excess utility,” or that you may be able to beat the market return, but you cannot do so without taking on more than market risk and hence sacrificing some utility.

The intuition behind market efficiency is that stock picking and market timing are both competitive industries. Therefore, their economic profit in the long run (by the competitive industry paradigm of economics) should be zero. If markets are not efficient, there must be profit opportunities out there. The market efficiency literature is mostly an empirical literature, but whether markets are efficient or not does have some theoretical implications. For the most part, we assume that markets are, in some sense, efficient.

## 1.2 Asset pricing

The second oldest subfield in finance is known as asset pricing. Asset pricing is the field that asks how the expected rates of return on various assets are determined. The answer that has emerged from asset pricing theory is that riskier assets must pay higher returns on average to compensate people for holding them. The big issue in asset pricing is how to measure risk properly. Put another way, asset pricing seeks to determine which risks people care about and demand compensation for, and which risks do not matter to people.

Asset pricing began with portfolio theory, the theory that describes the optimal way to combine assets into a portfolio. Markowitz (Journal of Finance, 1952) noticed that when assets are combined into portfolios, the variance of the resulting portfolio return is much smaller than the average of the variances of the returns of the assets in the portfolio. Further, the variance of a portfolio's return depends much more on the covariances of its assets returns with each other than on their individual variances. Portfolio theory explained why people were forming portfolios. The intuition that you “should not put all your eggs in one basket” was around long before Markowitz came up with a mathematical model to describe diversification. A few years after Markowitz's findings became known, several authors simultaneously derived what is known as the Capital Asset Pricing Model, or the CAPM.

As soon as the CAPM was developed, eager Ph.D. students began testing the theory with stock return data. The CAPM has been rejected for various reasons through the years. Nevertheless, it is still the model that many academics and practitioners think of when they think of asset pricing. Since researchers rejected the CAPM, asset pricing theory has largely been concerned with generalizing the model to come up with something more realistic.

The first generalization that asset pricers came up with is called no-arbitrage pricing. Pricing rules can be derived simply by not allowing people to make infinite trading

profits. Since these pricing rules rely on fewer assumptions than the CAPM, they are somewhat more believable. Examples of no-arbitrage models include the APT and the Black-Scholes option pricing model, both of which we will discuss this semester.

Asset pricers have come up with a number of models that vary by the strength of the assumptions that they require and the generality of their implications. Typically, stronger (or more restrictive) assumptions lead to more specific models. Most of the models that theorists have come up have been tested empirically. The long-run goal of all researchers is to determine which models describe the data and which fail. When a particular model fails empirically, we can always relax some of the assumptions underlying the model and move to the next level of generality. Most asset pricing models have failed empirically, so models have become more and more general over time.

### **1.3 Corporate Finance**

Corporate finance is the subfield of finance that deals with the interrelation between investors and firms. It originally dealt with the optimal way to finance a firm and the optimal dividend policy to follow. Modigliani and Miller (1958, *American Economic Review*) started corporate finance by asking whether or not it matters if corporations raise money by issuing bonds rather than selling stocks. They were perhaps the first economists to use a no-arbitrage argument to establish an important theory (asset pricers later borrowed the no-arbitrage idea). The Modigliani-Miller theorem says that it should not matter how firms raise capital, and it should not matter if they pay out large or small dividends. To some extent, all the reasons that Modigliani and Miller might be wrong has preoccupied corporate finance ever since.

Researchers in corporate finance have raised agency problems (the manager of the firm might not act in stockholders' interest), information issues, institutional details (like the legal system and tax effects), and all sorts of other reasons to reject the

Modigliani-Miller theorem.

Corporate finance is related to the other subfields of finance by the firm valuation techniques that it uses. As explained above, the Modigliani-Miller theorem relies on a no-arbitrage argument. The corporate topic of capital budgeting (deciding which investments you should take and which you should avoid) requires making an appropriate valuation of both the firm and the project under consideration. To value firms and projects, you must forecast their cash flows and discount those flows appropriately. The discount rate you use to value a firm (or project) should depend on the firm's risk. Measuring risk properly is the goal of asset pricing.

## 1.4 Microstructure

The newest subfield of finance is called market microstructure. Microstructure is the study of how assets are actually traded. It asks questions about the workings of the stock exchange rather than about how investors feel about risks. It relies heavily on information economics, asking how a market maker sets prices when faced with buyers and sellers that may have superior information. Microstructure theory helps to explain why there is such large variation in stock prices from day to day. It has the potential to answer asset pricing questions as well. For example, it may be the case that lots of the difference in average returns between assets is driven by differences in liquidity, or the ease with which different assets can be traded. Whether microstructure can explain expected returns remains to be seen.

While microstructure has been around for a while, it took off as a field with the theoretical models created by Kyle (*Econometrica*, 1985) and Amihud and Mendelson (1986, *JFE*) There is now a large empirical and theoretical literature on microstructure. It is covered fairly well by Campbell, Lo and MacKinlay.

## 2 General Properties of Asset Pricing Models

We are interested in pricing assets. Assets are either financial securities or actual means of production (e.g. land, machines) that people buy in order to earn a profit. The form of profit that we will analyze is called a return, and is generally defined as the profit earned on an asset divided by the original price of the asset. We want to explain the fact that some assets earn high returns and others earn negative returns. We want to be able to write down an equation that says  $E(r_i) = f(x_i)$ , where  $E$  is the expectations operator,  $r_i$  is the return on asset  $i$ , and  $x_i$  is a vector of characteristics of asset  $i$ . To figure out which asset characteristics should matter and which function of characteristics makes the most sense, we make up asset pricing models.

In this section we describe a generic asset pricing model. It turns out that all asset pricing models can be described as special cases of one single equation. In general, we want a way to describe the expected return for any asset. The equation we discuss in this section describes expected returns.

### 2.1 $1 = E(RM)$

The “mother of all asset pricing models” (my term) is a convenient way to summarize all of asset pricing. We will derive it in a number of ways, with a number of different assumptions. The model is:

$$E(R_i M) = 1, \quad i = 1, 2, \dots, n. \quad (1)$$

where  $E$  stands for expectation,  $R_i$  is the gross return on asset  $i$  (gross returns are equal to net returns plus one,  $R = r + 1$ ), and  $M$  is known as either the **pricing kernel**, the **stochastic discount factor**, or the **intertemporal marginal rate of substitution**. Almost all asset pricing models can be expressed as a special case of  $E(R_i M) = 1$ . Models are distinguished by their specification of  $M$ . Notice that while

$R_i$  is a firm-specific quantity, a single pricing kernel works for all assets. If we assume that there is a risk free asset, then the model can be expressed as:

$$E(R_i^e M) = 0, \quad R^e = R_i - R_f, \quad (2)$$

since equation 1 must hold for the risk free rate,  $R_f$ , as well as  $R_i$ .

Much of the class will be spent motivating various forms for the pricing kernel,  $M$ . For now, think of  $M$  as an aggregate measure of discomfort. In many of the models we will consider,  $M$  is the marginal utility of consumption. In bad states of the world,  $M$  is high;  $M$  is low in good states. Viewing  $M$  this way, think about a security that pays off relatively well in bad states. Such a security will have a high value when  $M$  is high and a low value when  $M$  is low. This is a low risk security, perhaps even an insurance contract of some sort. Now think of a security that pays off well when  $M$  is low, but does poorly when  $M$  is high. This second security is fairly risky. In order for equation 2.1 to hold, the second security will have to have higher payoffs on average than the first security. This is the intuition behind all of asset pricing. Assets that help to smooth consumption are relatively safe and hence do not pay high expected returns.

## 2.2 Alternative Representations

There are several different ways to express the  $E(RM) = 1$  result. Which representation is preferred depends on the application at hand.

### 2.2.1 Factor-Based Models

One common (and traditional) representation is the factor-based model. Demonstrating the link between factor-based models and the pricing kernel model should help you see that the fairly risky security described above must pay higher returns than the low

risk security. You may be familiar with factor-based models like the CAPM or the APT.

Using the fact that

$$\text{COV}(A, B) = E(AB) - E(A)E(B), \quad (3)$$

a simple relation between factor-based models and stochastic discount models can be derived. The fact that

$$E(R_i M) = 1 = \text{COV}(R_i, M) + E(R_i)E(M) \quad (4)$$

implies that

$$E(R_i) = \frac{1}{E(M)} - \frac{\text{COV}(R_i, M)}{E(M)} \quad (5)$$

$$= \frac{1}{E(M)} - \frac{\text{VAR}(M)}{E(M)} \left[ \frac{\text{COV}(R_i, M)}{\text{VAR}(M)} \right] \quad (6)$$

$$= \gamma_0 - \gamma_M \beta_{iM}. \quad (7)$$

Thus, any stochastic discount factor model can be expressed as a factor-based model by realizing that the factor in the factor-based model is the pricing kernel,  $M$ . Now we can verify the intuition of the previous section. An asset with returns that are positively correlated with  $M$  pays relatively low returns.

The risk premium in the factor-based model is equal to the variance of  $M$  divided by the mean of  $M$ , and the intercept equals one over the mean of  $M$ . Interpreting these terms requires some thought about the expected and permissible values of the pricing kernel, which happens to be discussed in section 2.4.

### 2.2.2 State Prices

If we are willing to assume that there are a finite number of possible future states of the world, we can express the pricing kernel in other useful ways. Assume that there are  $S$  possible states of the world, indexed by  $s$ , and that there are  $N$  assets, indexed by  $i$ . Let the  $N \times S$  matrix  $\mathbf{D}$  represent the payoffs of the  $N$  securities in the  $S$  states, and let security prices be given by the  $N$ -vector  $\mathbf{q}$ . Denoting portfolio weights by the  $N$ -vector  $\theta$ , the portfolio  $\theta$  costs  $\mathbf{q} \cdot \theta$  and it pays off  $\mathbf{D}'\theta$ .

Within this framework, a **state-price** vector is a strictly positive  $S$ -vector,  $\psi$ , with  $\mathbf{q} = \mathbf{D}\psi$ . We will interpret state prices,  $\psi_j$ , later. If state prices exist, then the  $1 = E(RM)$  results described in section 2 hold. Remember that all uncertainty in this model is about which state will be revealed to be true next period. If we use the symbol  $\pi_s$  to mean the probability of state  $s$  then the stochastic discount factor result is the same as

$$\sum_{s=1}^S \pi_s M_s R_{is} = 1. \quad (8)$$

The state price relation to prices can be written as

$$q_i = \sum_{s=1}^S D_{is} \psi_s \quad \text{or} \quad \sum_{s=1}^S R_{is} \psi_s = 1. \quad (9)$$

So, all we need to do to get the pricing kernel result from the state-price representation of the model is to define

$$M_s = \frac{\psi_s}{\pi_s}. \quad (10)$$

So the existence of state prices means that there is a pricing kernel that makes the mother of all asset-pricing models fit.



### 2.2.3 Risk-neutral Probabilities

We can perform the same trick that produced a pricing kernel in the previous section with any sort of “probability” that we care to consider. Many results use what is sometimes called “risk-neutral probabilities” which are constructed as

$$\psi_0 = \sum_{s=1}^S \psi_s, \quad (11)$$

$$\hat{\psi}_s = \frac{\psi_s}{\psi_0}. \quad (12)$$

The elements of  $\hat{\psi}$  can be considered probabilities because they sum to one. With these risk-neutral “probabilities”, the price of any security is the discounted expected payoff with these “probabilities,”

$$q_i = \psi_0 \hat{E}(D_i) = \psi_0 \sum_{s=1}^S \hat{\psi}_s D_{is}. \quad (13)$$

Note that this construction requires that  $\psi_0 = \frac{1}{R_f}$ .

Sometimes in dynamic contexts people refer to a “change of measure” or “equivalent martingale measure” result that employs probabilities like these. We will discuss these results in more detail later.

Each of these representations (the pricing kernel model, the factor-based model, state prices and risk-neutral probabilities) is essentially equivalent, and each will appear as an implication that we derive from some asset pricing model.

## 2.3 When Does $1 = E(RM)$ Hold?

Why is  $1 = E(RM)$  such a convenient model? Because we can expect it to hold under very general circumstances. Supposing again that all the uncertainty about the economy can be summarized by a discrete set of  $S$  states, some sort of pricing kernel

will always exist as long as the problem

$$\mathbf{q} = \mathbf{D}\eta, \tag{14}$$

or equivalently,

$$\mathbf{R}\eta = \mathbf{1}_S \tag{15}$$

has a solution. Again, the  $N \times S$  matrices  $\mathbf{R}$  and  $\mathbf{D}$  represent the gross returns and the payoffs of the  $N$  securities in the  $S$  states, respectively. Within this framework, we can argue that some sort of pricing kernel will always exist.

First of all, consider the case in which  $S = N$  and  $\mathbf{D}$  is of full rank. In this case, we say that markets are complete. In a world with complete markets, we can construct **Arrow-Debreu** securities or state contingent claims. A state contingent claim pays off one unit of consumption if a particular state is realized, and it pays zero otherwise. It is easy to create Arrow-Debreu securities from ordinary securities simply by inverting the payoff matrix. To create Arrow-Debreu securities, we need an  $S \times S$  matrix of weights,  $\mathbf{W}$ , such that

$$\mathbf{W}\mathbf{D} = \mathbf{I}. \tag{16}$$

Of course, the matrix  $\mathbf{W} = \mathbf{D}^{-1}$  fits this description.

Knowing this, we can see now why the state-price vector has its name. If we apply the weight matrix that we just found to the definition of a state-price vector given in section 2.2.2,

$$\mathbf{q} = \mathbf{D}\psi \Rightarrow \mathbf{D}^{-1}\mathbf{q} = \psi, \tag{17}$$

we see that the cost of one unit of payoff in state  $j$  is just  $\psi_j$ . Furthermore, the price of any security is just the sum of its payoffs in particular states times the cost of a unit of payoff in each state. Agents can use state contingent claims to insure against all future states of the world, so they should smooth consumption perfectly in complete market worlds.

What happens when  $N > S$  and  $\mathbf{D}$  is of full rank? Then we have what are called redundant securities. If we assume that a very weak form of no-arbitrage holds then redundant securities will be priced correctly by a pricing kernel in complete markets.

**The Law of One Price.** *Two bundles with exactly the same characteristics (e.g. portfolios with equal payoffs in all states) have to sell for the same price.*

In linear algebra terms, the  $N - S$  rows of redundant asset payoffs have to be linear combinations of the first  $S$  rows of  $\mathbf{D}$ . In other words, redundant securities are basically portfolios of state contingent claims. As such, they must be correctly priced by state prices for the law of one price to hold.

Now consider the case where  $N < S$ . In this case, we say that markets are not complete. People cannot insure against all possible future states because a complete set of contingent claims is not available. When this is the case, relying on simple linear algebra results, we know that there are an infinite number of solutions to problem 15.

Thus, under fairly general circumstances, we know that a pricing kernel exists. This is not a very interesting result, however, since we have no way of identifying pricing kernels and there is not a unique pricing kernel. We need to know more about the pricing kernel's properties in order to get anything useful out of the  $1 = E(RM)$  type of model. We can get more specific properties for  $M$  by making stronger assumptions to set up our model.

## 2.4 Empirical Restrictions on $M$

Without any additional assumptions, we can derive some empirical restrictions on the pricing kernel,  $M$ . First of all, since  $E(R_i M) = 1$  holds for the risk-free asset, it is true that

$$E(M) = \frac{1}{R_f}. \tag{18}$$

This implies that the intercept in the factor-based model implied by  $E(R_i M) = 1$  is the risk-free rate. This is consistent with the CAPM and other factor-based models,

and it satisfies the requirement that  $\psi_0 = \frac{1}{R_f}$  implied by the risk-neutral probability representation. Equation 18 also implies that the risk premium term,  $\gamma_M = \frac{\text{VAR}(M)}{E(M)}$ , is positive. Since the risk premium is positive, it must be the case that riskier securities have negative betas on  $M$  in the factor-based model. This is consistent with the description of  $M$  in section 2.1. Assets with payoffs that are positively correlated with  $M$  could actually have negative expected returns (e.g. insurance contracts).

Second, we can make statements about the variability of  $M$ . We will use the fact that

$$\text{COV}(A, B) = \rho(A, B)\sigma(A)\sigma(B), \quad (19)$$

where  $\rho(A, B)$  is the correlation coefficient between  $A$  and  $B$ , and  $\sigma(j)$  is the standard deviation of  $j$ . Applying this fact to the excess returns version of our stochastic discount factor model,

$$0 = E(MR_i^e) = E(M)E(R_i^e) + \text{COV}(M, R_i^e) \quad (20)$$

implies that

$$\sigma(M) = \frac{-E(M)E(R_i^e)}{\rho(M, R_i^e)\sigma(R_i^e)}. \quad (21)$$

Since  $|\rho| \leq 1$ ,

$$\sigma(M) \geq \frac{E(M)E(R_i^e)}{\sigma(R_i^e)}. \quad (22)$$

Furthermore, since  $E(m) = \frac{1}{R_f}$ ,

$$\sigma(M) \geq \frac{1}{R_f} \left[ \frac{E(R_i^e)}{\sigma(R_i^e)} \right]. \quad (23)$$

Notice that the term in brackets is security  $i$ 's **Sharpe Ratio**. This relation is used to construct the Hansen-Jaganathan bound, a region of permissible values for the moments of  $M$ .<sup>1</sup> The HJ bound is often used by empiricists to examine whether

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<sup>1</sup>Hansen, L. P., and R. Jaganathan, 1991, "Implications of security market data for models of dynamic economies," *Journal of Political Economy*, 99, 225-261.

particular models for  $M$  are reasonable.

## 2.5 Conditional Models

The models discussed above have all involved unconditional expectations. Many of the models we will consider will be conditional models, taking the form

$$E_t(R_{i,t+1}M_{t+1}) = 1, \tag{24}$$

where  $E_t$  means “the expected value conditional on all information available at time  $t$ .” Conditional expectations are defined carefully in Duffie (pg. 224) and elsewhere.

In general, conditional and unconditional models have different empirical implications. The conditional model above can be written as a factor-based model:

$$E_t(r_{t+1}) = r_f + \gamma_t\beta_t \tag{25}$$

where the  $i$  and  $M$  subscripts have been dropped to avoid confusion, and the net return,  $r$ , replaces the gross return,  $R$ . The  $t$  subscripts that appear here denote conditional moments. So, for example, the  $\beta_t$  in this model consists of a conditional covariance divided by a conditional variance. The  $t$  subscripts mean that both betas and risk premiums can change with time and information.

People often convert conditional models to unconditional models using something called the **law of iterated expectations**. The law of iterated expectations states that if the set of possible events  $\mathbf{B}$  is a subset of the set of possible events  $\mathbf{A}$  ( $\mathbf{B}$  represents more information than  $\mathbf{A}$ ) then  $E[E(x|\mathbf{B})|\mathbf{A}] = E(x|\mathbf{A})$ . Applying this law to our conditional model, the expected value of  $E_t(r)$  is just  $E(r)$ . Applying it to the right hand side is more complicated -  $\gamma_t$  and  $\beta_t$  are both random variables at time  $t$ . The

unconditional model turns out to be,

$$E(r_{t+1}) = r_f + E(\gamma_t)E(\beta_t) + \text{COV}(\gamma_t, \beta_t). \quad (26)$$

## 2.6 Homework Problems

1. Write the CAPM as a stochastic discount factor model. The CAPM is usually written as

$$E(r_i) = r_f + \gamma\beta_i, \quad (27)$$

where  $\beta_i$  is security  $i$ 's beta on the market portfolio and  $\gamma$  is some positive risk premium. Verify that you can get a relation for expected returns in the same form as (27) by using the stochastic discount factor,

$$M = a + bR_m, \quad (28)$$

where  $R_m$  is the gross return on the market portfolio. You probably want to start with,

$$0 = E(R_i^e M) \Rightarrow? \quad (29)$$

2. Show mathematically that if the law of one price holds, all redundant assets are priced by the same pricing kernel as the primitive assets. Hint: Use the discrete-state notation and partition the payoff matrix into primitive and redundant security payoffs.
3. If the law of one price holds, when will there be a unique pricing kernel? (markets not complete, exactly complete, or more than complete?) Carefully explain why.

### 3 The Fundamental Theorem of Asset Pricing

In this section we will discuss what is sometimes called the fundamental theorem of asset pricing. Much of the material in this section is from Duffie (pp. 3-13). Duffie's proof of the central no-arbitrage result uses a nasty construction called the separating hyperplane theorem. Ingersoll (pp. 52-58) derives similar results with some theorems from linear programming. Varian (pp. 383-385) also uses linear programming results. The article by Dybvig and Ross from the *New Palgrave* is another useful reference.

Why is no-arbitrage an attractive principle to start with?

#### 3.1 Notation

As assumed above, there are  $S$  possible states of the world, indexed by  $s$ . There are also  $N$  assets, indexed by  $i$ . The  $N \times S$  matrix  $\mathbf{D}$  represent the payoffs of the  $N$  securities in the  $S$  states. Security prices are given by the  $N$ -vector  $\mathbf{q}$ , and portfolio weights are given by the  $N$ -vector  $\theta$ . The portfolio  $\theta$  costs  $\mathbf{q} \cdot \theta$  and it pays off  $\mathbf{D}'\theta$ . A **state-price** vector is a strictly positive  $S$ -vector,  $\psi$ , with  $\mathbf{q} = \mathbf{D}\psi$ . A state price,  $\psi_j$ , is basically the marginal cost of a unit of payoff in state  $j$ .

#### 3.2 The Theorem

An **arbitrage** is a portfolio that satisfies one of two criteria:

**Arbitrage of first type.**  $\mathbf{q} \cdot \theta = 0$ , and  $\mathbf{D}'\theta > 0$ , where the  $>$  sign means that at least one element of  $\mathbf{D}'\theta$  is positive.

**Arbitrage of second type.**  $\mathbf{q} \cdot \theta < 0$ , and  $\mathbf{D}'\theta \geq 0$ , where the  $<$  sign means that the portfolio must have a negative price.

What do these arbitrage conditions mean?

**The Fundamental Theorem of Asset Pricing.** *The following three conditions are equivalent:*

- *The absence of arbitrage*
- *The existence of a positive linear pricing rule (state prices)*
- *The existence of an optimal portfolio for some agent who prefers more to less.*

This is such an important result that we need to take the time to prove it. First, we will prove that the absence of arbitrage is equivalent to the existence of state prices. Before delving into the proof, however, we need to know the separating hyperplane theorem.

### 3.2.1 Separating Hyperplane Theorem

We need to use a theorem known as the separating hyperplane theorem to prove our result. We will not prove the theorem, but we will discuss the intuition behind it. The theorem is discussed in Duffie (pages 275-277) and in references cited by Duffie.

What is a hyperplane? Let's think first about a "regular" plane. The equation of a 3 dimensional plane is  $F(\mathbf{x}) = a + b_1x_1 + b_2x_2$ . We can draw a regular plane:



The equation of an  $n$ -dimensional hyperplane is  $F(\mathbf{x}) = a + \sum_{i=1}^n b_i x_i$ . It is difficult to draw hyperplanes with  $n > 2$ . Notice that the equation for a hyperplane is a “linear functional.”

**Separating Hyperplane Theorem.** *Suppose that  $A$  and  $B$  are convex disjoint subsets of  $\mathfrak{R}^n$ . There is some nonzero linear functional  $F$  such that  $F(\mathbf{x}) < F(\mathbf{y})$  for each  $\mathbf{x}$  in  $A$  and  $\mathbf{y}$  in  $B$ . Moreover, if  $\mathbf{x}$  is in the interior of  $A$  or  $\mathbf{y}$  is in the interior of  $B$ , then  $F(\mathbf{x}) < F(\mathbf{y})$ .*

So what does the theorem mean? Draw a picture:

We need a special version of the separating hyperplane theorem for our result. This version holds for cones, which are spaces that include all the points described by  $\lambda \mathbf{x}$ ,  $\forall \lambda > 0$  for each  $\mathbf{x}$  in the set. What does a cone look like? Some examples:

**Linear Separation of Cones.** *Suppose  $M$  and  $K$  are closed convex cones in  $\mathfrak{R}^n$  that intersect precisely at zero. if  $K$  is not a linear subspace,<sup>2</sup> then there is a nonzero linear functional  $F$  such that  $F(\mathbf{x}) < F(\mathbf{y})$  for each  $\mathbf{x}$  in  $M$  and nonzero  $\mathbf{y}$  in  $K$ .*

### 3.2.2 Proof

Now that we have the separating hyperplane theorem, we can prove the first part of the fundamental theorem stated above. Let the sets

$$\begin{aligned} L &= \mathfrak{R} \times \mathfrak{R}^S, \\ M &= \{(-q \cdot \theta, \mathbf{D}'\theta) : \theta \in \mathfrak{R}^N\} \subset L, \\ K &= \mathfrak{R}_+ \times \mathfrak{R}_+^S \end{aligned} \tag{30}$$

Note that both  $K$  and  $M$  are closed, convex subsets of  $L$ . They are also both cones.  $M$  is a linear space, but  $K$  is not linear since it does not contain any negative elements. No arbitrage means that  $K$  and  $M$  intersect precisely at zero. Why? Notice first that

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<sup>2</sup>A nonempty set  $L$  of elements  $x, y, z, \dots$  is said to be a **linear space** (or **vector space**) if it satisfies the following three axioms:

1. Any two elements  $x, y \in L$  uniquely determine a third element  $x + y \in L$ , called the sum of  $x$  and  $y$ , such that
  - $x + y = y + x$  (commutativity);
  - $(x + y) + z = x + (y + z)$  (associativity);
  - There exists an element  $0 \in L$ , called the zero element, with the property that  $x + 0 = x$  for every  $x \in L$ ;
  - For every  $x \in L$  there exists an element  $-x$ , called the negative of  $x$ , with the property that  $x + (-x) = 0$ .
2. Any number  $\alpha$  and any element  $x \in L$  uniquely determine an element  $\alpha x \in L$ , called the product of  $\alpha$  and  $x$ , such that  $\alpha(\beta x) = (\alpha\beta)x$  and  $1x = x$ .
3. The operations of addition and multiplication obey two distributive laws,  $(\alpha + \beta)x = \alpha x + \beta x$ , and  $\alpha(x + y) = \alpha x + \alpha y$ .

$M$  must contain  $\{0\}$  to avoid arbitrage. Now consider Duffie's pictures:

Suppose  $K \cap M = \{0\}$ . The separating hyperplane theorem says that a linear functional exists,  $F : L \rightarrow \Re$  such that  $F(z) < F(x)$  for all  $z \in M$  and nonzero  $x \in K$ . Since  $M$  is a linear space,

$$\begin{aligned} F(z) &= 0, \quad \forall z \in M, \\ F(x) &> 0 \quad \forall x \neq 0, \in K \end{aligned} \tag{31}$$

Since  $M$  is a linear space,  $F(z)$  must equal zero for all  $z \in M$ . Suppose, for example, that  $F(z^*) < 0$  for some  $z^* \in M$ . Since the element  $\lambda z^* \in M$  exists for any scalar  $\lambda$ , we could find  $F(\lambda z^*)$ , of any arbitrary value, contradicting the separating hyperplane theorem.

The fact that  $F(x) > 0$  for all nonzero  $x \in K$  implies that  $F$  is represented by some  $\alpha > 0 \in \Re$  and a strictly positive  $\psi \in \Re^S$  by  $F(v, c) = \alpha v + \psi \cdot c$  for any  $(v, c) \in L$ . This implies that  $-\alpha q \cdot \theta + \psi \cdot (D'\theta) = 0 \quad \forall \theta \in \Re^N$ . Thus, the vector  $\frac{\psi}{\alpha}$  is a state price vector.

Conversely, if a state price vector exists, then there is clearly no arbitrage. Why not? Suppose that  $\eta$  is an arbitrage opportunity. This means that the  $(S + 1)$ -vector  $\eta'[-q : \mathbf{D}]$  is strictly nonnegative, with at least one element greater than zero.

Using our no-arbitrage result,

$$q = \mathbf{D}\psi, \tag{32}$$

we know that,

$$0 = \eta'[-q : \mathbf{D}][1 : \psi]. \tag{33}$$

This last statement is a contradiction since we assumed that  $\eta$  was an arbitrage and  $\psi$  is strictly positive.

### 3.3 The Existence of an Optimal Portfolio

Why do we care about the second part of the fundamental theorem? The second part of the theorem emphasizes why we are so concerned about arbitrage in the first place. We will show that if there are arbitrage opportunities, there is no optimal portfolio for agents to hold. When there are arbitrage opportunities, agents cannot maximize expected utility because they will always prefer to take unbounded positions in order to take advantage of the arbitrage opportunity. Thus, stating that the absence of arbitrage is equivalent to an optimum existing is just another way to say that arbitrage opportunities cannot exist in equilibrium.

We will not prove the second part of the theorem exactly as it is stated. Rather, we will prove a slightly less general but much more tractable version that appears in Duffie. This version requires assuming the existence of an agent with an increasing utility function over consumption,  $U(c)$ , and a random endowment,  $e$ . This agent's problem can be stated as:

$$\begin{aligned} & \text{Max } U(c) \\ & \text{Subject to: } c \cdot e + \mathbf{D}'\theta, \quad q'\theta = 0, \quad c \geq 0 \text{ in all states,} \end{aligned} \tag{34}$$

where the utility function  $U(c)$  is defined on the vector of consumptions in the  $S$  possible states. We also need to assume that at least one asset exists that has a positive payoff

and a positive price. Now we are ready to state the simpler version that we will prove.

**Existence of an Optimum in the Absence of Arbitrage.** *If there is a solution to (34) then there is no arbitrage. If  $U(\cdot)$  is continuous and there is no arbitrage, then there is a solution to (34).*

You should prove this result as an exercise.

### 3.4 Homework Problems

1. Suppose that there are two assets in a two state economy. Let the payoff matrix be:

$$\begin{array}{cc} & S_1 & S_2 \\ \begin{array}{l} P_1 \\ P_2 \end{array} & \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, & \end{array} \quad (35)$$

where  $S_1$  and  $S_2$  are the possible states of the world and  $P_1$  and  $P_2$  are possible payoffs. Let the probabilities of the two states be 0.2 for state 1 and 0.8 for state 2. let the security prices be 0.6 and 1.0 for assets 1 and 2 respectively. What does the risk-free rate of return have to be to avoid arbitrage in this economy? Calculate state prices, the value of the pricing kernel in each state, and the risk-neutral probabilities that correspond to this example (assuming the no-arbitrage risk-free rate).

2. What is the difference between the law of one price and the absence of arbitrage? Think of an example that violates one but not the other.
3. Prove the second part of the fundamental theorem of asset pricing (stated in section 3.3). You may want to (but don't have to) use the results listed below in your proof:

**The Theorem of the Maximum.** *Let the set  $D \in \mathfrak{R}^N$  be closed and bounded (closed + bounded = compact), and let  $f : D \rightarrow \mathfrak{R}$  be continuous. Then  $f$  assumes its*

maximum (and its minimum) values at some point  $x_0$  (and  $x_1$ ) of  $D$ .

**Definition of Bounded.** A set  $D \subset \mathfrak{R}^N$  is said to be bounded if there is a number,  $Q > 0$ , such that  $\|x\| < Q \quad \forall x \in D$ . A set is **closed** if it contains all its boundary points.

**Contrapositive Argument.** For any statements  $p$  and  $q$ ,  $(p \Rightarrow q) \Leftrightarrow (\text{not } q \Rightarrow \text{not } p)$ .

## 4 The APT

The no-arbitrage results that we just derived are very general. They are so general that they are very difficult to use. There is a notion among economists that "you get what you pay for" in terms of assumptions and results. We made very weak assumptions in the previous section, and we got fairly weak results. What would an MBA say to you if you told her "there is some positive pricing kernel out there but I have no idea what it is?"

We will turn now to the APT. The APT is an older model (published by Ross in 1976) that has been used quite a bit by both economists and practitioners. It makes slightly stronger assumptions and gets, as a result, stronger predictions. Like the no-arbitrage results of the last section, the APT does not rely on much economic argument. It only assumes that returns are driven by a linear factor model and that there are no "asymptotic arbitrage opportunities."

The stuff we will talk about today is covered in chapter 7 of Ingersoll.

### 4.1 Notation and Assumptions

Suppose that returns are driven by a set of factors,  $f_1, f_2, \dots, f_k$ , such that

$$r_{it} = \alpha_i + \beta_{i1}f_{1t} + \dots + \beta_{ik}f_{kt} + \varepsilon_{it}, \quad (36)$$

or, in vector notation,

$$\mathbf{r} = \alpha + \beta\mathbf{f} + \varepsilon. \quad (37)$$

Notice that this looks very much like a regression equation. The APT is loosely based on multivariate regression analysis. Notice also that the factors are macroeconomic aggregates rather than firm-specific characteristics. This is what is called a "factor model." The APT assumes that returns are generated by a factor model and then it derives the expected returns relation that follows from that assumption. Figuring out

what the APT really means is a little bit tricky. At first glance, it seems that we are assuming a factor model and then doing a lot of math to arrive at the same factor model. The APT assumes that returns are generated by a factor model and then it shows that, with no arbitrage, each asset's expected return is a linear function of the asset's "sensitivities" to the factors or its "factor loadings."

To make the derivation of the APT simple we need to make some assumptions about the factor model. Let

$$\begin{aligned}
 E(\varepsilon) &= \mathbf{0}, \\
 E(\mathbf{f}) &= \mathbf{0}, \\
 E(\mathbf{f}\mathbf{f}') &= \mathbf{I}, \\
 E(\varepsilon\mathbf{f}') &= \mathbf{0}, \\
 E(\varepsilon\varepsilon') &= \mathbf{G}
 \end{aligned} \tag{38}$$

where  $\mathbf{G}$  is a diagonal matrix with bounded diagonal elements,  $\{G_{ii}\} \equiv s_i^2 < S^2$ . These assumptions are mostly innocuous. The only assumptions with teeth are that a factor structure describes the returns generating process and that  $\mathbf{G}$  is diagonal.

## 4.2 Ideas in APT

The APT is derived by combining two separate ideas. The first idea is the law of large numbers, an asymptotic statistical concept.

**The Law of Large Numbers.** *Let  $z_i$  represent a sequence of iid. random variables with finite expectation,  $\mu_z$ . Then for any  $\epsilon > 0$*

$$\lim_{n \rightarrow \infty} \text{Prob} \left( \left| \frac{1}{n} \sum_{i=1}^n z_i - \mu_z \right| < \epsilon \right) = 1, \tag{39}$$

*or, equivalently,*

$$\text{plim} \left( \frac{1}{n} \sum_{i=1}^n z_i \right) = \mu_z. \tag{40}$$



You may have run across the law of large numbers in an econometrics class before - it is frequently used to prove consistency, for example. Here we will use it to argue that if we have enough assets in our portfolios, we can create portfolios that have no residual variance. In other words, we can use the LLN to make our portfolios have returns that look like

$$\mathbf{r} = \alpha + \beta \mathbf{f}, \quad (41)$$

which is just like the form that we assumed above except that it does not contain  $\varepsilon$ , the error term.

The second idea in the APT is that in the absence of arbitrage opportunities, securities that satisfy the exact factor structure outlined above must have expected returns that are linear in  $\beta$ . To see this, it is useful to think of the one factor case. Suppose that there is only one factor and we can find two different well diversified portfolios that are both sensitive to this factor,

$$\begin{aligned} r_1 &= \alpha_1 + \beta_1 f, \\ r_2 &= \alpha_2 + \beta_2 f \end{aligned} \quad (42)$$

By our assumptions,  $E(r_1) = \alpha_1$ , and likewise for asset 2. We will invest  $w$  in portfolio 1 and  $1 - w$  in portfolio 2. This gives us a return of

$$r_p = w\alpha_1 + (1 - w)\alpha_2 + [w\beta_1 + (1 - w)\beta_2]f. \quad (43)$$

We can construct a risk-free portfolio by setting the term in brackets to zero. We choose

$$w^* = \frac{\beta_2}{\beta_2 - \beta_1}, \quad (44)$$

and we know that the resulting portfolio must have a return equal to the risk-free rate since it is riskless. Thus,

$$w^* \alpha_1 + (1 - w^*) \alpha_2 = r_f, \quad (45)$$

or

$$\left(\frac{\beta_2}{\beta_2 - \beta_1}\right)(\alpha_1 - \alpha_2) = r_f - \alpha_2, \quad (46)$$

or

$$\frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2} = \frac{\alpha_2 - r_f}{\beta_2}. \quad (47)$$

By symmetry, the same thing holds if we interchange the 1 and 2 subscripts. Notice that

$$\frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2} = \frac{\alpha_2 - \alpha_1}{\beta_2 - \beta_1}, \quad (48)$$

so that

$$\frac{\alpha_2 - r_f}{\beta_2} = \frac{\alpha_1 - r_f}{\beta_1}. \quad (49)$$

Finally, this implies that

$$\alpha_i = r_f + \lambda\beta_i = E(r_i). \quad (50)$$

The APT math holds for any number of securities, of course. Suppose we try it with two factors and three assets, so that

$$\begin{aligned} r_1 &= \alpha_1 + \beta_{11}f_1 + \beta_{12}f_2, \\ r_2 &= \alpha_2 + \beta_{21}f_1 + \beta_{22}f_2, \\ r_3 &= \alpha_3 + \beta_{31}f_1 + \beta_{32}f_2. \end{aligned} \quad (51)$$

Now if we find weights such that  $\sum_{i=1}^3 w_i\beta_{i1} = 0$  and  $\sum_{i=1}^3 w_i\beta_{i2} = 0$  then we will have formed a riskless portfolio again. This portfolio will have to satisfy  $\sum_{i=1}^3 w_i\alpha_i = r_f$ . In matrix form,

$$\begin{pmatrix} \alpha_1 - r_f & \alpha_2 - r_f & \alpha_3 - r_f \\ \beta_{11} & \beta_{21} & \beta_{31} \\ \beta_{12} & \beta_{22} & \beta_{32} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (52)$$

The vector of weights cannot equal a zero vector because it must sum to one. The matrix in this problem must be singular (that's from a matrix algebra theorem), so

the first row must be a linear combination of the last two rows, which we assume are linearly independent. Thus,

$$E(r_i) = \alpha_i = r_f + \lambda_1\beta_{i1} + \lambda_2\beta_{i2} \quad (53)$$

in this case, and

$$E(r_i) = r_f + \sum_{j=1}^k \lambda_j\beta_{ij} \quad (54)$$

in the more general case of  $k$  factors.

### 4.3 Derivation

To derive the APT, we need to define an “asymptotic arbitrage.” We will then assume that no asymptotic arbitrage exists and derive the implication of that assumption.

**Asymptotic Arbitrage.** *An asymptotic arbitrage is a sequence of portfolios,  $\theta^n = \{\{\theta_1^n, \theta_2^n\}, \{\theta_1^n, \theta_2^n, \theta_3^n\}, \dots, \{\theta_1^n, \theta_2^n, \dots, \theta_n^n\}\}$ , that satisfy:*

$$\begin{aligned} \sum_{i=1}^n \theta_i^n &= 0, \\ \sum_{i=1}^n \theta_i^n E(r_i) &\geq \delta > 0, \\ \sum_{i=1}^n \sum_{j=1}^n \theta_i^n \theta_j^n \sigma_{ij} &\rightarrow 0. \end{aligned} \quad (55)$$

Why should no asymptotic arbitrage exist? How could you exploit such an arbitrage?

**The Arbitrage Pricing Theory (APT).** *If returns are generated by the factor model defined in (37) and (38) and there are no asymptotic arbitrage opportunities, then there exists a linear pricing model that gives expected returns with a mean squared error of zero,*

$$\alpha_i - \lambda_0 - \sum_{j=1}^k \lambda_j\beta_{ij} = \nu_i \quad (56)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \nu_i^2 = 0. \quad (57)$$

Think of a linear projection (or a least squares regression) of the vector  $\alpha$  onto the space spanned by the matrix  $\beta$  and a vector of ones. We could express the result of this “regression” as

$$\alpha_i = \lambda_0 + \sum_{j=1}^k \lambda_j \beta_{ij} + \nu_i \quad (58)$$

Since the error terms in regressions are orthogonal to the regressors, the conditions:

$$\sum_{i=1}^n \nu_i = 0, \quad (59)$$

$$\sum_{i=1}^n \nu_i \beta_{ik} = 0, \quad \forall k, \quad (60)$$

must hold. Now consider the arbitrage portfolio (a portfolio with zero investment),

$$\theta_i = \frac{\nu_i}{\sqrt{n \sum_{i=1}^n \nu_i^2}} = \psi \nu_i, \quad (61)$$

where  $\psi = \frac{1}{\sqrt{n \sum_{i=1}^n \nu_i^2}}$  for notational convenience. The profit on this portfolio is

$$\psi \sum_{i=1}^n \nu_i r_i = \psi \sum_{i=1}^n \nu_i [\alpha_i + \sum_{j=1}^k \beta_{ij} f_j + \varepsilon_i] = \psi \sum_{i=1}^n \nu_i (\alpha_i + \varepsilon_i), \quad (62)$$

while the expected profit is,

$$\psi \sum_{i=1}^n \nu_i \alpha_i = \psi \left[ \lambda_0 \sum_{i=1}^n \nu_i + \sum_{j=1}^k \lambda_j \sum_{i=1}^n \nu_i \beta_{ij} + \sum_{i=1}^n \nu_i^2 \right] = \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^n \nu_i^2}. \quad (63)$$

The variance of the profit is

$$\frac{\sum_{i=1}^n \nu_i^2 s_i^2}{n \sum_{i=1}^n \nu_i^2} \cdot \frac{S^2}{n} \quad (64)$$

Now suppose that the APT theorem above is false. If it is false then the expected profit of this arbitrage portfolio is non-zero while the variance goes to zero with  $n$ . This would be an arbitrage opportunity, so it can't exist in equilibrium. Therefore, the expected profit term must also vanish, meaning that  $\frac{\sum_{i=1}^n \nu_i^2}{n} \rightarrow 0$  and the theorem holds. This concludes our proof of the APT theorem.

There is quite a large literature about the APT. People have worried about whether or not the APT is testable, how precisely it is likely to fit, and other issues. We will not review all of these developments here. You can read about them in Ingersoll if you want to.

## 4.4 Connections

The APT can be made a little more transparent with the tools that we have developed in previous sections. For example, consider the result of applying our pricing kernel result to the APT equations without error,

$$\begin{aligned}
 \text{APT : } r_{it} &= \alpha_i + \beta_{i1}f_{1t} + \dots + \beta_{ik}f_{kt}, \\
 E[M(1 + r_{it})] &= 1 = E[M](1 + \alpha_i) + \beta_{i1}E[Mf_{1t}] + \dots + \beta_{ik}E[Mf_{kt}], \\
 1 + r_f &= (1 + \alpha_i) + \beta_{i1}E[Mf_{1t}](1 + r_f) + \dots + \beta_{ik}E[Mf_{kt}](1 + r_f), \\
 \alpha_i &= r_f - \beta_{i1}E[Mf_{1t}](1 + r_f) - \dots - \beta_{ik}E[Mf_{kt}](1 + r_f), \\
 \alpha_i &= r_f + \lambda_1\beta_{i1} + \dots + \lambda_k\beta_{ik}
 \end{aligned} \tag{65}$$

where  $\lambda_j = -E[Mf_{jt}](1 + r_f)$ ,  $j = 1, 2, \dots, k$ . Thus, using our previously derived arbitrage results, we can prove the APT result much more easily.

It is also useful to me to think of what the APT assumptions mean in terms of the finite state world that we have assumed in previous sections. The APT assumptions are basically equivalent to the condition that the  $N \times S$  matrix of gross returns in

various states,  $\mathbf{R}$  can be decomposed as follows:

$$\mathbf{R} = \sum_{j=1}^k \beta_j f_j' + \epsilon \quad (66)$$

where  $\beta_j$  is an  $N$ -vector of security betas on factor  $j$ ,  $f_j$  is an  $S$ -vector that contains factor  $j$ 's realizations in each state, and  $\epsilon$  has the property that  $\epsilon' \epsilon$  is a diagonal matrix. The outer product terms above,  $\beta_j f_j'$ , each consist of an  $N \times S$  matrix of rank 1. Thus, the middle term of equation (66) is (at most) of rank  $k$ .

This representation illustrates the truly important assumption of the APT. If  $k$  is reasonably small, the APT assumes that the almost all returns can be described by a returns matrix of low rank. According to the APT, all assets have returns that are essentially linear combinations of a few important factors. But there is really nothing special about the way that the APT decomposes the matrix  $\mathbf{R}$ . So if a simpler way to decompose  $\mathbf{R}$  is available, it may be possible to develop a more parsimonious APT. There is a literature on a “nonlinear” APT that attempts to exploit this fact.<sup>3</sup>

## 4.5 Homework Problems

1. Suppose you found that an APT model with the market return and an unexpected inflation factor fits expected returns quite well:

$$\begin{aligned} r_{it} &= \alpha_i + \beta_{im} r_{mt} + \beta_{i\tilde{l}} \tilde{l}_t + \varepsilon_{it} \\ E[r_{it}] &= \alpha_i = r_f + \lambda_m \beta_{im} + \lambda_{\tilde{l}} \beta_{i\tilde{l}}. \end{aligned} \quad (67)$$

Suppose, furthermore, that you found  $\lambda_{\tilde{l}} = -0.4$ . What does the fact that  $\lambda_{\tilde{l}} < 0$  mean? What sign do you expect  $\lambda_m$  to have?

2. What is the stochastic discount factor implied by the APT model if the APT has

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<sup>3</sup>Bansal, R. and S. Viswanathan, 1993, No Arbitrage and Arbitrage Pricing: A New Approach, *Journal of Finance*, XLVIII, 1231-1262.

only two factors? You should be able to express the stochastic discount factor as a function of the two factor realizations and the parameters of the APT model.

## 5 Expected Utility Theory

We have been talking about arbitrage models in discrete time. Now we are going to begin talking about utility-based models in discrete time. In this section of the notes, we review some results from the economics of uncertainty. We are going to say that people maximize expected utility subject to budget constraints. This material is covered in several places, including Varian's chapter 11. It should be review for most of you so we will cover it fairly quickly.

### 5.1 Expected Utility

A consumer's expected utility function is not a primitive in economics. We make assumptions about an agent's preferences in order to derive an expected utility function for him or her. We define expected utility over the space of lotteries,  $\mathcal{L}$ . Using Varian's notation,  $(p \circ x \oplus (1-p) \circ y)$  means receiving  $x$  with probability  $p$  and  $y$  with probability  $(1-p)$ . The operator  $\sim$  implies indifference while  $\succeq$  implies weak preference. We assume:

1. Getting a prize with probability = 1 is the same as getting the prize for certain.

$$(1 \circ x \oplus (1-1) \circ y \sim x)$$

2. The consumer doesn't care about the order in which the lottery is described.

$$(p \circ x \oplus (1-p) \circ y \sim (1-p) \circ y \oplus p \circ x)$$

3. A consumer's perception of a lottery depends only on the net probabilities in the lottery, not on how the lottery is packaged.  $(q \circ (p \circ x \oplus (1-p) \circ y) \oplus (1-q) \circ y \sim$

$$(qp) \circ x \oplus (1-qp) \circ y)$$

4. Consumers' preferences over lotteries are:

- complete, (either  $x \succeq y$  or  $y \succeq x$  or both  $\forall x, y$ )



- reflexive, ( $x \succeq x \forall x$ )
  - and transitive. (if  $x \succeq y$  and  $y \succeq z$  then  $x \succeq z$ )
5. Preferences are continuous. ( $\{p \in [0, 1] : p \circ x \oplus (1 - p) \circ y \succeq z\}$  and  $\{p \in [0, 1] : z \succeq p \circ x \oplus (1 - p) \circ y\}$  are closed sets for all  $x, y,$  and  $z$  in  $\mathcal{L}$ .)
6. If people are indifferent about two goods they will be indifferent about lotteries over those goods. ( $x \sim y \Rightarrow p \circ x \oplus (1 - p) \circ z \sim p \circ y \oplus (1 - p) \circ z$ )

**Existence of Expected Utility Function.** *If  $(\mathcal{L}, \succeq)$  satisfy the above axioms then there is a utility function,  $u$  that ranks lotteries according to preferences,*

$$\begin{aligned} p \circ x \oplus (1 - p) \circ y \succ q \circ w \oplus (1 - q) \circ z &\Leftrightarrow \\ u(p \circ x \oplus (1 - p) \circ y) &> u(q \circ w \oplus (1 - q) \circ z), \end{aligned} \tag{68}$$

*and satisfies the expected utility property*

$$u(p \circ x \oplus (1 - p) \circ y) = pu(x) + (1 - p)u(y). \tag{69}$$

You can read the proof of the theorem in Varian or in other references. It can also be shown that expected utility functions are unique up to an affine (or linear) transformation. These properties make expected utility maximization an extremely useful way to think about people's behavior under uncertainty.

Why do we worry about the existence of an expected utility function? Some of the attractiveness of expected utility maximization is driven by its mathematical tractability. We don't want our models to be determined by tractability alone - we want them to reflect reality as well. Several economists have proposed alternatives to the expected utility paradigm. Mark Machina has made a career out of his alternative to expected utility. He drops some of the assumptions we made above and finds something like

local expected utility maximization. Kahnemann and Tversky, two psychologists, have also been trying to replace expected utility with something that is more consistent with behavior. Whenever economic models fail, it is possible that people are simply not maximizing expected utility functions like we want them to. For this reason, many economists are more comfortable assuming that there is no arbitrage in the market than assuming that all agents are maximizing expected utility somehow.

## 5.2 Risk aversion

What is it about expected utility that makes it so useful for finance? Besides assuming that people are maximizing expected utility functions, we usually assume that their utilities make them *risk averse*. A risk averse person would rather take a certain amount of money than take a gamble with an expected payoff that is slightly larger than the certain amount. People can also be risk neutral or risk loving, of course.

It turns out that people with concave utility functions are risk averse. This result is expressed with an oft-used inequality

**Jensen's Inequality.** *If  $f(x)$  is a strictly concave function (like a risk-averse utility function) then  $E[f(x)] < f(E[x])$ .*

Again, you can see the proof of this result in Varian. The intuition behind this result is what comes out of the diagram that is usually explained in intermediate economics classes.

To say more about risk aversion, we are going to have to define a risk premium. Suppose we were thinking about a random consumption bundle  $\tilde{x} = \bar{x} + \tilde{\varepsilon}$ , where  $\bar{x}$  is a constant and  $\tilde{\varepsilon}$  is a random variable with an expected value of zero. For now, a risk premium is defined as the value of  $\rho$  that makes true the statement:

$$E[u(\tilde{x})] = u(\bar{x} - \rho). \quad (70)$$

Now for a particular realization of  $\tilde{\varepsilon}$ , we can use a Taylor series expansion<sup>4</sup> to argue that

$$u(\bar{x} + \varepsilon) \approx u(\bar{x}) + \varepsilon u'(\bar{x}) + \frac{\varepsilon^2}{2} u''(\bar{x}). \quad (71)$$

Therefore,

$$E[u(\tilde{x})] \approx u(\bar{x}) + \frac{\sigma_\varepsilon^2}{2} u''(\bar{x}). \quad (72)$$

Furthermore, if  $\sigma_\varepsilon^2$  is “small” then  $\rho$  is also small, so using a Taylor expansion again,

$$u(\bar{x} - \rho) \approx u(\bar{x}) - \rho u'(\bar{x}) \quad (73)$$

which means that we can express our risk premium as

$$\rho \approx -\frac{\sigma_\varepsilon^2}{2} \frac{u''(\bar{x})}{u'(\bar{x})} = \frac{\sigma_\varepsilon^2}{2} R_A, \quad (74)$$

where  $R_A$  is known as the absolute risk aversion coefficient. The absolute risk aversion coefficient is a nice way to measure risk aversion. People with higher coefficients are more risk averse than people with lower coefficients.

If you replace the additive error term that we assumed above with a multiplicative

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<sup>4</sup>  $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^k \frac{(x-t)^k}{k!} f^{(k+1)}(t) dt.$

error term, the measure of risk aversion that results is known as relative risk aversion

$$R_R = \bar{x}R_A = -\frac{\bar{x}u''(\bar{x})}{u'(\bar{x})}. \quad (75)$$

You can find a derivation for relative risk aversion in Varian or elsewhere. Next we are going to state (but not prove) an important theorem.

**Pratt's theorem.** *Given 2 utility functions,  $u^1$  and  $u^2$ , that are twice differentiable, strictly concave and increasing, the following are equivalent:*

1.  $R_A^1(\bar{x}) \geq R_A^2(\bar{x})$
2.  $\rho^1(\bar{x}, \varepsilon) \geq \rho^2(\bar{x}, \varepsilon)$
3.  $u^1$  is more concave than  $u^2$ .

Proofs of this theorem can be found in lots of places, including Varian. Pratt's theorem tells us three different but equivalent ways to determine if one person is more risk averse than another.

### 5.3 Utility Functions

There are several utility functions that are used very frequently by economists. We will discuss three of them here. The first type of utility function we will discuss is what is known as the constant relative risk aversion (CRRA) or power utility function. It is parameterized as:

$$u(x) = x^\alpha, \quad \alpha \in (0, 1). \quad (76)$$

As the exponent of a particular version of the power utility function goes to zero, it becomes the log utility function,

$$\lim_{\alpha \rightarrow 0} \frac{x^\alpha - 1}{\alpha} = \log(x), \quad (77)$$

where the logarithm in the function is a natural log. This family of utility functions is called CRRA because its coefficient of absolute risk aversion is

$$R_A = (1 - \alpha)x^{-1}, \quad (78)$$

giving it a relative risk aversion that is constant.

A second family of utility functions that is commonly used in research is the constant absolute risk aversion family (CARA). This family is parameterized

$$u(x) = 1 - e^{-\lambda x}, \quad \lambda > 0. \quad (79)$$

The absolute risk aversion coefficient for this utility function is just  $\lambda$ .

The last type of utility function we will discuss is the quadratic utility function. This function is written

$$u(x) = a + bx - cx^2, \quad b, c > 0. \quad (80)$$

You can calculate the risk aversion coefficients for this utility function as a homework assignment. A quadratic utility function looks like an inverted parabola. There is always a point at which marginal utility,  $u'(x)$ , becomes negative. You get to solve for this as a homework problem as well.

## 5.4 Stochastic Dominance

We have talked about ways to determine whether one person is more or less risk averse than another person. Now we will shift our emphasis to asking whether a particular lottery is more or less risky than another lottery. Probably the most general way to compare the risk of lotteries is in terms of what is called stochastic dominance.

**First Order Stochastic Dominance.** *The cumulative distribution of payoffs  $F$  first*

order stochastic dominates (FOSD)  $G$  iff  $G(x) \geq F(x) \forall x \in I$ , where  $I$  is the sample space of  $x$ .

First order stochastic dominance is an attractive property because it has been shown that, for all increasing utility functions,  $u(x)$ ,

$$F \text{ FOSD } G \Leftrightarrow E_F[u(x)] \geq E_G[u(x)], \quad (81)$$

where  $E_F$  is the expectation taken under the assumption that  $F$  is the distribution of payoffs. To interpret FOSD, remember that  $G(x) = \Pr(\tilde{x} \cdot x)$  and draw a picture:

A weaker concept than FOSD is second order stochastic dominance (SOSD). To define SOSD, we need to define the function

$$T(x) = \int_I [G(x) - F(x)] dx \quad (82)$$

**Second Order Stochastic Dominance.**  $F$  SOSD  $G$  iff  $T(x) \geq 0 \forall x \in I$  and  $E_G[x] = E_F[x]$ .

Second order stochastic dominance is a weaker concept than FOSD in the sense that FOSD implies SOSD but SOSD does not imply FOSD. For all increasing and

strictly concave utility functions,

$$F \text{ SOSD } G \Leftrightarrow E_F[u(x)] \geq E_G[u(x)]. \quad (83)$$

Since SOSD is a weaker concept than FOSD, we need the additional condition that  $u(x)$  is concave to get the result that people should prefer payoffs that second order dominate.

Second order stochastic dominance is an attractive property to work with because it corresponds to a frequently used abstraction known as a mean preserving spread. Economists often add a mean preserving spread to their models in order to introduce uncertainty. They are sometimes described as a “sprinkling” of risk. If we define the random variable  $\tilde{y}$  as  $\tilde{x}$  plus a mean preserving spread then

$$\tilde{y} = \tilde{x} + \tilde{v}, \quad (84)$$

where

$$\begin{aligned} E[\tilde{v}] &= 0 \\ E[\tilde{v}|x] &= 0 \\ \text{Var}[\tilde{v}] &> 0 \end{aligned} \quad (85)$$

In this case, the distribution of  $\tilde{x}$  will second order stochastic dominate the distribution of  $\tilde{y}$ . A plot of a mean preserving spread can be instructive:

A useful theorem for interpreting SOSD is the Rothschild-Stiglitz theorem.

**Rothschild-Stiglitz Theorem.** *The following conditions are equivalent:*

1.  $F$  SOSD  $G$ .
2.  $G = F$  plus noise (mean preserving spread)
3.  $F$  and  $G$  have the same mean and all risk averters prefer  $F$  to  $G$ .



## 5.5 Homework Problems

1. Solve for the absolute risk aversion coefficient and the relative risk aversion coefficient for the quadratic utility function, (80). Solve also for the point at which utility is maximized if there is no constraint on consumption,  $x$ . Assuming that there exists some maximum possible value of  $x$ , what assumption could you make to rule out satiated consumers.
2. (Number 11.6 from Varian) Esperanza has been an expected utility maximizer ever since she was five years old. As a result of the strict education she received at an obscure British boarding school, her utility function  $u$  is strictly increasing, strictly concave, twice differentiable and bounded. Now, at the age of thirty-something, Esperanza is evaluating an asset with stochastic outcome  $R$  which is normally distributed with mean  $\mu$  and variance  $\sigma^2$ . Thus, its density function is given by

$$f(R) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{R-\mu}{\sigma}\right)^2\right]. \quad (86)$$

- Show that Esperanza's expected utility from  $R$  is a function of  $\mu$  and  $\sigma^2$  alone. Thus, show that  $E[u(R)] = \phi(\mu, \sigma^2)$ .
- Show that  $\phi(\cdot)$  is increasing in  $\mu$ .
- Show that  $\phi(\cdot)$  is decreasing in  $\sigma^2$ .

Hint: define a new variable  $\epsilon = (R - \mu)/\sigma$ , write down the expression for  $E[u(\epsilon)]$  and sign the derivatives - you may need integration by parts.

3. (Number 11.7 from Varian) Let  $R_1$  and  $R_2$  be the random returns on two assets. Assume that  $R_1$  and  $R_2$  are independently and identically distributed. Show that an expected utility maximizer will divide her wealth between both assets provided she is risk averse; and invest all her wealth in one of the assets if she's risk loving.

## 6 The CAPM

In this section we begin to look at models that have a little more economic content than pure arbitrage models. The first model we explore is the Capital Asset Pricing Model (CAPM). The CAPM is of mostly historical interest - it is used by practitioners and it is generally taught to MBA students, but very few active researchers still believe that it describes expected returns well. The CAPM is developed under fairly strong assumptions, meaning that it should only hold in specific situations. Understanding the CAPM does help to develop intuition for more complicated asset-pricing models.

We will derive the CAPM with a risk-free asset. There is a version of the CAPM that does not use a risk-free asset, but it is probably not worth our time in class to derive all the possible CAPM relations.

### 6.1 Assumptions and Notation

#### Assumptions:

1. A Representative Agent Exists

- common time horizon

- homogeneous beliefs

2. Mean-Variance Analysis is Optimal

- normally distributed returns

- quadratic utility

3. Perfect Markets

- perfect competition

- unlimited short sales and margin positions

- no transactions costs

assets are perfectly divisible

4. One Period World

5. There are  $n$  Risky Assets, 1 Risk-Free Asset, No Other Assets.

**Notation:**

$\bar{r}^e$  =  $n$ -vector of expected **excess** returns on risky assets ( $\bar{r} - r_f$ )

$\bar{r}_p^e$  = the (scalar) expected excess portfolio return desired

$r_f$  = the risk-free rate

$w$  =  $n$ -vector of portfolio weights (risky assets)

$w_0$  = the weight on the risk-free asset

$\Sigma$  =  $n \times n$  variance-covariance matrix of returns on risky assets

$\mathbf{1}$  =  $n$ -vector of ones

## 6.2 Mean-Variance Math

The mean-variance optimization problem can be stated as:

$$\text{minimize } \frac{1}{2}\sigma_p^2 = \frac{1}{2}w'\Sigma w \tag{87}$$

$$\text{subject to: } w'\bar{r}^e = \bar{r}_p^e. \tag{88}$$

What does this problem mean?

Note that  $\Sigma$  is positive definite (as are all well-specified variance-covariance matrices), so the objective function is convex and a first order condition will be necessary and sufficient for optimization. Forming a lagrangian function,

$$L = \frac{w'\Sigma w}{2} - \lambda(w'\bar{r}^e - \bar{r}_p^e) \tag{89}$$

The first order conditions are:

$$\frac{\partial L}{\partial w} = \Sigma w - \lambda \bar{r}^e = \mathbf{0}, \quad (90)$$

$$\frac{\partial L}{\partial \lambda} = w' \bar{r}^e - \bar{r}_p^e = 0, \quad (91)$$

By multiplying through by  $\Sigma^{-1}$ , (90) can be written as

$$w = \lambda \Sigma^{-1} \bar{r}^e. \quad (92)$$

The value of the lagrangian multiplier,  $\lambda$ , can be found by substituting the expressions for  $w$  into (91),

$$\lambda = \frac{\bar{r}_p^e}{\bar{r}^{e'} \Sigma^{-1} \bar{r}^e}. \quad (93)$$

If we define:

$$\begin{aligned} A &= \mathbf{1}' \Sigma^{-1} \mathbf{1}, \\ B &= \mathbf{1}' \Sigma^{-1} \bar{r} = \bar{r}' \Sigma^{-1} \mathbf{1}, \\ C &= \bar{r}' \Sigma^{-1} \bar{r}, \end{aligned} \quad (94)$$

then we can express,

$$\lambda = \frac{\bar{r}_p^e}{(\bar{r}' - r_f \mathbf{1}') \Sigma^{-1} (\bar{r} - r_f \mathbf{1})} = \frac{\bar{r}_p^e}{[C - 2r_f B + r_f^2 A]}. \quad (95)$$

Now we can solve for the optimal weights by substituting the value of  $\lambda$  into (92).

$$w = \frac{\bar{r}_p^e}{[C - 2r_f B + r_f^2 A]} \Sigma^{-1} \bar{r}^e, \quad w_0 = 1 - \sum_{i=1}^N w_i. \quad (96)$$

Any set of portfolio weights that satisfy (96) are mean-variance efficient. Substituting

(92) into the definition for  $\sigma_p^2$  yields

$$\begin{aligned}\sigma_p^2 &= w' \Sigma (\lambda \Sigma^{-1} \bar{r}^e) \\ &= \lambda w' \bar{r}^e \\ &= \lambda \bar{r}_p^e\end{aligned}\tag{97}$$

$$\sigma_p^2 = \frac{\bar{r}_p^{e2}}{[C - 2r_f B + r_f^2 A]},\tag{98}$$

which is the equation of a parabola. In mean-standard deviation space, this becomes

$$\sigma_p = \frac{|\bar{r}_p - r_f|}{\sqrt{C - 2r_f B + r_f^2 A}}.\tag{99}$$

To plot this, we would draw two rays extending from  $r_f$  to the right. The rays would have slopes of positive and negative  $\sqrt{C - 2r_f B + r_f^2 A}$ . If we were to solve the same problem without a risk-free asset, the corresponding plot would be a hyperbola. These rays and this hyperbola are the shapes commonly depicted in the familiar “minimum-variance set” pictures.

We can identify the “tangency” portfolio by finding the minimum variance portfolio

for which  $\sum_{i=1}^N w_i = 1$ . Equation (96) says that the optimal weights should be a constant multiplied by  $\Sigma^{-1}\bar{r}^e$ . If we sum up the elements of  $\Sigma^{-1}\bar{r}^e$ , they come to  $B - r_f A$ , so the tangency portfolio has weights,

$$w_t = \frac{\Sigma^{-1}\bar{r}^e}{B - r_f A}. \quad (100)$$

The tangency portfolio has a mean return equal to

$$\bar{r}_t = \frac{C - r_f B}{B - r_f A}, \quad (101)$$

and a variance equal to

$$\sigma_t^2 = \frac{C - 2r_f B + r_f^2 A}{(B - r_f A)^2}. \quad (102)$$

We can construct any portfolio on the minimum variance frontier with the tangency portfolio and the risk-free asset. Actually, we can use any two portfolios to create all other portfolios on the frontier, but it is intuitive to think of using the tangency portfolio and the risk-free asset.

### 6.3 Equilibrium Conditions

We have shown that mean-variance optimization leads the investor to choose from a set of minimum variance portfolios that are described by equation (96). Now we need to make some equilibrium arguments to finish our derivation of the CAPM.

Given our assumptions, we can state that investors will only want to hold minimum variance portfolios. Since all minimum variance portfolios can be generated with two minimum variance portfolios, the holdings of all investors can be generated by combining the weights of just two portfolios. This is an example of a mutual fund theorem. There are several mutual fund theorems in Ingersoll's chapter 6 if you are interested in them. Since all investors want to hold just two funds, we need to identify two minimum

variance funds to let them hold. The easy fund to identify is just the risk-free asset. The other fund can be identified by noting that the risk-free asset is in zero net supply. This means that the aggregate value for  $w_0$  is zero. Thus, the aggregate value for  $w' \mathbf{1}$  is one. The other minimum variance portfolio we will use is the tangency portfolio defined above. Since the tangency portfolio contains all risky assets in proportion to their market weights, it is usually called the market portfolio.

We can define a vector of covariances with the market portfolio as

$$\sigma_{im} = \Sigma w_m = \frac{\bar{r}^e}{(B - r_f A)}. \quad (103)$$

By using the relation

$$\sigma_m^2 = w'_m \sigma_{im} = w'_m \Sigma w_m = \frac{\bar{r}_m^e}{(B - r_f A)}, \quad \text{or} \quad (B - r_f A) = \frac{\bar{r}_m^e}{\sigma_m^2}, \quad (104)$$

we can derive the CAPM relation

$$\bar{r}^e = \beta_{im} \bar{r}_m^e, \quad \beta_{im} = \frac{\sigma_{im}}{\sigma_m^2}. \quad (105)$$

This relation can, of course, also be written in a more familiar form,

$$E(r_i) = r_f + \beta_{im} [E(r_m) - r_f], \quad (106)$$

which is the equation most often associated with the CAPM.

Notice, however, that any mean-variance efficient portfolio's weights can be expressed as

$$w_\theta = c \Sigma^{-1} \bar{r}^e. \quad (107)$$

Thus, the covariance of each asset with this mean-variance efficient portfolio is

$$\sigma_{i\theta} = \Sigma w_{\theta} = \frac{\bar{r}^e}{(B - r_f A)} \quad (108)$$

By now, this equation looks a lot like equation (103). It should not be surprising that we can derive an equation just like (106) for any mean-variance efficient portfolio. The observation that for any mean-variance efficient portfolio there is a linear relation between expected returns and covariances with the portfolio is a tautology. Furthermore, there will always be a mean-variance efficient portfolio that can be identified after the fact. The only real significance in the CAPM is the idea that the market portfolio must be mean-variance efficient. Put another way, the only important implication of the CAPM is that the pricing kernel should be a linear function of the market return and no other factors. Since the “true” market portfolio return is never observed, tests of the CAPM are quite difficult to perform. This idea is known as Roll’s critique because Richard Roll was the first person to point this out.

The CAPM makes some very strong assumptions in order to conclude that

1. All investors hold just the market portfolio and the risk-free asset.
2. The market portfolio is mean-variance efficient.
3. Expected returns are proportional to market betas.

Other asset-pricing models are similar in spirit to the CAPM, so it is worthwhile knowing how it works.

We have already seen how the CAPM fits into the  $1 = E(RM)$  framework above, so we won’t return to that problem.



## 6.4 Homework Problems

1. Suppose that all of the assumptions underlying both the APT and the CAPM hold. Suppose, in particular, that returns are generated by a  $k$ -factor model and that equation (106) describes expected returns. Suppose also (for simplicity) that there are no error terms in the factor models that describe the generation of returns. What can you say about the risk premia (the  $\lambda$ s) in the APT?
2. (Huang and Litzenberger, problem 3.1) In a CAPM world, let there be two securities with (random) returns  $r_i$  and  $r_j$ . Suppose that these securities have identical expected rates of return and identical variances. The correlation coefficient between  $r_i$  and  $r_j$  is  $\rho$ . Show that an equally-weighted portfolio of assets  $i$  and  $j$  achieves the minimum possible variance regardless of the value of  $\rho$ .
3. (Huang and Litzenberger, problem 3.3) Let  $p$  be a mean-variance efficient portfolio and let  $q$  be any portfolio having the same expected return. Show that  $\text{COV}(r_p, r_q) = \text{VAR}(r_p)$  and, as a consequence, the correlation coefficient of  $r_p$  and  $r_q$  lies in  $(0, 1]$ .

## 7 Representative Agent Theorems

Most of the economic models that we will apply can be considered representative agent models. A representative agent model is a model in which all agents act in such a manner that their cumulative actions might as well be the actions of one agent maximizing its expected utility function. Economists construct representative agents in order to deal with the complicated issue of aggregation. It is relatively simple to model the behavior of one person given some preferences and constraints. It is more difficult to model the behavior of a group of people, or an entire economy. Aggregation is what economists call the summing up of individuals' behavior to derive the behavior of a market or an economy. Models that aggregate individuals without using a representative agent device are sometimes called heterogeneity models.

The representative agent only exists under certain circumstances, and those circumstances are the subject of this section. Most of the material in this section comes from Huang and Litzenberger (1987), chapter 5.

### 7.1 Complete Markets

The biggest assumption involved in creating a representative agent is that markets are complete. A complete market is a market in which there are at least as many assets with linearly independent payoffs as there are states. Returning to the notation used in section 3, we will think again of an  $N \times S$  matrix  $\mathbf{D}$  that represent the payoffs of  $N$  securities in  $S$  states. A complete market in this notation is characterized by the condition that the rank of  $\mathbf{D}$  is  $S$ . This requires that  $N \geq S$ . It also means that we can assume that  $\mathbf{D}$  is square. If there are more securities than states then at least  $S - N$  of the securities must have payoffs that are linear combinations of other securities. As long as the law of one price holds, these redundant securities can be ignored.

Some of the implications of complete markets are discussed in section 2. For example, in complete markets worlds, we can create Arrow-Debreu securities and every agent

can use these securities to smooth consumption. When markets are not complete, they can sometimes be made complete by the use of derivatives. Derivatives generally have payoffs that are nonlinear functions of their underlying assets, so adding derivatives to the set of assets available can increase the rank of  $\mathbf{D}$ .

## 7.2 Pareto Optimality

In order to see how to construct a representative agent, we are going to need a slightly more general asset pricing model than the CAPM. We will consider a consumption-based one period model. Throughout this section we will assume that there is one perishable consumption good that serves as our numeraire. All the uncertainty in our model is about which state will be revealed at the end of period one. Agents can consume at time zero and at time one, but for their consumptions to be feasible, it must be true that

$$\sum_{i=1}^I c_{i0} = C_0, \quad (109)$$

and

$$\sum_{i=1}^I c_{i\omega} = C_\omega \quad \forall \omega \in \Omega, \quad (110)$$

where there are  $I$  individuals indexed by  $i$ ,  $C_0$  denotes aggregate time zero consumption available, and  $C_\omega$  means the total amount of consumption possible in state  $\omega$ . A set of state contingent consumption allocations is *pareto optimal* if it is feasible and if there do not exist other feasible allocations that can strictly increase the utility of one individual without decreasing the utilities of others.

There is a well know result that is associated with the second welfare theorem of economics (described in Varian, pp. 329-335) that states that corresponding to every Pareto optimal allocation, there exist a set of non-negative numbers,  $\{\lambda_i\}_{i=1}^I$ , such that the same allocation can be achieved by a social planner maximizing a linear combination of individuals' utility functions using  $\{\lambda_i\}_{i=1}^I$  as weights. The social planner solves the

problem

$$\max \sum_{i=1}^I \lambda_i \left[ \sum_{\omega \in \Omega} \pi_{i\omega} u_{i\omega}(c_{i0}, c_{i\omega}) \right] \quad (111)$$

subject to the constraints (109) and (110) listed above, where  $\pi_{i\omega}$  denotes individual  $i$ 's subjective probability of state  $\omega$  occurring and  $u_{i\omega}(c_{i0}, c_{i\omega})$  is individual  $i$ 's utility over  $c_{i0}$  and  $c_{i\omega}$ .

What is the intuition behind this result? We will show that  $\lambda_i$  can be interpreted as the reciprocal of the marginal utility of income of agent  $i$ . Agents that have relatively large incomes therefore have more weight in the maximization than agents with low incomes. Thus, how the problem turns out depends on initial endowments. Since we have assumed that utility is strictly increasing, the weights are strictly positive,  $\lambda_i > 0$ ,  $i = 1, 2, \dots, I$ .

Forming the Lagrangian for the social planner, we obtain

$$\max_{c_{i0}, c_{i\omega}} L = \sum_{i=1}^I \lambda_i \left[ \sum_{\omega \in \Omega} \pi_{i\omega} u_{i\omega}(c_{i0}, c_{i\omega}) \right] + \phi_0 \left[ C_0 - \sum_{i=1}^I c_{i0} \right] + \sum_{\omega \in \Omega} \phi_\omega \left[ C_\omega - \sum_{i=1}^I c_{i\omega} \right]. \quad (112)$$

Since the utility functions used here are strictly concave and the  $\lambda_i$  are strictly positive, the first order conditions are necessary and sufficient for maximization in this problem.

Those conditions are:

$$\lambda_i \sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i0}} = \phi_0, \quad i = 1, 2, \dots, I \quad (113)$$

$$\lambda_i \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i\omega}} = \phi_\omega, \quad \omega \in \Omega, \quad i = 1, 2, \dots, I \quad (114)$$

plus the constraints, (109) and (110). We can get rid of the weights,  $\lambda_i$ , by examining the ratio of these two conditions:

$$\frac{\pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i\omega}}}{\sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i0}}} = \frac{\phi_\omega}{\phi_0}, \quad \omega \in \Omega, \quad i = 1, 2, \dots, I. \quad (115)$$

From this condition for maximization, it is clear that a feasible allocation of state contingent consumption is Pareto optimal if and only if, for each state, marginal rates of substitution between present consumption and future state contingent consumption are equal across individuals. In other words, a Pareto optimal outcome is one in which all individuals share risk perfectly. This does not mean, of course that each individual has equal consumption or equal utility in all states. Agents with a larger endowment will have higher consumption in all states than agents with a smaller endowment. It just means that everyone's relative unhappiness in a bad state is the same.

Pareto optimal allocations are always possible in competitive economies with complete securities markets. Suppose that markets are complete and that  $\psi_\omega$  is the price of the Arrow-Debreu security that provides one unit of consumption in state  $\omega$ . Then the individual's problem can be stated as:

$$\max_{c_{i0}, c_{i\omega}} \sum_{\omega \in \Omega} \pi_{i\omega} u_{i\omega}(c_{i0}, c_{i\omega}) \quad (116)$$

$$\text{s.t. } c_{i0} + \sum_{\omega \in \Omega} \psi_\omega c_{i\omega} = e_{i0} + \sum_{\omega \in \Omega} \psi_\omega e_{i\omega}, \quad (117)$$

where  $e_{i0}$  and  $e_{i\omega}$  represent agent  $i$ 's endowment in period zero and state  $\omega$ . We assume that these endowments are such that wealth is strictly positive at time zero. The Lagrangian for each agent is

$$\max_{c_{i0}, c_{i\omega}} L = \sum_{\omega \in \Omega} \pi_{i\omega} u_{i\omega}(c_{i0}, c_{i\omega}) + \theta_i \left[ e_{i0} - c_{i0} + \sum_{\omega \in \Omega} \psi_\omega e_{i\omega} - \sum_{\omega \in \Omega} \psi_\omega c_{i\omega} \right], \quad (118)$$

which has first order conditions,

$$\sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i0}} = \theta_i, \quad (119)$$

$$\pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i\omega}} = \theta_i \psi_\omega, \quad \omega \in \Omega, \quad (120)$$

plus the budget constraint. Once again, we can get rid of  $\theta_i$  by forming the ratio

$$\frac{\pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i\omega}}}{\sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i0}}} = \psi_\omega, \quad \omega \in \Omega. \quad (121)$$

In a market equilibrium, the feasibility constraints, (109) and (110), are always satisfied. If we set  $\phi_0 = 1$ ,  $\phi_\omega = \psi_\omega$ , and  $\lambda_i = \frac{1}{\theta_i}$  then we can see that the conditions for the optimality of a single agent are equivalent to the conditions for a Pareto optimal allocation discussed above. Conversely, to achieve a Pareto optimal allocation from this competitive economy, the social planner assigns a weight of  $\theta^{-1}$  to each individual.

This derivation reinforces what we already know about the weights,  $\lambda_i$ . The weight of agent  $i$  is the reciprocal of agent  $i$ 's "shadow price of the budget constraint." The shadow price of the budget constraint is set equal to the marginal utility of consumption in period zero here. This all makes intuitive economic sense - the social planner considers people with more at stake (people with a higher initial wealth) more important in the total maximization than people with a smaller market weight.

Where does this derivation break down if markets are not complete? We need state prices to evaluate the agent's endowment. But we have assumed that we have an equilibrium, so there must not be any arbitrage opportunities available in our economy. This means, of course, that a state price vector exists regardless of whether or not markets are complete. The problem is that the state price vector is not unique if markets are not complete. If state prices are not unique then we can sort of think of one representative agent existing for each set of state prices that we assume people use to value their claims. This construction, however, assumes that all agents use the same state prices to value their endowments. If different people use different state price vectors then the whole representative agent idea breaks down.

Duffie has a nice, concise discussion of the representative agent in his first chapter. He says that it is not always necessary for markets to be complete for Pareto optimality, and thus for a representative agent to exist. Complete markets (and other assumptions)

are just sufficient for the representative agent, they are not necessary. However, Duffie also points out that “...it can be shown that, with incomplete markets and under natural assumptions on utility, for almost every endowment, the equilibrium allocation is not Pareto optimal.” Thus, we should not expect to have a representative agent with incomplete markets.

### 7.3 Constructing the Representative Agent

Now we are ready to derive the representative agent result. We assume that markets are complete and that the economy is competitive. We also assume that individuals have homogeneous beliefs and time-additive, state-independent utility functions that are strictly concave, increasing, and differentiable. This means that the conditions for maximization by a single agent, (119) and (120), can be stated as:

$$\frac{\partial u_{i0}(c_{i0})}{\partial c_{i0}} = \theta_i, \quad (122)$$

and

$$\pi_\omega \frac{\partial u_{i1}(c_{i\omega})}{\partial c_{i\omega}} = \theta_i \psi_\omega, \quad \omega \in - , \quad (123)$$

where  $\pi_\omega$  is the subjective probability of state  $\omega \in -$  that all agents agree on.

Let  $\psi_\omega$  represent state prices as defined in section 2 of the notes. Define  $u_0$  and  $u_1$  to be:

$$\begin{aligned} u_0(z) &= \max_{\{z_i\}_{i=1}^I} \sum_{i=1}^I \lambda_i u_{i0}(z_i) \\ \text{s.t. } &\sum_{i=1}^I z_i = z, \end{aligned} \quad (124)$$

and

$$\begin{aligned} u_1(z) &= \max_{\{z_i\}_{i=1}^I} \sum_{i=1}^I \lambda_i u_{i1}(z_i) \\ \text{s.t. } &\sum_{i=1}^I z_i = z, \end{aligned} \quad (125)$$

where  $\lambda_i = \frac{1}{\theta_i}$  is the social planner’s weight for each individual. An immediate conse-

quence of these definitions is that

$$u'_0(C_0) = \sum_{i=1}^I \lambda_i u'_{i0}(c_{i0}) \frac{\partial c_{i0}}{\partial C_0} = \sum_{i=1}^I \lambda_i \theta_i \frac{\partial c_{i0}}{\partial C_0} \quad (126)$$

$$= \sum_{i=1}^I \frac{\partial c_{i0}}{\partial C_0} = 1, \quad (127)$$

and

$$u'_1(C_\omega) = \sum_{i=1}^I \lambda_i u'_{i1}(c_{i\omega}) \frac{\partial c_{i\omega}}{\partial C_\omega} = \sum_{i=1}^I \lambda_i \theta_i \frac{\psi_\omega}{\pi_\omega} \frac{\partial c_{i\omega}}{\partial C_\omega} \quad (128)$$

$$= \frac{\psi_\omega}{\pi_\omega} \sum_{i=1}^I \frac{\partial c_{i\omega}}{\partial C_\omega} = \frac{\psi_\omega}{\pi_\omega}. \quad (129)$$

These results use the constraints (109) and (110) to conclude that

$$\sum_{i=1}^I \frac{\partial c_{i0}}{\partial C_0} = 1 \quad (130)$$

and

$$\sum_{i=1}^I \frac{\partial c_{i\omega}}{\partial C_\omega} = 1. \quad (131)$$

Now think of a representative agent that has endowments of  $C_0$  and  $C_\omega$ ,  $\omega \in -$  in periods one and two, respectively. Let the representative agent's subjective probabilities be  $\pi_\omega$ , and let its utility for period zero and one consumption be  $u_0(C_0)$  and  $u_1(C_\omega)$  respectively. The state prices in this type of economy must be equal to  $\psi_\omega$  for the representative agent to exist.

In this economy, for the markets to clear, the representative agent must not want to trade away from its endowment. Thus, using time zero consumption as the numeraire, the state price for state  $\omega$  must be equal to the representative agent's marginal rate of substitution (MRS) between time zero consumption and time one, state  $\omega$  consumption. This MRS is:

$$\frac{\pi_\omega u'_1(C_\omega)}{u'_0(C_0)}. \quad (132)$$



After substituting (127) and (129) into (132), we know that the representative agent's MRS is equal to

$$\frac{\pi_\omega u'_1(C_\omega)}{u'_0(C_0)} = \frac{\pi_\omega \psi_\omega}{\pi_\omega} = \psi_\omega. \quad (133)$$

Thus, the representative agent exists in this case.

This representative agent, however, never wants to trade. Equilibriums in which this agent is satisfied will never involve trading. This has led many to look for models that say something about the quantity of trade going on.

## 7.4 Other Aggregation Results

The representative agent's utility function in the previous results depended on the initial endowments of individuals through the parameter  $\lambda_i$ . Therefore, the prices in the economy will vary with the initial endowments of agents. For some utility functions, the utility function of the representative agent does not depend on the endowments of the agents in the economy. Such utility functions have what is called the *aggregation property*. Two utility functions that satisfy the aggregation property are the power utility function,

$$u_i(z) = \frac{1}{1-B} (A_i + Bz)^{1-\frac{1}{B}}, \quad (134)$$

and the negative exponential utility function,

$$u_i(z) = -A_i \exp\left(-\frac{z}{A_i}\right). \quad (135)$$

The power function includes log utility,

$$u_i(z) = \ln(A_i + z), \quad (136)$$

for the case when  $B = 1$ .

In the case of negative exponential utility, it can even be shown that if agents have

different time preference parameters and different subjective probabilities than the representative agent's time preference and probabilities are composites of the individuals' parameters. See Huang and Litzenberger for more discussion of these issues.

## 7.5 Thoughts on the Representative Agent

We have shown that under fairly general conditions a representative agent exists. We have not shown, however, that the representative agent is always an interesting construction. It may be that while a representative agent exists in most circumstances, a simpler way to characterize prices is possible. However, most of the economic models that are prominent in Finance use some sort of representative agent formulation.

A few models that don't rely on the representative agent argument have been suggested in recent years. Most of these models rely on some sort of simulation in order to determine prices and things because they are fairly complicated. Heterogeneity models or models with incomplete markets seem like a fairly fertile ground for future study. For examples of heterogeneity models, see Heaton and Lucas (1995),<sup>5</sup> Aiyagari (1993),<sup>6</sup> Telmer (1993),<sup>7</sup> and Constantinides and Duffie (1996).<sup>8</sup>

## 7.6 Homework Problem

1. Show that  $u_0$  and  $u_1$  defined in (124) and (125) are both strictly concave and increasing (Huang and Litzenberger problem 5.1).
2. Provide some intuition for equation (115). In particular, without using any math,

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<sup>5</sup>Heaton, J. and D. Lucas (1995) "The Importance of Investor Heterogeneity and Financial Market Imperfections for the Behavior of Asset Prices," *Carnegie-Rochester Conference Series on Public Policy* 42 1-32.

<sup>6</sup>Aiyagari, S. R. (1993) "Explaining Financial Market Facts: The Importance of Incomplete Markets and Transaction Costs," *Federal Reserve Bank of Minneapolis Quarterly Review*, Winter, 17-31.

<sup>7</sup>Telmer, C. (1993) "Asset-Pricing Puzzles and Incomplete Markets," *Journal of Finance*, XLVIII, 1803-1832.

<sup>8</sup>Constantinides, G., and D. Duffie (1996) "Asset Pricing with Heterogeneous Consumers," *Journal of Political Economy*, 219-240.

explain why agents will want to equalize (across states) marginal rates of substitution between current consumption and future consumption.

3. What kind of asset pricing model does equation (133) imply? To answer, write down the representative agent's pricing model in the form  $E(Rm) = 1$ . Your expression for  $m$  should involve the representative agent's utility function. Is your expression consistent with our previous interpretations of  $m$ ?

## 8 Dynamic Programming

We are going to discuss multiperiod models that are more general than the CAPM, the APT, and the arbitrage results that we have already studied. In order to study these models, we need to understand a technique called dynamic programming. Dynamic programming is used to solve problems that involve optimization over time. It has been used extensively by economists in all fields. Our optimization problem involves choosing consumption through time to maximize expected utility, so we will use dynamic programming to solve it.

You can read Kreps' appendix (distributed in class) to gain some intuition for the dynamic programming technique. Kreps gives several references to more advanced texts on dynamic methods. If you are interested in working with these types of models, you will probably want to consult a more complete text. There are several alternatives to dynamic programming for solving multiperiod problems. For example, there is a technique known as optimal control that is sort of a continuous-time counterpart to dynamic programming.

Dynamic programming problems can always be categorized as either finite or infinite horizon problems. While the techniques for solving these two types of problems are somewhat different, the inferences obtained from the two types are usually the same. For simplicity, we will discuss the finite horizon case first. After understanding finite horizon problems, we will examine the implications of allowing the horizon to go to infinity.

Dynamic programming works well for problems in which agents make their decisions based on just a few variables that we will call state variables. In statistical terms, it works well when just a few state variables are sufficient statistics for predicting the future. It is fairly common in dynamic programming models to assume that all state variables are current values, such as current wealth or current prices. When only current state variables are assumed to matter, we say that the model is a *Markovian*

model. A Markov process is a stochastic process that satisfies,

$$\text{Markov Process : } f(x_t|x_{t-1}, x_{t-2}, \dots) = f(x_t|x_{t-1}). \quad (137)$$

## 8.1 Dynamic Programming with a Finite Horizon

We need to set up some notation before proceeding with our discussion of dynamic programming. Let  $I = \{1, 2, 3, \dots\}$  be the set of possible future states and let  $A$  be a finite set of feasible actions that you can take. Define  $R(i,a)$  as the expected current reward when the state is  $i \in I$  and the action chosen is  $a \in A$ . Define the value function,  $V_n(i)$ , as the maximum attainable sum of expected current and future rewards when  $n$  periods remain and the current state is  $i \in I$ .

In finite horizon problems, we always start with optimization in the last period and then work backwards to get to the present decision. We begin by thinking about what the value function will be when we have just one period left,

$$V_1(i) = \max_{a \in A} R(i, a), . \quad (138)$$

The optimal policy in the last period is to just maximize your reward given the state. Now let  $p_{ij}(a)$  equal the probability that state  $j$  occurs next period given that state  $i$  describes today and that you choose action  $a$ . We can express your value function with two periods left as a function of your final value function,

$$V_2(i) = \max_{a \in A} \{R(i, a) + \sum_j p_{ij} V_1(j)\} \quad (139)$$

If we define  $a_2(i)$  as your optimal policy when the current state is  $i$  and you have two periods to go, then we want to find a function,  $a_2(i)$  that solves (139). We can do this by first finding the optimal policy with one period left,  $a_1(i)$ , by solving (138). Second, we plug our values for  $a_1(i)$  into (139) and in a third step we solve (139). This is what

dynamic programming is all about. We always solve multiperiod problems by thinking about what will be optimal at the end of the problem and then working backwards to determine what is optimal at the beginning of the problem.

Of course, we can generalize this to more than two periods. In general, we take the value function with  $n$  periods remaining to be the maximum of the reward function defined over future states and actions,

$$V_n(i) = \max_{a \in A} R(i_n, i_{n-1}, \dots, i_1, a_n, a_{n-1}, \dots, a_1), \quad (140)$$

where  $a_\tau$  is the action chosen with  $\tau$  periods left and  $i_\tau$  is the state of the world with  $\tau$  periods left. Because problems like this are usually intractable, we often assume that the reward function is additively separable,

$$V_n(i) = \max_{a \in A} \sum_{t=1}^n R_t(i, a), \quad (141)$$

where  $R_t(i, a)$  indicates the reward function at period  $t$ . Additive separability is a fairly strong assumption that may not always be warranted. Some recent work on habit persistence and alternative utility formulations has challenged the assumption that utility is additively separable.<sup>9</sup>

Using a result known as the principle of optimality, the value function can be restated by the *Bellman equation*:

$$V_n(i) = \max_{a \in A} \{R(i, a) + \sum_j p_{ij} V_{n-1}(j)\}. \quad (142)$$

The principle of optimality basically states that *if a particular strategy is optimal for each point in time at that point in time and if an optimal strategy is going to be followed*

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<sup>9</sup>See Epstein, L. and S. Zin (1991) "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis," *Journal of Political Economy*, 99, p. 263-86.

for all future points in time then the particular strategy is optimal. Kreps motivates this with a little math. He says we can easily convert the problem

$$\max_{x,y} f(x, y) \tag{143}$$

into the equivalent problem

$$\max_x [\max_y f(x, y)]. \tag{144}$$

This mathematical operation is essentially what we have done above in converting the value equation, (141), into the Bellman equation, (142). We use the Bellman equation to learn about the optimal policy function,  $a_n(i)$ . Once again, the optimal policy function is a rule that describes the optimal choice of action when the state is  $i$  and there are  $n$  periods left.

There are three principal ways to use Bellman equations:

1. We can sometimes use Bellman equations to obtain explicit analytic solutions for  $V_n(i)$  and  $a_n(i)$ .
2. We often use Bellman equations to characterize the properties of  $V_n(i)$  and  $a_n(i)$ .
3. We can sometimes solve for  $V_n(i)$  and  $a_n(i)$  numerically. We can always do this in principle, but some problems become too large to be tractable.

The Bellman equation is a fundamental building block in a dynamic programming model. Deriving the appropriate Bellman equation is usually the first step in analyzing a dynamic, finite horizon model.

Once we have a Bellman equation, we typically look at the first order conditions that solve the equation's maximization problem. These first order conditions are often rich enough to provide us with the elements of an interesting model. We also use a condition called an envelope condition at times. Envelope conditions are derived by applying the envelope theorem. The envelope theorem can be understood as follows.

Suppose that we want to maximize  $f(x, a)$  over  $x$ . We can think of  $a$  as being a state variable and  $x$  as being a choice variable. For every value of  $a$  in this problem there will be a maximizing value of  $x$ . In what Varian calls “sufficiently regular” cases, we can think of defining a function,  $x(a)$  that gives the optimal  $x$  value for each value of  $a$ . We can also think of the value function in these terms as  $V(a) = f(x(a), a)$ . If we take the derivative of the value function with respect to the state variable, we obtain

$$\frac{\partial V(a)}{\partial a} = \frac{\partial f(x(a), a)}{\partial x} \frac{\partial x(a)}{\partial a} + \frac{\partial f(x(a), a)}{\partial a}. \quad (145)$$

But we know that  $x(a)$  is the value of  $x$  that maximizes  $f$ , so

$$\frac{\partial f(x(a), a)}{\partial x} \frac{\partial x(a)}{\partial a} = 0, \quad (146)$$

and

$$\frac{\partial V(a)}{\partial a} = \frac{\partial f(x(a), a)}{\partial a} \Big|_{x=x(a)}. \quad (147)$$

This is a very simple statement of the envelope theorem. In the dynamic programming context, if we take the derivative of the value function with respect to the state variables and if we hold the choice variables (the actions) at their optimal levels, then we can consider the derivatives of the value function with respect to the choice variables to be equal to zero.

## 8.2 Example: The Gambler’s Problem

Let’s work through a simple example to illustrate the method. Suppose that in each of  $T$  periods a gambler can bet up to his entire wealth. With probability  $p$  the gambler wins and the size of his reward is equal to the size of his bet. With probability  $(1 - p)$  the gambler loses the amount of his bet. The gambler’s objective is to maximize  $E[\ln(\text{final wealth})]$ . Let  $x$  equal the gambler’s current wealth and  $\alpha \in [0, 1]$  equal the



fraction that he chooses to bet this period. So in this problem,  $x$  is the state variable and  $\alpha$  is the action or the choice variable. The gambler's Bellman equation is

$$V_n(x) = \max_{\alpha \in [0,1]} \{pV_{n-1}(x + \alpha x) + (1 - p)V_{n-1}(x - \alpha x)\} \quad (148)$$

This Bellman equation is very simple because there is no current reward. Since the gambler has log utility, we also know that

$$V_0 = \ln(x). \quad (149)$$

We will show that when  $p \cdot \frac{1}{2}$ , the optimal policy function is  $\alpha_n(x) = 0 \forall n, x$ .

With one gamble left, the gambler has the value function,

$$V_1(x) = \max_{\alpha \in [0,1]} \{p \ln(x + \alpha x) + (1 - p) \ln(x - \alpha x)\} \quad (150)$$

We can solve for the first order condition for this problem,

$$\frac{\partial}{\partial \alpha} = \frac{px}{x + \alpha x} - \frac{(1 - p)x}{x - \alpha x} \quad (151)$$

which implies that the optimal value for  $\alpha$  is

$$\alpha = 2p - 1 \quad (152)$$

When  $p \cdot \frac{1}{2}$ , the optimal fraction of wealth to gamble is less than or equal to zero. Thus, for  $p \cdot \frac{1}{2}$ , it is never optimal for the gambler to gamble in the last period.

We can prove that this is true for any  $n$ . We will use a proof of induction, which contains the following steps:

1. Basis step - prove the hypothesis for  $n = 1$ , the starting point.
2. Induction hypothesis step - suppose that the hypothesis is true for  $n - 1$ .

3. Induction step - under the supposition, prove the hypothesis for  $n$ .

If we follow these steps then our proof by induction will be complete.

We showed above that  $V_1(x) = \ln(x)$  because it will never be optimal for the gambler to gamble in the last period. We will now hypothesize that  $V_{n-1}(x) = \ln(x)$ . We need to show that  $V_n(x) = \ln(x)$  under this supposition. This must be true since we can express our value function as,

$$V_n(x) = \max_{\alpha \in [0,1]} \{p \ln(x + \alpha x) + (1 - p) \ln(x - \alpha x)\}, \quad (153)$$

$$= \max_{\alpha \in [0,1]} \ln(x) + \{p \ln(1 + \alpha) + (1 - p) \ln(1 - \alpha)\}, \quad (154)$$

and because the maximum value of the term in brackets is zero. Thus, the value function with  $n$  periods to go is

$$V_n(x) = \ln(x) \quad (155)$$

The problem in every period is the same and our proof by induction is finished.

What about the case when  $p > \frac{1}{2}$ ? When  $p > \frac{1}{2}$  the optimal  $\alpha$  with one period remaining will still be given by  $2p - 1$ . If we substitute this value of  $\alpha$  into the value function with one period left, we obtain

$$V_1(x) = p \ln[x + (2p - 1)x] + (1 - p) \ln[x - (2p - 1)x] \quad (156)$$

$$= \ln[x] + \ln[2] + p \ln[p] + (1 - p) \ln[1 - p] = \ln[x] + C. \quad (157)$$

We will leave as a homework problem the task of deriving the value function in this problem and showing that the optimal policy function is always

$$a_n(x) = 2p - 1 \quad (158)$$

for each period,  $n$ . Note that this policy rule is a stationary rule - it does not change with time. This is a nice property for optimal policy functions to have.

### 8.3 Dynamic Programming with an Infinite Horizon

The models that we will examine all have finite horizons. It is useful, however, to know what happens to dynamic programming when the horizon is not assumed to be finite. There are two big changes that occur when going from finite problems to infinite problems.

First, in finite problems, we can always start at the last period and then work forward to derive our answer. In infinite horizon problems, there is no last period to begin at. Thus, we cannot usually just write down a Bellman equation and derive an optimal policy. Rather, we have to conjecture a form for the value function and an optimal policy rule and then we have to determine whether these conjectures are correct. We usually make conjectures that seem reasonable - people often “tweak” results that others have found in the past. We validate our conjectures by showing that it is not possible to improve upon our policy functions. Something very similar to the induction proof outlined above is implemented for this purpose.

Second, in infinite horizon problems we usually need some sort of convergence result that is commonly referred to as a *transversality condition*. What do transversality conditions look like? They can look something like

$$\lim_{t \rightarrow \infty} E_t \beta^t \frac{\partial V_t(\mathbf{x})}{\partial \mathbf{x}} \mathbf{x} = 0 \quad (159)$$

Intuitively, they can involve restrictions like the restriction that the discounted terminal value of a stock goes to zero as the time to liquidation goes to infinity. If you want to know more about infinite horizon methods you can consult one of the references in Kreps.

## 8.4 Homework Problems

1. Solve the gambler's problem for the case when  $p > \frac{1}{2}$ . Derive the value function at each point in time,  $V_n$ . Show by induction that the optimal policy is always given by (158).
2. Do problem number 1 in Kreps' appendix.

## 9 Consumption-Based Models

In this section we will derive a typical consumption-based asset pricing model in discrete time with dynamic programming. These models have been developed both by macroeconomists and financial economists. Macroeconomists want to understand the behavior of consumption under uncertainty while financial economists want to understand asset returns. Consumption and asset returns turn out to be intricately related to each other. The model that we will examine is similar to that of Lucas.<sup>10</sup> The consumption-based model has motivated lots of empirical research, including the development of GMM.<sup>11</sup>

### 9.1 Assumptions and Notation

To begin, we make several assumptions:

1. There is a representative agent.
2. Markets are perfect - no rationing constraints, transactions costs, etcetera.
3. There exists a (real) riskless asset.
4. Individuals can freely borrow and lend at the risk-free rate.
5. labor income is given exogenously and it is diversifiable.
6. The representative investor has additively separable Von Neumann-Morgenstern utility.
7. The representative investor lives until period  $T$ .

We also need to describe the notation to be used:

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<sup>10</sup>Lucas, R. (1978), "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429-45.

<sup>11</sup>Hansen, L. and K. Singleton (1982), "Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models," *Econometrica*, 50, 1269-1288.

- $u(c_t)$  is the agent's (undiscounted) utility of consumption at period  $t$ .
- $\beta$  is the representative agent's discount factor for future utility.
- $E_t$  is the expectations operator conditional on all information available at  $t$ .
- $A_t$  is the agents total wealth in period  $t$ .
- $y_t$  is the agent's exogenous labor income in period  $t$ .
- $r_{it+1}$  is the return on asset  $i$  from  $t$  to  $t + 1$ . It is not known until  $t + 1$ .
- $r_{0t+1} = r_{ft}$  is the risk-free return from  $t$  to  $t + 1$ . It is known at  $t$ .
- $w_{it}$  is the fraction of the agent's wealth invested in asset  $i$  from  $t$  to  $t + 1$ . It is chosen in period  $t$ .

## 9.2 Derivation

Under these assumptions, the representative investor solves the problem,

$$\max_{c_t, w_{it}} E_0 \sum_{t=0}^T \beta^t u(c_t), \quad (160)$$

$$\text{s.t. } A_{t+1} = (A_t + y_t - c_t) \left[ \sum_{i=0}^n (1 + r_{it+1}) w_{it} \right], \quad (161)$$

$$\sum_{i=0}^n w_{it} = 1. \quad (162)$$

The term in brackets in equation (161) is just the return on the agent's portfolio. The budget constraint just says that wealth next period is equal to the wealth that we start with this period minus what we consume plus what we earn times our portfolio return. The additive utility assumption expressed in (160) is significant. Assuming additive separability ties together the concepts of risk aversion and intertemporal substitution. A big point in the work of Epstein and Zin (cited earlier) is that we may want to be

able to identify risk aversion and intertemporal substitution separately. Using the fact that  $r_{0t+1} = r_{ft}$ , we can combine (161) and (162) to obtain

$$A_{t+1} = (A_t + y_t - c_t) \left[ 1 + r_{ft} + \sum_{i=1}^n (r_{it+1} - r_{ft}) w_{it} \right]. \quad (163)$$

Now the summation term is the excess return on our portfolio of risky assets. We can express the value function in period  $t$  as

$$V_t(A_t) = \max_{c_t, w_{it}} E_t \sum_{s=t}^T \beta^{s-t} u(c_s), \quad \text{s.t. (163)}, \quad (164)$$

or as the bellman equation,

$$V_t(A_t) = \max_{c_t, w_{it}} [u(c_t) + \beta E_t V_{t+1}(A_{t+1})], \quad \text{s.t. (163)}. \quad (165)$$

We can substitute (163) into (165) to get the final objective function:

$$V_t(A_t) = \max_{c_t, w_{it}} \left\{ u(c_t) + \beta E_t V_{t+1} \left( (A_t + y_t - c_t) \left[ 1 + r_{ft} + \sum_{i=1}^n (r_{it+1} - r_{ft}) w_{it} \right] \right) \right\}. \quad (166)$$

Given our objective function, we can derive first order conditions for the representative investor. The first order condition for consumption is,

$$\text{FOC for } c_t : \quad u'(c_t) = \beta E_t \left( V'_{t+1}(A_{t+1}) [1 + r_{ft} + \sum_{i=1}^n (r_{it+1} - r_{ft}) w_{it}] \right), \quad (167)$$

$$t = 1, 2, 3, \dots, T.$$

This condition characterizes the optimal consumption-savings path for our agent. It says that she should always set the marginal utility of consumption today equal to the discounted expected return of her portfolio times the marginal value of wealth next period. This demonstrates another type of consumption smoothing behavior predicted by economic models. In intertemporal models, agents smooth their consumptions across

time. It would probably not be optimal, for example, for our agent to consume 90% of her wealth in period 0 and leave the remaining 10% for the rest of her lifespan. The first order condition for portfolio weights look like,

$$\text{FOC for } w_{it} : E_t[V'_{t+1}(A_{t+1})(r_{it+1} - r_{ft})] = 0, \quad i = 1, 2, 3, \dots, N, \quad t = 1, 2, 3, \dots, T. \quad (168)$$

Besides these first order conditons, we can also use the envelope condition,

$$V'_t(A_t) = \beta E_t \left( V'_{t+1}(A_{t+1}) \left[ 1 + r_{ft} + \sum_{i=1}^n (r_{it+1} - r_{ft}) w_{it} \right] \right). \quad (169)$$

Combining (167) and (169), we obtain

$$V'_t(A_t) = u'(c_t), \quad (170)$$

so the FOC for the portfolio weights, (168), can be written as

$$E_t[u'(c_{t+1})(r_{it+1} - r_{ft})] = 0, \quad i = 1, 2, 3, \dots, N, \quad t = 1, 2, 3, \dots, T. \quad (171)$$

We immediately recognize that (171) is of the form

$$E_t[M_{t+1} r_{it+1}^e] = 0, \quad (172)$$

and we are done. The consumption-based model basically implies that the pricing kernel,  $M_{t+1}$ , is equal to the marginal utility of consumption in the next period,  $u'(c_{t+1})$ .

How do we interpret this model's results? Where are the demand and supply curves? The representative agent in this model is actually setting prices and choosing consumption simultaneously to make markets clear. We could probably derive demand and supply curves for this model, but their forms would be messy. Like most representative agent models, this model predicts no trading.



This asset pricing model provides some intuition about what a model with heterogeneous agents and incomplete markets might look like. The first order conditions of the representative agent are the same conditions that any ordinary consumer would solve in the same circumstances. Thus, we can think of consumers choosing their consumptions to meet the conditions above. If we aggregate consumers in some way without a representative agent, it is not clear what sort of equilibrium we will get.

Models similar to this have been developed with the first order conditions of firms rather than consumers. Production-based asset pricing models basically solve a similar problem in which firms are maximizing the present value of future cash flows with the constraints that output is produced by a production function and capital in period  $t$  is equal to depreciated capital from period  $t - 1$  plus investment. These models basically stipulate that expected stock returns should be set equal to the expected returns on investment.

### 9.3 Homework Problems

1. Derive the consumption based model without assuming the existence of a risk free asset. You will have to add a constraint that we relaxed in class. You may want to give that constraint a lagrange multiplier and proceed.
2. Suppose that the representative agent has quadratic utility. What sort of asset pricing model results (this is a famous model - you may not know the name yet). Now suppose that the representative agent's consumption is a linear function of his return. What sort of asset pricing model results?
3. Suppose that  $\beta = \frac{1}{1+r_{ft}}$  and suppose that you have no labor income. Using all the first order conditions we derived, derive an expression that relates consumption today to future consumption only (you should get portfolio returns to cancel out somehow). What does this expression mean in economic terms?