

Portfolio Rebalancing in General Equilibrium *

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Abstract

This paper develops an overlapping generations model of optimal rebalancing where agents differ in age and risk tolerance. Equilibrium rebalancing is driven by a leverage effect that influences levered and unlevered agents in opposite directions, an aggregate risk tolerance effect that depends on the distribution of wealth, and an intertemporal hedging effect. After a negative macroeconomic shock, relatively risk tolerant investors sell risky assets while more risk averse investors buy them. Owing to interactions of leverage and changing wealth, however, all agents have higher exposure to aggregate risk after a negative macroeconomic shock and lower exposure after a positive shock.

Keywords: household finance, portfolio choice, heterogeneity in risk tolerance and age

JEL codes: E21, E44, G11

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1 Introduction

Understanding how different people trade in response to large market movements, or how they rebalance their portfolios, is central to finance and macroeconomics. Studying rebalancing in a meaningful way requires a model that incorporates investor heterogeneity because if all investors are identical then all will hold the same portfolio and there will be no trade. We study optimal rebalancing behavior with a simple general equilibrium overlapping generations model that features heterogeneity in both age and risk tolerance. The model is focused on asset allocation, with one risky asset in fixed supply and a risk-free asset in zero net supply. We find that both general equilibrium and lifecycle considerations have large effects on optimal rebalancing, and thus should be inherent parts of asset allocation models. Optimal rebalancing is fairly complex, with trading driven by leverage, aggregate risk tolerance, and intertemporal hedging effects.

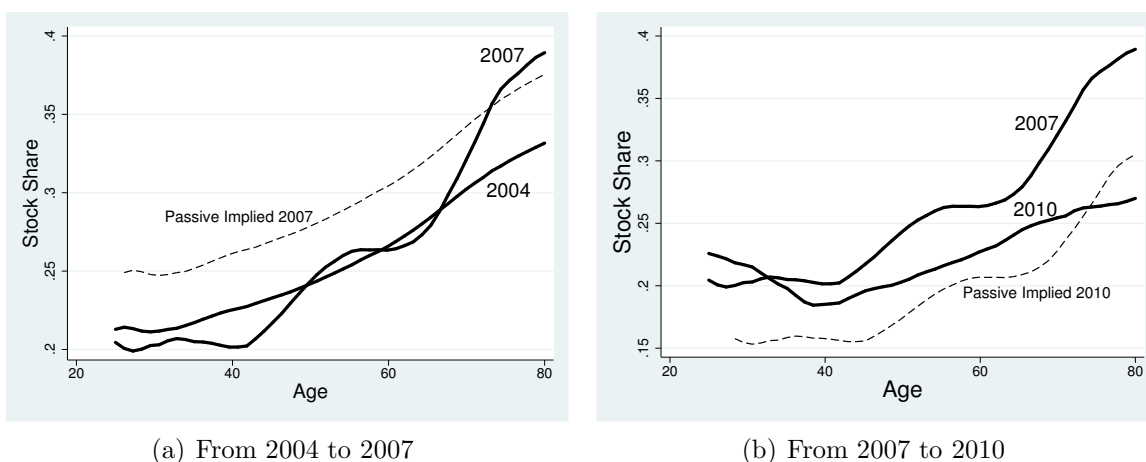
Understanding the effects that drive optimal rebalancing behavior is particularly important for two reasons. First, as more empirical work is done on the rebalancing behavior of individuals it is important to have a benchmark for equilibrium behavior in an economy in which investors behave optimally. While our model is quite simple, it helps to provide such a benchmark. Second, to meet the needs of investors with different horizons and preferences, the mutual fund industry has created a number of products called “lifecycle” or “target-date” funds that are designed to serve as the whole portfolio for a large group of investors over their lifespans. To the extent that these funds follow simple linear rules of thumb for allocating assets across time their allocations will be both suboptimal and generally infeasible if universally adhered to because simple rules ignore general equilibrium constraints. Our model helps to illustrate how “universal target-date” funds can be structured. Universal target-date funds follow an optimal strategy over time and are feasible in general equilibrium.

Empirical work on observed rebalancing is still relatively rare. Its primary finding is that most investors appear to display inertia in their asset allocations.¹ To motivate our analysis

¹Calvet, Campbell and Sodini (2009) use four years of Swedish data to show that households only partially rebalance their portfolios, or they only partially reverse the passive changes in their asset allocations. Brunnermeier and Nagel (2008) find similar results using data from the Panel Study of Income Dynamics.

along these lines, we plot nonparametric estimates of stock share as a function of age from three waves of the Survey of Consumer Finances around the recent financial crisis. The left panel of Figure 1 depicts stock shares in the 2004 and 2007 waves, between which stock prices rose about 30 percent. The right panel covers 2007 and 2010, between which prices fell about 28 percent. Both plots also give the “passive” share implied by market returns and initial shares. Average rebalancing by age is therefore given by the difference between the passive curve and the actual curve for the second wave in each plot.²

Figure 1: Rebalancing, SCF Data



Several interesting features are immediately apparent when looking at the plots. First, older investors have consistently higher stock shares in these data, which may be due to cohort effects such as those described in Malmendier and Nagel (2011). Second, there is substantial rebalancing in the data. While many individual investors may display inertia in their asset allocations, in aggregate investors do appear to react to market movements. Third, most age groups do not simply revert to their initial share. Rather, older investors, who on average hold more stock to begin with, buy more stock than required to return to their initial positions when the market rises. They also sell more stock than required to

²The curves are estimated with simple nonparametric regressions using Epanechnikov kernels. The passive curves are plotted assuming the stock returns reported in the text and total average non-stock returns of 6% between 2004 and 2007 and 4% between 2007 and 2010. To be included in the sample each individual must report at least \$10,000 of financial wealth and some positive stock holding. There are 1389, 1260, and 1313 observations in 2004, 2007 and 2010, respectively.

return to their initial position when the market falls. Younger investors display the opposite pattern, selling more than required to return to their initial position when prices rise, and buying shares as markets fall. In other words, rather than the stock share by age function either reverting back to the initial share or shifting to the passive share, it appears to pivot in each wave. While our model will not capture every feature of these plots, it highlights the importance of the general equilibrium constraints that the plots suggest and it allows older investors to behave differently than younger investors, as they appear to do in the data.

Though our model is intentionally simple, it sheds light on the counter-intuitive implications of general equilibrium. Our first set of implications is for the direction of rebalancing in equilibrium. We show that after a negative macroeconomic (and return) shock, more risk averse investors buy risky assets while less risk averse investors sell. After a positive shock the opposite happens. This finding is consistent with the pivots in the plots above. Our second set of implications is for portfolio shares in equilibrium. After a negative shock all investors hold a higher risky asset share than they held before the shock. Risk tolerant investors hold levered positions that makes their risky asset share rise when prices fall. At the same time, because expected returns are higher after prices fall their desired risky asset shares after a negative shock are higher than before the shock. Thus, while the risk tolerant investors sell shares after a negative shock, their selling only partially offsets the increase in their risky asset shares. Again, after a positive shock the opposite happens. Finally, we show that the effect of intertemporal hedging is quite small in our general equilibrium setting.

Section 2 describes intuitively the three effects that drive our equilibrium outcomes: the leverage, aggregate risk tolerance, and intertemporal hedging effects. Section 3 discusses how our work relates to other research. In Section 4 we describe the overlapping generations model in detail. The model has a partially closed-form solution, which makes the intuition more transparent and substantially eases the solution of the model. Section 5 presents our quantitative solution of the model. Section 6 presents our conclusions.

2 Preview

Most of the earliest models of asset allocation in general equilibrium feature homogeneous agents, with the implication that agents will never trade or rebalance, as in Lucas(1978). We refer to such an implication as the buy and hold rule. Other models, when considered in partial equilibrium, imply that agents should always revert to their initial stock share, as in Merton (1971). We refer to this implication as the constant share rule. Optimal portfolio rebalancing deviates from these two very different rules for at least three reasons in our model and more generally. First, there is a *leverage effect*. Since the risk-free asset is in zero net supply in our model, there will always be some investors that hold a levered position in the risky asset. For those with a levered position, their passive risky asset share actually declines as the market rises, and vice versa. Levered investors need to buy when the market rises and sell when it falls in order to keep their risky-asset shares constant, which is the opposite of standard rebalancing advice. Second, there is an *aggregate risk tolerance effect*. As the wealth of different types of agents shifts, aggregate risk tolerance also shifts, which in turn affects equilibrium prices and expected returns. Movements in aggregate risk tolerance induce people to trade in response to market movements in a way that does not maintain a constant share. In our specification, the aggregate risk tolerance effect is substantial. Third, there is an *intertemporal hedging effect*. Since future expected returns are time-varying, young agents have an incentive to hedge changes in their investment opportunities when they are middle aged. In principle, intertemporal hedging can have a significant effect on asset demand, but for reasons we will explain, in our specification it is quite difficult to generate enough mean reversion in asset returns to generate a substantial intertemporal hedging effect. We discuss hedging in Section 4.

A simple example can help clarify the leverage effect. Suppose an investor wants to maintain a levered portfolio with a share of 200% in the risky asset. With a \$100 stake to begin with, the investor borrows \$100 to add to that stake and buys \$200 worth of the risky asset. If the risky asset suddenly doubles in value, the investor then has \$400 worth of the risky asset and an unchanged \$100 of debt, for a net financial wealth of \$300. The constant

share rule generally has investors sell assets when their prices rise. After this increase in the price of the risky asset, however, the share of the investor's net financial wealth in the risky asset is only $400/300 = 133\%$, since the investor's net financial wealth has risen even more than the value of the risky asset. Therefore, if the goal is to maintain a 200% risky asset share, the investor needs to borrow more to buy additional shares. Hence, levered investors rebalance in a direction that is the opposite of unlevered investors. The general equilibrium constraint that each buyer must trade with a seller is partially met by having levered investors trade with unlevered investors.

The aggregate risk tolerance effect is a consequence of general equilibrium adding up. It arises because the share of the risky asset held by an individual is a function of that individual's risk tolerance relative to aggregate risk tolerance; and aggregate risk tolerance must move in general equilibrium when there are shocks that affect the distribution of wealth. We will now show that this feature of general equilibrium makes it impossible for all investors to maintain a constant portfolio share. Returns adjust to make people willing to adjust their portfolio shares in a way that allows the adding-up constraint to be satisfied.

To see this, consider the aggregation of an economy characterized by individuals who have portfolio shares determined as in Merton (1971). That is, an individual i 's share of investable wealth \mathcal{W}_i held in the risky asset would be

$$\theta_i \equiv \frac{PS_i}{\mathcal{W}_i} = \tau_i \frac{\mathcal{S}}{\sigma}, \quad (1)$$

where P is the price of the risky asset and S_i is the amount of the risky asset. The only heterogeneity across individuals here comes from individual risk tolerance τ_i and the only heterogeneity over time comes from a time-varying expected Sharpe ratio (expected excess return over volatility) \mathcal{S} or from volatility σ . After aggregating this expression and imposing adding up constraints, we can derive an expression for the risky asset share that does not

depend on the distribution of the asset's return:

$$\theta_i = \tau_i \left[\sum_i \frac{\mathcal{W}_i \tau_i}{\mathcal{W}} \right]^{-1} = \frac{\tau_i}{\mathcal{T}}, \quad (2)$$

where $\mathcal{T} = \sum_i \frac{\mathcal{W}_i \tau_i}{\mathcal{W}}$ is aggregate risk tolerance.³

This equation implies that neither the buy and hold rule nor the constant share rule makes sense in general equilibrium. As prices change, wealth effects cause aggregate risk tolerance to change. This induces rebalancing for all types of investors. This result also appears in Dumas (1989), Wang (1996) and other papers. In our model we make precise the exact nature of this rebalancing.

3 Relation to the Literature

There is a great deal of research about asset allocation and portfolio choice in finance and economics. These topics have become particularly interesting recently as economists have begun to study household finance in detail following Campbell (2006). Three papers that discuss optimal trading behavior are Dumas (1989), Longstaff and Wang (2012) and Wang (1996). Dumas (1989) solves for the equilibrium wealth sharing rule in a dynamic equilibrium model with more and less risk tolerant investors. This paper nicely illustrates the importance of the distribution of wealth as a state variable. Longstaff and Wang (2012) discuss the leverage effect that we describe. Like our model, these papers find that more risk tolerant investors sell after negative market returns while less risk tolerant investors buy. All three of these papers feature continuous time models, in which prices evolve according to Brownian motion processes and agents trade every instant. As Longstaff and Wang (2012) explain,

³Recall that the risk free asset is in zero net supply, so the total value of all investable wealth must equal the total value of the risky asset. Multiplying (1) through by each type's wealth and aggregating across types yields

$$\sum_i \theta_i \mathcal{W}_i = \sum_i PS_i = \left[\sum_i \mathcal{W}_i \tau_i \right] \frac{\mathcal{S}}{\sigma} = \mathcal{W}. \quad (3)$$

Dividing equation (3) through by aggregate wealth and comparing the resulting expression to equation (1) allow us to derive (2)

more risk tolerant investors trade in a dynamic way to create long positions in call options, while less risk tolerant investors create short call positions. As we discuss below, that paper's main contribution is about the size of the credit market. Dumas (1989) shows that more risk tolerant investors would like to hold perpetual call options. Since investors are continuously hedging to create options positions, these models generally imply infinite trading volume. Our model differs from these by featuring lifecycle effects, discrete time and relatively large price movements, which will be more realistic for many investors. None of these papers focuses on rebalancing behavior.

Two papers that show how optimal portfolio choice changes when some of the assumptions of classic models are weakened are Liu (2007) and Kim and Omberg (1996). Kim and Omberg (1996) show that optimal behavior can be quite complex when investors are not myopic, with some investors hedging while others speculate. Liu (2007) shows that the properties of dynamic portfolio choice models can be quite different from those of static choice models.

Relatively little of the research on portfolio choice, however, considers the general equilibrium constraints that we apply in our model. Three recent papers that feature general equilibrium are Cochrane, Longstaff and Santa-Clara (2008), Garleanu and Panageas (2015), and Chan and Kogan (2002). Cochrane, Longstaff and Santa-Clara (2008) has heterogeneous assets rather than heterogeneous agents, so it is not about trade. Garleanu and Panageas (2015), like our paper, models investor heterogeneity in risk preference and age in an overlapping generations model. Chan and Kogan (2002) examines the aggregate risk tolerance effect in detail. All of these papers focus on the behavior of asset prices, showing that general equilibrium can generate return predictability that is consistent with empirical evidence. None of these papers explores the implications of market returns for either trading or the portfolio choice of different types of individuals, which is the focus of our paper.

The focus of our paper is much closer to the focus of Campbell and Viceira (2002). They find an important intertemporal hedging component of portfolio choice. In our model the leverage effect and the time-varying aggregate risk tolerance effect are much more important. The intertemporal hedging effect in our model is intentionally weak because we do not build

in features that might lead to hedging such as mean reversion in fundamentals or noise traders. We have experimented with several different sets of parameter values to see if we can make the intertemporal hedging effect larger in our context, but we have not been able to find a scenario in which it matters very much. Approaches such as Campbell and Viceira that use actual asset returns, which might reflect mispricing, have greater scope for finding intertemporal hedging.

There is a large and important literature about labor income risk and portfolio choice, including Bodie, Merton and Samuelson (1992), Jagannathan and Kocherlakota (1996), Cocco, Gomes and Maenhout (2005), and Gomes and Michaelides (2005). In a recent paper that is related to our work, Glover, Heathcote, Krueger and Rios-Rull (2014) find that young people can benefit from severe recessions if old people are forced to sell assets to finance their consumption, causing market prices to fall more than wages. We purposely abstract from human capital considerations by assuming that agents only earn labor income before they have an opportunity to rebalance. Our aim is thus to focus on the general equilibrium effects while suppressing the important, but complex effects of human capital.

The leverage effect that we find is of course related to the effect discussed by Black (1976). Geanakoplos (2010) has recently developed a model emphasizing how shocks to valuation shift aggregate risk tolerance. Risk tolerant investors, who take highly levered risky positions, may go bankrupt after bad shocks, thereby decreasing aggregate risk tolerance and further depressing prices. This analysis hence features both the leverage and aggregate risk tolerance effects. The details of his model are quite different from ours. Moreover, the focus of his analysis is defaults, which will not occur in our setting. Nevertheless, both his model and our model feature time varying aggregate risk tolerance driven by changes in the distribution of wealth.

As mentioned in the introduction, Calvet, Campbell and Sodini (2009) and Brunnermeier and Nagel (2008) examine portfolio rebalancing with administrative holdings data and survey data, respectively. Both papers find that people do not completely rebalance, or that investors do not completely reverse the passive changes in their asset allocations.

This finding is consistent with optimal rebalancing according to our model. Moreover, the leverage effect implies that investors with high initial shares will change their allocations in the opposite direction from that predicted by full rebalancing, or in other words that risk tolerant investors will chase returns. In fact, regressions in the online appendix of Calvet, Campbell and Sodini (2009) indicate that the coefficients on passive changes in allocation are increasing in initial stock share. This is exactly what we would expect to find.⁴

When academic researchers think about asset allocation, they generally include all risky assets (including all types of corporate bonds) in their definition of the risky mutual fund that all investors should hold. In our model, regardless of how one might divide assets into safe or risky categories, we define the “risky asset” or the “tree” to be the set of all assets in positive net supply and we define the risk-free asset to be in zero net supply. The levered position in trees in our model encompasses both stock holdings and the risky part of the corporate bonds.

By contrast, when practitioners think about portfolio choice they generally treat stocks, bonds, and cash-like securities as separate asset categories in their allocation and rebalancing recommendations, as Canner, Mankiw and Weil (1997) point out. To understand the size of the leverage effect, it is important to remember that stock is a levered investment in underlying assets. Thus, for example, if the corporate sector is financed 60% by stock and 40% by debt, an investor with more than 60% in stock is actually levered relative to the economy as a whole. If some corporate debt can be considered essentially risk-free then it should be possible to express such an investor’s position as a portfolio of a levered position in the underlying assets of the economy and some borrowing at the risk-free rate. The leverage effect will be important for all investors that hold a levered position in the underlying assets of the economy—which is everyone who has an above-average degree of leverage (though typically not everyone who has an above-median degree of leverage).

⁴Since Calvet, Campbell and Sodini (2009) have detailed data on investors’ actual holdings and returns, some portion of the returns their investors experience are idiosyncratic. This rationalizes the fact that the coefficients in the online appendix of Calvet, Campbell and Sodini (2009) increase in initial share but they do not become positive for investors with high initial shares.

4 Model

We include in the model only the complications necessary to study rebalancing meaningfully. The economy has an infinite horizon and overlapping generations. There is heterogeneity in preferences to generate trade within generations. Specifically, there are two types of agents with different degrees of risk tolerance. The level of risk tolerance is randomly assigned at the beginning of the agent's life and remains unchanged throughout life. Each agent lives for three equal periods: young (Y), middle-aged (M) and old (O). Agents work in a representative firm and receive labor income when they are young, and live off their savings when they are middle-aged and old. All agents start their life with zero wealth, and do not leave assets to future generations; that is, they have no bequest motive. There is no population growth, so there is always an equal mass across the three cohorts. In this section, we provide the details of how we model and specify the technology, the asset markets, and the agents' preferences and their resulting decision problems.

4.1 Technology and Production

Our setup has a character very similar to that of an endowment economy. There is a fixed supply of productive assets or “trees.” Hence, we have a Lucas (1978) endowment economy, though unlike Lucas, we will have trade because not all agents are identical, and the endowment includes labor as well as “trees.” The young supply labor inelastically. Trees, combined with labor, produce output that is divided into labor income and dividend income. The labor income stream serves to provide resources to each generation. The dividend income provides a return to risky saving. This setup allows us to provide resources to each generation without inheritance. As noted above, by providing labor income only at the start of the life, we intentionally abstract from the interaction of human capital with the demand for risky assets.

The production function is Cobb-Douglas $Y_t = \frac{1}{\alpha} D_t K_t^\alpha L_t^{1-\alpha}$, where K_t is the fixed quantity of trees, L_t is labor, α is capital's share in production, and D_t is the stochastic shock to technology. We normalize the quantity of trees K_t to be one. The quantity of labor is also

normalized to be one. Therefore, total output is $Y_t = \frac{D_t}{\alpha}$, total labor income is $\frac{(1-\alpha)D_t}{\alpha}$, and total dividend income is D_t . Given our normalizations, technology levels will equal stochastic dividends. All labor income goes to the young, since they are the only ones supplying labor. Dividend income will be distributed to the middle-aged and old agents depending on their holdings of the risky tree.

In order not to build in mean reversion mechanically, we assume the growth rate $d_{t+1} = D_{t+1}/D_t$ is serially uncorrelated. In our calculations, we parameterize the dividend growth as having finite support $\{G_n\}_{n=1}^N$, where G_n occurs with probability π_n each period. The dividend growth shocks are realized at the beginning of each period before any decisions are made.

4.2 Assets

Before turning to agents' decision problems, it is helpful to define assets and returns. In our economy, there are two assets available to agents: one risk-free discount bond and one risky tree security. The risk-free bond is in zero net supply, and the risky tree has a fixed supply of one. We parameterize the dividend growth process to have two states. These two assets provide a complete market.

One unit of a bond purchased at period t pays 1 unit of consumption in period $t + 1$ regardless of the state of the economy then. A bond purchased at time t has a price of $1/R_t$, where R_t is the gross risk-free interest rate. One unit of the tree has a price P_t in period t and pays $D_{t+1} + P_{t+1}$ if sold in period $t + 1$. Thus, the gross tree return R_t^{Tree} is $\frac{P_{t+1} + D_{t+1}}{P_t}$, and the excess tree return is $Z_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t R_t}$.

4.3 Agents' Decision Problems

Since this paper focuses on asset allocation, we specify preferences so that there is meaningful heterogeneity in risk tolerance, but the consumption-saving tradeoff is simple. We use the Epstein-Zin-Weil preferences with a unit elasticity of intertemporal substitution (Epstein and Zin 1989, Weil 1990). The utility of agents born in date t with risk tolerance $\tau_j = \frac{1}{\gamma_j}$ is

given by

$$\rho_Y \ln(C_{Yt}^j) + \frac{1 - \rho_Y}{1 - \gamma_j} \ln E_t \exp \left((1 - \gamma_j) \rho_M \ln(C_{Mt+1}^j) + (1 - \rho_M) \ln \left(E_{t+1} (C_{Ot+2}^j)^{1 - \gamma_j} \right) \right), \quad (4)$$

where C_{Yt}^j , $C_{M,t+1}^j$ and $C_{O,t+2}^j$ denote consumption at young, middle and old ages, and ρ_Y and ρ_M govern time preference when young and when middle-aged. These parameters ρ_Y and ρ_M will turn out to be the average propensities to consume. With an underlying discount factor equal to δ , we have $\rho_Y = \frac{1}{1 + \delta + \delta^2}$ and $\rho_M = \frac{1}{1 + \delta}$. The degree of risk tolerance can be either high ($j = H$) or low ($j = L$). Because this utility has a recursive form, we work backwards.

Old Agents

The problem for old agents at date t is trivial. They simply consume all of their wealth, so total consumption of the old C_{Ot}^j equals the total wealth of the old W_{Ot}^j . Note that capitalized W denotes wealth per person.

Middle-aged Agents

A middle-aged agent at date t chooses consumption when middle-aged and a probability distribution of consumption when old by choosing bond holding B_{Mt} and tree holding S_{Mt} to maximize intertemporal utility

$$\rho_M \ln(C_{Mt}^j) + (1 - \rho_M) \ln \left(E_t (C_{Ot+1}^j)^{1 - \gamma_j} \right)^{\frac{1}{1 - \gamma_j}}, \quad (5)$$

subject to the budget constraints

$$C_{Mt}^j + B_{Mt}^j / R_t + P_t S_{Mt}^j = W_{Mt}^j, \quad (6)$$

$$C_{Ot+1}^j = W_{Ot+1}^j = B_{Mt}^j + S_{Mt}^j (P_{t+1} + D_{t+1}). \quad (7)$$

Recall that the middle-aged agents have no labor income, so they must live off of their wealth W_{Mt}^j .

Consider the portfolio allocation of this middle-aged agent. It is convenient to define the *certainty equivalent total return*

$$\phi_{Mt}^j = \max_{\theta \in \Theta} \left\{ E_t \left\{ (1 - \theta)R_t + \theta R_t Z_{t+1} \right\}^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}}, \quad (8)$$

where θ is the share of savings (invested wealth) in the risky asset. Given the homotheticity of preferences and the absence of labor income when youth is past, the optimal portfolio share θ_{Mt}^j that yields ϕ_{Mt}^j also maximizes expected utility from consumption when old.

Thus, the middle-aged agent's problem is equivalent to solving the following decision problem:

$$\max_{C_{Mt}^j, \theta \in \Theta} \rho_M \ln(C_{Mt}^j) + (1 - \rho_M) \ln(W_{Mt}^j - C_{Mt}^j) + (1 - \rho_M) \ln \left\{ E_t \left\{ (1 - \theta)R_t + \theta R_t Z_{t+1} \right\}^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}}, \quad (9)$$

which has the familiar special structure found with an intertemporal elasticity of substitution equal to one. Since only the first two terms depend on C_{Mt}^j , the first order condition for C_{Mt}^j is

$$\frac{\rho_M}{C_{Mt}^j} = \frac{1 - \rho_M}{W_{Mt}^j - C_{Mt}^j}, \quad (10)$$

which has the solution

$$C_{Mt}^j = \rho_M W_{Mt}^j. \quad (11)$$

That is, log utility in the intertemporal dimension leads to a simple consumption function and makes saving and portfolio choices independent. Thus, the maximized utility of the middle aged is given by

$$V_{Mt}^j(W_{Mt}^j) = \rho_M \ln(\rho_M W_{Mt}^j) + (1 - \rho_M) \ln \left((1 - \rho_M) W_{Mt}^j \phi_{Mt}^j \right). \quad (12)$$

Bond holdings B_{Mt}^j are given by $(1 - \theta_{Mt}^j)(1 - \rho_M)W_{Mt}^j R_t$ and tree holdings S_{Mt}^j are given by $\theta_{Mt}^j(1 - \rho_M)W_{Mt}^j/P_t$.

Young Agents

With no initial wealth, the young have only their labor income for consumption and saving. Labor income of a young agent in period t is $W_{Yt} = (1 - \alpha)D_t/\alpha$. A young agent of type j decides on consumption, bond and tree holdings (B_{Yt}^j, S_{Yt}^j) (that together determine middle-aged wealth $W_{M,t+1}^j$) to maximize the utility

$$\rho_Y \ln(C_{Yt}^j) + \frac{1 - \rho_Y}{1 - \gamma_j} \ln(E_t \exp((1 - \gamma_j)V_{M,t+1}^j(W_{M,t+1}^j))) \quad (13)$$

subject to the budget constraints

$$C_{Yt}^j + B_{Yt}^j/R_t + P_t S_{Yt}^j = W_{Yt}^j \quad (14)$$

and

$$W_{M,t+1}^j = B_{Yt}^j + S_{Yt}^j(P_{t+1} + D_{t+1}). \quad (15)$$

Substituting the maximized middle-aged utility given by equation (12) into (13) yields the following decision problem for a young agent:

$$\begin{aligned} \max_{C_{Yt}^j, \theta \in \Theta} \quad & \rho_Y \ln(C_{Yt}^j) + (1 - \rho_Y) \ln(W_{Yt}^j - C_{Yt}^j) \\ & + (1 - \rho_Y) \ln \left\{ \left[E_t \left((1 - \theta)R_t + \theta R_t Z_{t+1} \right) (\phi_{M,t+1}^j)^{1 - \rho_M} \right]^{1 - \gamma_j} \right\}^{\frac{1}{1 - \gamma_j}} \\ & + (1 - \rho_Y) [\rho_M \ln(\rho_M) + (1 - \rho_M) \ln(1 - \rho_M)]. \end{aligned} \quad (16)$$

Only the first two terms depend on C_{Yt}^j . Again because of log utility, optimal consumption is very simple. It is given by

$$C_{Yt}^j = \rho_Y W_{Yt}^j.$$

Since only the third term of equation (4) depends on θ , the optimal tree portfolio share θ_{Yt}^j for the young maximizes the certainty equivalent, time-aggregated total return function for the young ϕ_{Yt}^j :

$$\phi_{Yt}^j = \max_{\theta \in \Theta} \left\{ E_t \left\{ [(1 - \theta)R_t + \theta R_t Z_{t+1}] (\phi_{M,t+1}^j)^{1 - \rho_M} \right\}^{1 - \gamma_j} \right\}^{\frac{1}{1 - \gamma_j}}. \quad (17)$$

Comparing the middle-aged portfolio problem in (8) with the young portfolio problem in (17), we find that the young solve a more complicated problem given their longer life span. Middle-aged agents with only one more period of life simply maximize the certainty equivalent return of their portfolio investment next period, as in equation (8). Young agents with two more periods to live need to pay attention to the covariance between portfolio returns from youth to middle-age and the certainty equivalent of the returns going forward beyond middle age.

Value of a Universal Target-Date Fund as Numeraire

Since future expected returns are time-varying, the multi-period decision problem of the young creates an opportunity for intertemporal hedging (Merton 1973). To sharpen the interpretation of intertemporal hedging, define

$$Q_{M,t+1}^j = \frac{1}{\rho_M^{\rho_M} (1 - \rho_M)^{1-\rho_M} (\phi_{M,t+1}^j)^{1-\rho_M}}, \quad (18)$$

which prices the variable annuity or “universal target date fund” for a given age and risk aversion. It yields, at lowest cost, a lifetime utility equal to the lifetime utility from one unit of consumption at period $t + 1$ and one unit of consumption at period $t + 2$. That is, the agent will be indifferent between being limited to consuming one unit of consumption at both middle and old age and having wealth of $Q_{M,t+1}^j$ to deploy optimally.⁵ The sense in which the universal target date fund is “universal” is that it remains the optimal thing to do even if everyone else has a universal target date fund in equilibrium. Given this definition

⁵To see this, note that by (5), consumption of one at both middle and old age yields a lifetime utility of zero for the middle-aged agent. Unit middle-aged holdings of $Q_{M,t+1}$ must therefore make the middle-aged lifetime utility given by (9) equal to zero once the optimal consumption and portfolio share are substituted in, so

$$\rho_M \ln(\rho_M Q_{M,t+1}^j) + (1 - \rho_M) \ln((1 - \rho_M) Q_{M,t+1}^j) + (1 - \rho_M) \ln(\phi_{M,t+1}^j) = 0.$$

The value of $Q_{M,t+1}^j$ given in (18) is the solution to this equation. The universal target date fund itself implicitly replicates the optimal portfolio strategy for the agent.

of the universal target date fund, equation (17) can be rewritten as follows:

$$\phi_{Yt}^j = \rho_M^{-\rho_M} (1 - \rho_M)^{\rho_M - 1} \max_{\theta \in \Theta} \left\{ E_t \left\{ \frac{(1 - \theta)R_t + \theta R_t Z_{t+1}}{Q_{M,t+1}^j} \right\}^{1 - \gamma_j} \right\}^{\frac{1}{1 - \gamma_j}}. \quad (19)$$

Thus, a young agent maximizes the simple certainty equivalent of his or her portfolio's return relative to the return on the universal target date fund. In other words, this universal target date fund paying out in periods $t + 1$ and $t + 2$ serves as a numeraire for returns from t to $t + 1$.⁶

4.4 Market Clearing

In our economy, the safe discount bonds are in zero net supply, and the risky tree has an inelastic supply normalized to one. We now substitute the asset demands we derived in the previous subsection into the asset market clearing conditions. Recall that the individual-level risky tree (S) and safe bond (B) demands of the young are given by $S_{Yt}^j = (1 - \rho_Y)W_{Yt}^j \theta_{Yt}^j / P_t$ and $B_{Yt}^j = (1 - \rho_Y)W_{Yt}^j R_t (1 - \theta_{Yt}^j)$. The asset demands of the middle-aged are given by $S_{Mt}^j = (1 - \rho_M)W_{Mt}^j \theta_{Mt}^j / P_t$ and $B_{Mt}^j = (1 - \rho_M)W_{Mt}^j (1 - \theta_{Mt}^j) R_t$. The old have no asset demand. Summing over the type j and the generations, the asset market clearing conditions

⁶Note that, because of its definition as the amount of wealth yielding lifetime utility equal to the lifetime utility from consumption constant at one, equation (12) can be rewritten

$$V_{Mt}^j(W_{Mt}^j) = \ln(W_{Mt}^j) - \ln(Q_{Mt}^j).$$

Defining the corresponding price of the universal target date fund in youth that can give lifetime utility equivalent to the lifetime utility of one unit of consumption in each of the three periods of life, $[Q_{Yt}^j]^{-1} = \rho_Y^{\rho_Y} (1 - \rho_Y)^{1 - \rho_Y} [\rho_M^{\rho_M} (1 - \rho_M)^{1 - \rho_M}]^{1 - \rho_Y} (\phi_{Yt}^j)^{1 - \rho_Y}$, the maximized utility for a young agent that results from substituting in the optimal decisions can be similarly written as

$$V_{Yt}^j(W_{Yt}^j) = \ln(W_{Yt}^j) - \ln(Q_{Yt}^j).$$

Thus, the price of the universal target-date fund fully captures the dependence of the value function on the investment opportunity set. Note that, unless $\gamma = 1$, wealth and Q are no longer additively separable after the application of the appropriate curvature for risk preferences.

can be written as

$$1 = \frac{1}{P_t} \sum_{j=H,L} \psi_j [(1 - \rho_M)W_{Mt}^j \theta_{Mt}^j + (1 - \rho_Y)W_{Yt}^j \theta_{Yt}^j] \quad (20)$$

and

$$0 = R_t \sum_{j=H,L} \psi_j [(1 - \rho_M)W_{Mt}^j (1 - \theta_{Mt}^j) + (1 - \rho_Y)W_{Yt}^j (1 - \theta_{Yt}^j)], \quad (21)$$

where ψ_j is the constant fraction of young agents of type j (or in more general cases, the constant fraction of labor income going to young agents of type j).

In addition to the asset market clearing conditions, the goods market clearing condition is also instructive. It is

$$\frac{D_t}{\alpha} = \frac{\rho_Y(1 - \alpha)D_t}{\alpha} + \rho_M(\psi_H W_{Mt}^H + \psi_L W_{Mt}^L) + W_{Ot}, \quad (22)$$

where the wealth level of the old W_{Ot} is given by the sum across the two types. Note that the propensity to consume out of wealth is 1 for the old, so there is no parameter multiplying W_{Ot} . Since the asset demands for old agents are equal to zero regardless of the level of risk aversion, we do not need to keep track separately of the wealth of each old group.

The total financial wealth of the economy is the sum of the value of the tree P_t and the dividend D_t , since the risk-free asset is in net zero supply. At the beginning of the period all financial wealth is held by the middle-aged and the old. Hence, the total wealth of the old is $W_{Ot} = P_t + D_t - \psi_H W_{Mt}^H - \psi_L W_{Mt}^L$. Substituting this expression into equation (22) and solving for P_t gives rise to an equation for P_t based only on contemporaneous variables:

$$P_t = (1 - \rho_Y) \frac{1 - \alpha}{\alpha} D_t + (1 - \rho_M)(\psi_H W_{Mt}^H + \psi_L W_{Mt}^L). \quad (23)$$

The right-hand side of (23) can be interpreted simply as the sum of savings supplies, and therefore total asset demands, of the young and middle aged, determined by their propensities to save and initial levels of wealth.

Normalizing all the variables in the above equation by dividend D_t gives the expression

for the price-dividend ratio p_t ,

$$p_t = (1 - \rho_Y) \frac{1 - \alpha}{\alpha} + (1 - \rho_M)(\psi_H w_{Mt}^H + \psi_L w_{Mt}^L), \quad (24)$$

where lower case letters denote the corresponding variables divided by the dividend D_t . Thus, the price-dividend ratio p_t is linear in the total (per-dividend) middle-aged wealth $\psi_H w_{Mt}^H + \psi_L w_{Mt}^L$. The first term on the right-hand side of (24) corresponds to the demand for saving of the young, which is a constant share of dividends and is therefore a constant in the per-dividend expression.

Having an analytic expression for the price-dividend ratio both adds transparency to our analysis and simplifies the solution of the model. Analytic expressions for portfolio shares and expected returns for the risky and risk-free assets are not available. In Section 5, we will turn to numerical solutions of the model. Unlike the price-dividend ratio, which only depends on total middle-aged wealth, portfolio shares and expected returns will depend on the distribution of wealth among the middle-aged agents. This distribution depends on the history of shocks. For example, as we will discuss explicitly in Section 5, a good dividend growth shock raises the share of wealth held by the middle-aged risk-tolerant agents who invested heavily in the risky asset when young. This increased wealth share of the risk-tolerant agents in turn affects asset demands and expected returns.

4.5 Equilibrium Solution

4.5.1 State Variables

Consider the state variable vector at the moment prices are determined. At that moment, the wealth of each group of agents is known. Because all preferences are recursive, nothing from the past directly enters into the preferences induced over actions now and in the future. The sum of human wealth and non-human wealth for each agent, together with the stochastic process for returns from that moment on fully determines what actions maximize an agent's utility. Each agent's decision problem has a perfect scale symmetry because of the

homotheticity of preferences together with linear budget constraints. This scale symmetry aggregates perfectly. Thus, only wealth relative to D_t matters for behavior. Furthermore, the scale symmetry means that behavior can be smoothly aggregated across agents within a type, so that, in effect, there is a representative agent of each type. Thus, despite the complexity of the agents' asset allocation choices (particularly the choices of the young, who have to worry about intertemporal hedging), the set of possible equilibria is determined entirely by forward-looking considerations given the wealth of each type at that moment when prices and therefore pre-consumption wealth are known.

These considerations indicate that the dynamic stochastic equilibrium from any point on only needs to be a function of the vector of normalized wealth levels for each type of agent. (Recall that the normalization is dividing through by the dividend, D .) What is less obvious is that, once all old agents are considered as belonging to one type, since their risk aversion no longer matters for their behavior, the vector of possible normalized wealth levels spans only a two-dimensional space. For convenience, represent a point on this two-dimensional space by the normalized wealth of the daring and cautious middle-aged agents. By (24), the sum of middle-aged wealth, together with the constant value of the wealth of the young in relation to the dividend determines the price of the tree and hence the value of all wealth. Of the nonhuman wealth owned at the beginning of the period, everything not in the hands of middle-aged agents must be in the hands of the old, since the absence of inheritance means that the young begin their first period of economic life with zero nonhuman wealth. Thus, the wealth of the old can be deduced from the wealth of the two middle-aged types. And as mentioned already, the normalized human wealth of the young is constant at $\frac{1-\alpha}{\alpha}$. So we can take the state variable vector to be the two-dimensional vector of normalized wealth levels of the middle-aged types where wealth is measured prior to consumption.

Because the forward-looking behavior of each type (other than the old, who are simple) makes the dynamic stochastic general equilibrium so complex, it is not possible to give a closed-form expression for the Markov transition from one period to the next. But again, since each representative agent's range of behavior only depends on that type's wealth and

forward-looking expectations for returns given different possible states of the economy, no lagged variable can enter into the determination of the equilibrium. Taking as given the outcome of the complex decision problem of agents in one period (including the complexity of needing to figure out the entire forward-looking dynamic stochastic general equilibrium), it is possible to give a closed form solution for the price of the tree in the next period. We discuss this in the next subsection.

4.5.2 Markov Representation

To show that our model is Markov, we make two points. First, the optimal portfolios are entirely forward-looking. Individual agents only care about the intertemporal probabilistic pattern of returns over their lifetime. The expected return distribution is entirely determined by the state vector of the aggregate normalized wealth levels of the two types of middle-aged agents. (See Appendix A.) Second, given the forward-looking portfolio behavior of agents, what happens the next period is fully determined. We will show that in a way that also gives some insight into the dynamics we will discuss later.

Once agents have made their decisions at time $t - 1$, the key facts that go forward to the next period are the assets owned by the daring and cautious middle-aged agents ($j = \{L, H\}$). Denote the fraction of the aggregate tree owned by these groups as

$$x_{M,t-1}^j = \psi_j S_{Y,t-1}^j = \psi_j \frac{\theta_{Y,t-1}^j}{p_{t-1}} (1 - \rho_Y) \frac{1 - \alpha}{\alpha}. \quad (25)$$

Note that x is an aggregate quantity held by the middle-aged at the beginning of period t , that is, before realization of the dividend shock and before consumption. Therefore, it is entirely a function of decisions made when young. Note that $1 - x_{M,t-1}^H - x_{M,t-1}^L$ is the fraction of the tree owned by old agents at the beginning of period t . Because there is no inheritance, the young agents have only their human capital at the beginning of the period.

The level of middle-aged wealth for the high and low risk tolerance types, normalized by

the level of dividends, is then given by

$$\psi^H w_{Mt}^H = (p_t + 1)x_{M,t-1}^H + \psi^H \frac{R_{t-1}}{G_t} (1 - \theta_{Y,t-1}^H) (1 - \rho_Y) \frac{1 - \alpha}{\alpha} \quad (26)$$

and

$$\psi^L w_{Mt}^L = (p_t + 1)x_{M,t-1}^L + \psi^L \frac{R_{t-1}}{G_t} (1 - \theta_{Y,t-1}^L) (1 - \rho_Y) \frac{1 - \alpha}{\alpha} \quad (27)$$

where the second terms of both equations are the aggregate bond holdings of the two types of middle-aged agents. Note that w is measured before consumption, but after the dividend shock. Moreover, to calculate bond holding relative to the period t dividend, the debt relative to the period $t - 1$ dividend must be divided by the growth rate of the dividend between $t - 1$ and t . Because the wealth of the non-young must add up to p_t , the wealth of the old can be determined as a residual.

Now consider equilibrium in period t . It is convenient to denote the average tree share in the portfolio of young agents at the end of the period by

$$\theta_{Y,t-1} = \psi^H \theta_{Y,t-1}^H + \psi^L \theta_{Y,t-1}^L. \quad (28)$$

Then equation (24) for the overall supply and demand for all forms of wealth put together implies

$$p_t = (1 - \rho_Y) \frac{1 - \alpha}{\alpha} + (1 - \rho_M) \left[(x_{M,t-1}^H + x_{M,t-1}^L) (p_t + 1) + \frac{R_{t-1}}{G_t} (1 - \theta_{Y,t-1}) (1 - \rho_Y) \frac{1 - \alpha}{\alpha} \right] \quad (29)$$

or

$$p_t = \frac{(1 - \rho_Y) \frac{1 - \alpha}{\alpha} + (1 - \rho_M) \left[(x_{M,t-1}^H + x_{M,t-1}^L) + \frac{R_{t-1}}{G_t} (1 - \theta_{Y,t-1}) (1 - \rho_Y) \frac{1 - \alpha}{\alpha} \right]}{1 - (1 - \rho_M) (x_{M,t-1}^H + x_{M,t-1}^L)}. \quad (30)$$

This expression gives p_t as a function of past variables and current dividend growth, G_t .

From equations (26), (27), and (30), it is clear that (w_{Mt}^H, w_{Mt}^L) is the state vector for period t . Since the state vector at t depends only on variables known at period $t - 1$ and

the i.i.d. shock, our model is Markovian. Of course, it does not have a closed-form solution because the state vector depends on p_t , but as just shown, p_t can be represented in terms of past variables and the current shock.

4.5.3 Alternative State Variable Representation

The combination of the price-dividend ratio and aggregate risk tolerance is sufficient to fully characterize the state of the model economy at any point in time. In terms of our model, aggregate risk tolerance is

$$\mathcal{T}_t = \frac{\mathcal{W}_{Yt}^H \tau_H + \mathcal{W}_{Mt}^H \tau_H + \mathcal{W}_{Yt}^L \tau_L + \mathcal{W}_{Mt}^L \tau_L}{\mathcal{W}_t}, \quad (31)$$

where the savings or invested wealth of each type $j = \{H, L\}$ and $a = \{Y, M\}$ is $\mathcal{W}_{at}^j = (1 - \rho_a)\psi_j w_{at}^j D_t$ and the total invested wealth across all types is $\mathcal{W}_t = \mathcal{W}_{Yt}^H + \mathcal{W}_{Mt}^H + \mathcal{W}_{Yt}^L + \mathcal{W}_{Mt}^L$.⁷ Both the price dividend ratio and aggregate risk tolerance are simple functions of (w_{Mt}^H, w_{Mt}^L) .⁸ One could in fact view (p_t, \mathcal{T}_t) as the state vector for the economy as an alternative to the equivalent description of the state by (w_{Mt}^H, w_{Mt}^L) . We study the dynamics of both variables below.

5 Quantitative Results

In this section, we examine numerical solutions of a particular parameterization of our model focusing on the optimal portfolio allocations and rebalancing behavior of households in general equilibrium. Portfolios vary across agents with different levels of risk tolerance. In

⁷Note that the relevant wealth for this aggregation in our model is saving or invested wealth, i.e., wealth after consumption. In the continuous-time Merton model, there is no consumption during the investment period, so the distinction between before- and after-consumption wealth is absent.

⁸Recall (24) and rewrite (31) as

$$\mathcal{T}_t = \frac{(1 - \rho_Y)[(1 - \alpha)/\alpha][\psi_H \tau_H + \psi_L \tau_L] + (1 - \rho_M)[\psi_H w_{Mt}^H \tau_H + \psi_L w_{Mt}^L \tau_L]}{(1 - \rho_Y)[(1 - \alpha)/\alpha] + (1 - \rho_M)[\psi_H w_{Mt}^H + \psi_L w_{Mt}^L]}.$$

These two equations are easily invertible (indeed, given p_t and \mathcal{T}_t they translate into a pair of linear equations in w_{Mt}^H and w_{Mt}^L). Thus, there is a one-to-one mapping between (w_{Mt}^H, w_{Mt}^L) and the vector (p_t, \mathcal{T}_t) .

particular, the optimal portfolio share is larger than the market portfolio share (100%) for the high-risk-tolerance agents and lower than the market portfolio share for the low-risk-tolerance agents. The portfolio tree shares of all types comove in response to the dividend growth rate shock: declining after a good shock and rising after a bad shock. The underlying wealth dynamics ensure that such comovements in the portfolio shares are consistent with general equilibrium.

5.1 Parameterization and Solution

Table 1 summarizes the parameter values used in the quantitative analysis. A model period corresponds to 20 years in the data. The 20-year discount rate of 0.67 corresponds to an annual discount rate of 0.98. Consequently, ρ_M is 0.75 and ρ_Y is 0.69. Capital's share is 0.33. The cross-sectional heterogeneity in risk tolerance is set to match the findings in Kimball, Sahm and Shapiro (2008). They estimate the mean risk tolerance to be 0.21 and the standard deviation to be 0.17. The distribution they estimate has skewness of 1.84. We match these three moments by giving 92% of agents a low risk tolerance τ_L of 0.156 and 8% of agents a high risk tolerance τ_H of 0.797. We refer to the low-risk-tolerance agents as *cautious agents* and the high-risk-tolerance agents as *daring agents*.

The dividend growth shock process is assumed to be i.i.d with two states of equal probability. In the bad state, the detrended gross growth rate is $G_1 = 0.67 \approx (\frac{1.005}{1.025})^{20}$ and in the good state, it is $G_2 = 1.5 \approx (\frac{1.045}{1.025})^{20}$. With our 20-year time horizon, these parameters imply that the dividend growth rate is 2.5% per year in expected terms, 0.5% per year in the bad state and 4.5% per year in the good state. The dividend growth rate shocks capture generational risk. The mean scenario mimics the experience of the United States, the good scenario mimics that of South Korea, and the bad scenario mimics that of the Japan in the last two decades. Since all key variables below are expressed on a per-dividend basis and the elasticity of intertemporal substitution is equal to 1, the trend growth rate of 2.5% per year does not affect any of our analysis.

As noted above, our model does not have an analytical solution, so we solve the model

Table 1: Benchmark Parameter Values

Capital's Share	α	0.33
Discount Factor	β	0.67
Middle-aged Average Propensity to Consume	ρ_M	0.75
Young Average Propensity to Consume	ρ_Y	0.69
Low Risk Tolerance	τ_L	0.156
High Risk Tolerance	τ_H	0.797
Fraction of Low Risk Tolerance Agents	ψ_L	0.92
Fraction of High Risk Tolerance Agents	ψ_H	0.08
i.i.d. Dividend Growth Shock Process		
Detrended Growth Rate	(G_1, G_2)	(0.67, 1.50)
Probability	(π_1, π_2)	(0.50, 0.50)

numerically. To make the model stationary, we normalize relevant variables with same-dated dividends. Given six types of agents and two types of assets in our model, it might seem that the dimension of the state space would be large. As shown in Section 4.5, however, we are able to reduce the state space to two endogenous variables: the wealth levels of the middle-aged cautious and middle-aged daring agents (w_{Mt}^H, w_{Mt}^L) . We are able to omit states governing the dividend growth rate from the state space due to the i.i.d. assumption. We discretize the state space in our numerical solution, but we allow the decision variables to be continuous. We use the recursive dynamic technique to solve the model. For the detailed computational algorithm see Appendix A.

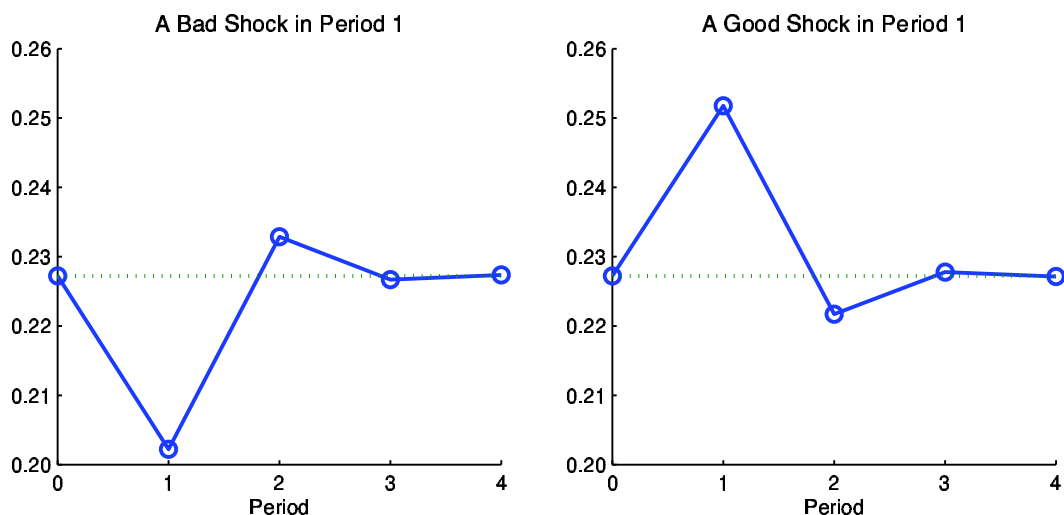
We now present the quantitative behavior of our model economy, focusing particularly on asset holdings. Our findings on price and return dynamics are not the main contribution of this paper, so we discuss them in Appendix B.

5.2 The Dynamics of Risk Tolerance

Understanding the endogenous wealth dynamics across agents is central to understanding the dynamics of asset holdings. The history of shocks matters because it affects the distribution of wealth and therefore aggregate risk tolerance. We define aggregate risk tolerance in Equation (31). We display impulse response functions, starting from the average steady state

and then show the average response of the economy after a good or bad shock in period 1. Figure 2 shows how aggregate risk tolerance responds to both positive and negative shocks. The plot is symmetric because it gives deviations from the mean and good shocks and bad shocks are equally likely.

Figure 2: Aggregate Risk Tolerance



The impulse response shown in Figure 2 shows aggregate risk tolerance rises as a result of a good shock and falls as a result of a bad shock. While the long-run average aggregate risk tolerance is about 0.227, it falls to 0.202 after a bad shock and rises to 0.25 after a good shock. This is because a good shock corresponds to a positive return and an increase in the wealth of daring agents. The shock in period 1 matters less for risk tolerance in period 2 because one of the affected cohorts dies, and another affected cohort reaches the simplicity of old age. Yet on average the period 1 movement in risk tolerance rebounds so that it is more than reversed in period 2.

What explains the dependence of aggregate risk tolerance in period 2 on the shock in period 1? A good shock at time 1 puts a large share of invested wealth in the hands of daring middle-aged agents, who then hold a large fraction of all available shares of the risky asset, leaving less for their more cautious peers and for the young cohort as a whole. Thus, the overall holdings of the young cohort are less aggressive when they enter economic life on the

heels of a good dividend-growth-rate shock than when they enter economic life on the heels of a bad dividend-growth-rate shock. This is particularly relevant for daring young agents, as it means that the fraction of all invested wealth in middle age that they hold is less than it would otherwise be. Thus, after a good shock in period 1 the wealth of daring middle aged agents in period 2 is unusually low, and therefore the aggregate risk tolerance in that state of the world is also unusually low. This explanation is symmetric, so that after a bad shock in period 1 risk tolerance is unusually high in period 2. Similar reasoning can explain how shocks affect risk tolerance in later periods. In principle each shock affects all future values of aggregate risk tolerance, but in practice the effect of any shock tends to average out after 3 periods.

Figure 3 plots the impulse responses for wealth shares after bad and good shocks and Figure 4 shows how the aggregate wealth shares of different types of agents sum to one. Consistent with the rest of our results, the wealth share of the middle aged daring goes down after a bad shock and up after a good shock. The wealth share of the middle aged cautious shows the opposite pattern, and the wealth share of the young does not vary much with shocks.

Figure 3: Wealth Shares by Type

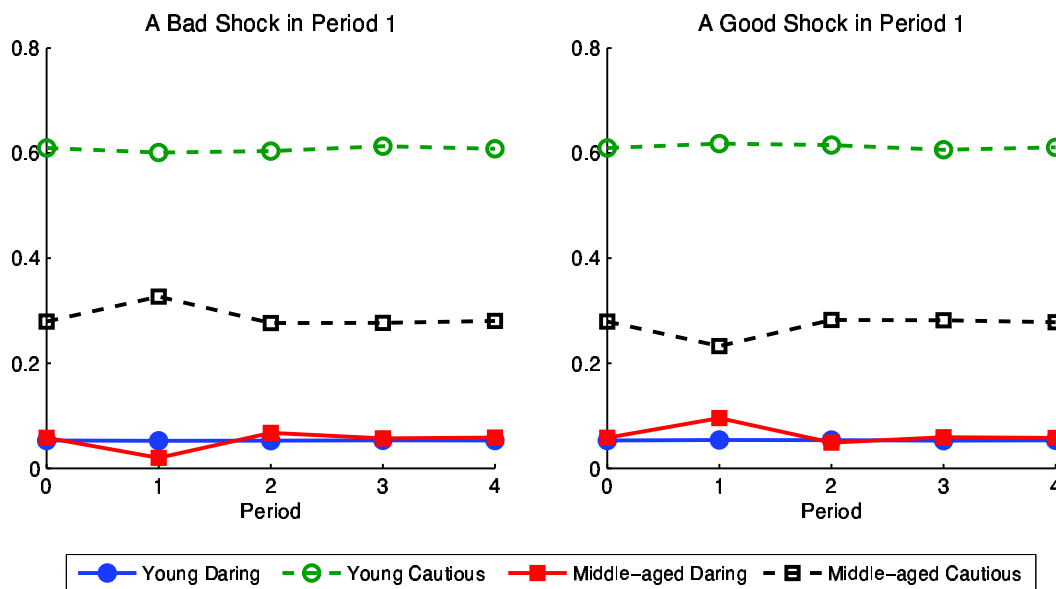
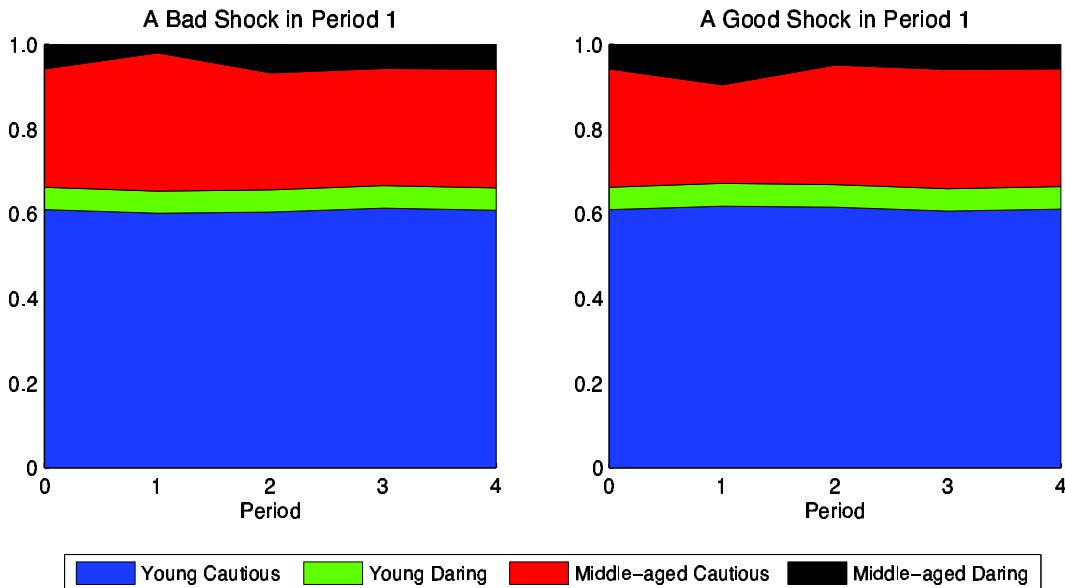


Figure 4: Aggregating Wealth Shares by Type



5.3 Portfolio Choice and Rebalancing

Most modern discussions of asset pricing in general equilibrium focus on the properties of asset prices and returns. Asset holdings are left implicit. Since we are particularly concerned with implications for individual asset demands, we look directly at portfolio choice and rebalancing in response to shocks in this subsection. Figure 5 plots the impulse response of the equilibrium tree holdings of different types of agents to both bad and good shocks. Figure 6 plots the same thing in a cumulative fashion to show explicitly that the holdings always sum to one.

Like the impulse response plot of aggregate risk tolerance in Figure 2, the period 0 value in these plots is the long-run mean tree holding for each group. Also like Figure 2, these figures are symmetric in the shock, in that the impulse response for positive shocks is the opposite relative to that mean of the impulse response for the negative shocks. Both figures show that tree holdings respond substantially to good and bad shocks. After a bad shock, the tree holdings of the young agents of both types increase. The holdings of daring middle aged investors fall substantially while those of cautious middle-aged investors rise. After a

Figure 5: Equilibrium Tree Holdings

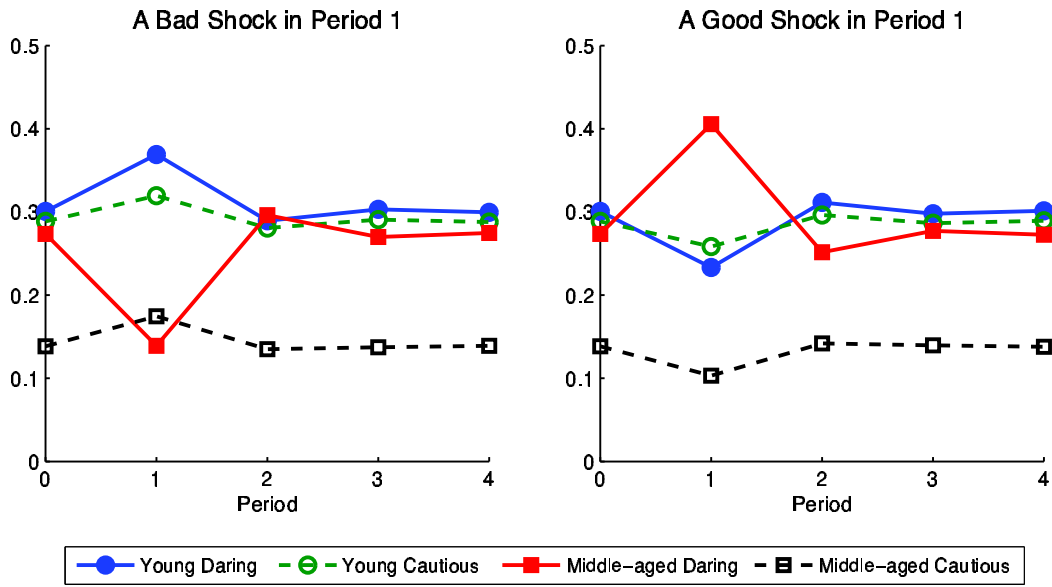
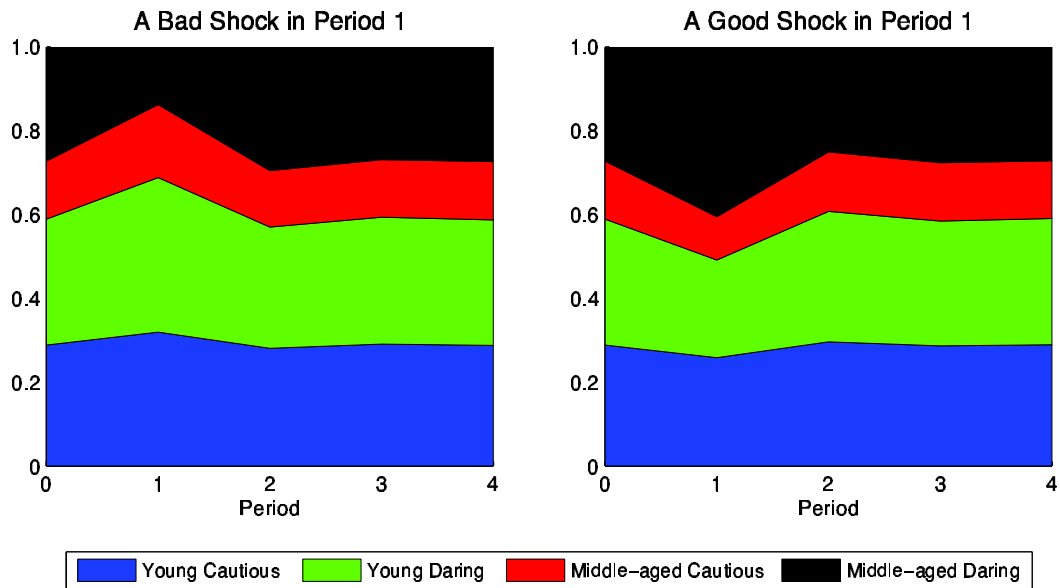


Figure 6: Tree Holdings in Aggregate



good shock the holdings of daring middle-aged investors rise substantially and those of the other types fall. By period 2 these effects are largely reversed.

The effects in Figures 5 and 6 are driven by the same type of logic used to explain aggregate risk tolerance above. After a bad shock the wealth of middle-aged daring investors has declined substantially and they hold less of the risky asset. Because the risky asset is relatively cheap, other investor types (young daring, young cautious, middle-aged cautious) are happy to hold more. The middle-aged cautious investors are rebalancing in the direction (but not the exact extent) a constant share rule would imply. The logic after a good shock is the reverse.

The findings of these figures are developed further in Table 2. The first three columns of Table 2 report the portfolio share of each type of agent. The statistics in the table help us to isolate three important effects: leverage, aggregate risk tolerance, and intertemporal hedging.

First, *leverage* arises endogenously, with the daring agents being highly levered and with the cautious agents holding a risky, but unlevered portfolio. As shown in the first column of Table 2, the cautious agents hold only about 48% of their savings (invested wealth) in the risky asset and the rest in the risk-free bonds. By contrast, the daring agents invest on average 568% of their invested wealth in the risky asset by selling the risk-free bonds to the cautious agents. This difference in portfolio shares across types enables shocks to change the distribution of wealth across types.

Second, as the second and third columns of Table 2 show, one of the most striking implications of the model is that all types have a higher mean portfolio tree share after a bad shock than after a good shock due to the *aggregate risk tolerance* effect. A bad shock pushes down aggregate risk tolerance and raises the expected excess return. As a result, all types of agents choose a larger portfolio share. After a good shock, aggregate risk tolerance rises, the expected excess return declines, and all agents decrease their portfolio shares. It may seem surprising that all agents are able to move their portfolio shares in the same direction in general equilibrium. Section 5.5 below shows that changes in the wealth

Table 2: Portfolio Choice, Savings Weight and Asset Demand by Type (Percent)

G_t	Portfolio Share θ			Savings Weight $\frac{Y_i}{W}$		Tree Amount $\frac{\theta Y_i}{W}$	
	Average	Bad	Good	Bad	Good	Bad	Good
Cautious	48	53	43	92.8	85.0	49	36
Young	47	53	42	60.1	61.7	32	26
Middle	49	53	44	32.7	23.3	17	10
Daring	568	704	432	7.2	15.0	51	65
Young	570	706	433	5.2	5.4	37	23
Middle	566	701	432	2.0	9.6	14	42
Total	90	105	74	100.0	100.0	100	100
Young	89	105	73	65.3	67.1	69	48
Middle	90	105	75	34.7	32.9	31	52

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. G_t denotes the dividend growth rate shock in period t .

distribution combined with the effects of leverage, in particular, changes in the share of all invested wealth in the hands of middle-aged daring agents, allow all portfolio shares to move in the same direction.

Third, Table 2 indicates that for each level of risk tolerance the portfolio share of the young is similar to that of the middle-aged. Recall that we abstract from the dynamic interaction between human capital and portfolio choice by having labor income only at the beginning of the life. The comparison between young and middle-aged portfolios for a given level of risk tolerance thus isolates the pure intertemporal hedging effect. Table 2 indicates that the intertemporal hedging effect is small quantitatively. Qualitatively, the cautious young have a lower portfolio share than the cautious middle-aged, while the daring young have a higher portfolio share than the daring middle-aged. Section 5.7 discusses these implications in detail.⁹

The average young portfolio share has important implications for the dynamics of the total middle-aged wealth and aggregate risk tolerance next period. Recall that the cautious

⁹Our results do not appear to be special to the parameterization that we have assumed. In Appendix C we both halve and double the variation in risk tolerance and the magnitude of economic shocks. The results we obtain are qualitatively the same.

agents account for 92% of the population and the daring agents account for the other 8%. The average young portfolio share is 105% ($= 0.92 \times 53 + 0.08 \times 704$) after a bad shock and 74% ($= 0.92 \times 43 + 0.08 \times 433$) after a good shock. Thus, the young hold a slightly more daring portfolio (105%) than the market portfolio (100%) after a good shock, but a much more conservative portfolio (74%) after a bad shock. These patterns for the portfolio share of the young contribute to the history dependence of aggregate risk tolerance that we discussed in the previous subsection.

5.4 Prices and Returns

In this section we describe the prices and returns implied by the model in order to make the intuition behind our rebalancing results clear. Table 3 presents the summary statistics based on a model simulation of 10,000 periods, which is a sufficiently long sample to eliminate the simulation uncertainty. Returns are expressed as annual rates. The log risk free rate has a mean of 1.9% and a standard deviation of 22 basis points. The log realized tree return is on average 3.63% with a standard deviation of 2%. Thus, the model economy generates a 1.7% risk premium with a standard deviation of 2%, resulting in an average Sharpe Ratio of 0.85. Although our model is not specifically designed to deliver a large risk premium, it goes a substantial fraction of the way toward explaining the historical premium. The model's Sharpe Ratio is quite high because the return volatility implied by the model is low. This results from our choice to match the volatility of macroeconomic shocks rather than market shocks.

Table 3: Summary Statistics on Prices, Annualized

Variable	Mean	Standard Deviation	Correlation with G	Auto Correlation
ln(Risk Free Rate)	1.92%	0.22%	0.808	-0.059
ln(Realized Tree Return)	3.63%	2.00%	0.996	0.028
ln(Realized Excess Return)	1.70%	2.00%	0.997	-0.057
Expected ln(Tree Return)	3.61%	0.16%	0.285	-0.353
Expected ln(Excess Return)	1.68%	0.13%	-0.968	-0.172
Tree Return Volatility	2.00%	0.05%	-0.985	-0.114
Sharpe Ratio	0.85	0.05	-0.955	-0.203
Tree Price-Dividend Ratio	18.94	0.54	-0.471	-0.246

Similarly, the model’s expected returns and price-dividend ratio are much less volatile than in historical data. For instance, the price-dividend ratio is on average 19, with a standard deviation of 0.54. Because shocks to the fundamentals are i.i.d. in the growth rate, movements in the realized return on the risky asset move essentially one-for-one with the dividend shock. Moreover, the i.i.d. growth shock means that any forecastability of returns comes from general equilibrium effects, not from the fundamentals *per se*. It turns out that these general equilibrium effects have a contribution to returns that is relatively small compared to sheer good and bad luck, so there is relatively little serial correlation in the realized risky return and the risk-free rate.

The expected excess return is critical for portfolio choice. It is strongly negatively correlated with the dividend growth rate shock and is somewhat mean reverting. Mechanically, these results are coming from movements in the risk-free rate, which is highly correlated with the shock to the dividend growth rate. Intuitively, the risk-free rate is moving because the shock to dividends changes the wealth of daring agents and therefore shifts the aggregate risk tolerance. We discuss the dynamics of the price dividend ratio in more detail in Appendix B.

We plot the Sharpe ratio of the risky asset in Figure 7. Consistent with the findings in Table 3, the Sharpe ratio increases after a bad shock and decreases after a good shock.

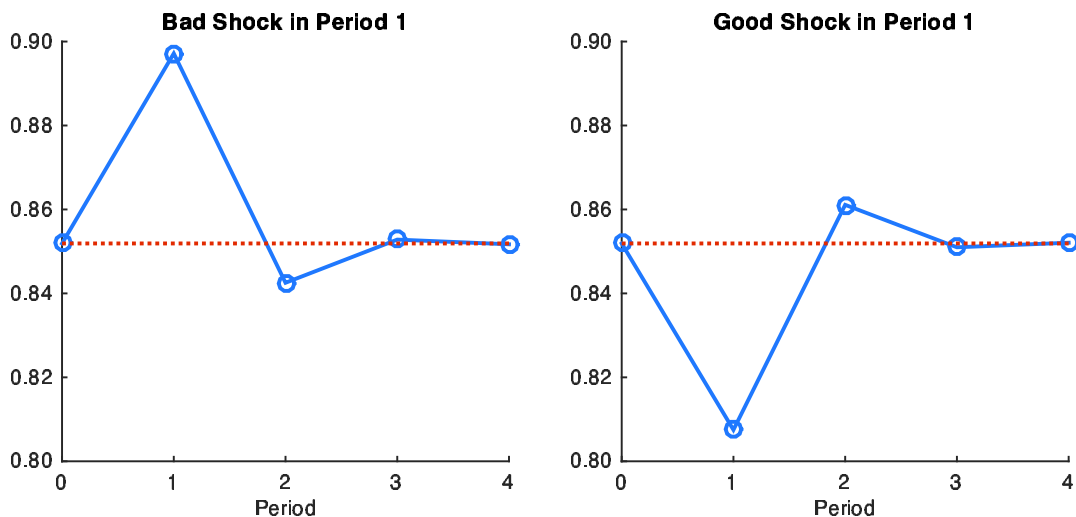
5.5 Correlated Movements in Portfolio Shares

It may seem surprising that all types of agents move their portfolios in the same direction in a general equilibrium. The key forces behind this finding are the changes in the total wealth and therefore in the invested wealth levels of the middle-aged agents that generate changes in the tree holdings $\frac{\theta \mathcal{W}_i}{\mathcal{W}}$ of each type to satisfy the adding-up constraint

$$1 = \frac{\mathcal{W}_{Yt}^L}{\mathcal{W}_t} \theta_{Yt}^L + \frac{\mathcal{W}_{Yt}^H}{\mathcal{W}_t} \theta_{Yt}^H + \frac{\mathcal{W}_{Mt}^L}{\mathcal{W}_t} \theta_{Mt}^L + \frac{\mathcal{W}_{Mt}^H}{\mathcal{W}_t} \theta_{Mt}^H. \quad (32)$$

To understand the implications of this constraint, look first at the “savings weights,” the

Figure 7: Sharpe Ratio



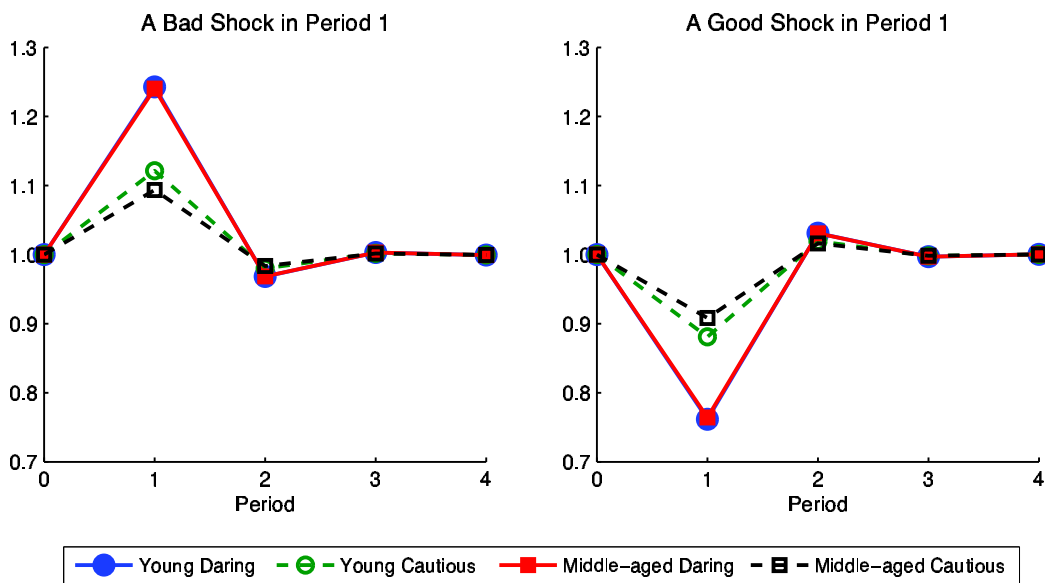
weights for invested wealth or after-consumption wealth $\frac{w_i}{W}$, reported in the middle columns of Table 2. The cautious agents have a larger savings weight than the daring because the cautious have a larger mass. The savings weight of the cautious is, however, lower after a good shock (85%) than after a bad shock (92.8%), so the aggregate risk tolerance increases after a good shock. Both the daring and cautious young agents have a slightly larger savings weight after a good shock than after a bad shock. The cautious middle-aged have a much smaller savings weight after a good shock (23.3%) than after a bad shock (32.7%). In contrast, the daring middle-aged have a much larger savings weight after a good shock (9.6%) than after a bad shock (2%).

Now, examine the quantity of trees held by each type of agent in the last two columns of Table 2. The quantity of trees held by each type equals that type's portfolio share multiplied by the savings weight of each type. Even though the economy has relatively fewer daring agents, the tree holdings of the daring agents are comparable in total size to those of the cautious agents due to their heavily levered position. After a good shock, the young agents have slightly higher savings weights but much lower portfolio shares, leading to much lower total tree holdings. After a good shock, the middle-aged cautious agents have a much lower savings weight and thus an even lower quantity of tree holdings given their decreased portfolio

share. On the other hand, the middle-aged daring agents have a much higher savings weight after a good shock, almost four times higher. The increase in their savings weight dominates the decline in their portfolio share, implying a larger total quantity of tree holdings by the middle-aged daring agents.

This effect can be illustrated with the impulse response plot in Figure 8. This figure plots the reaction of each investor type to bad and good shocks by comparing their portfolio shares to their long-run mean portfolio shares. The shares of the young daring and middle-aged daring are so similar that the lines are very difficult to distinguish in the plot. The figure clearly shows that after a good shock, all portfolio shares drop. After a bad shock they rise.

Figure 8: Portfolio Shares relative to Mean



In our parameterization all types of agents in the economy adjust their portfolio shares in the same direction in response to the dividend growth shocks. In particular, all increase their portfolio shares after a bad shock and decrease them after a good shock. This is also true for the all of the dramatically different alternative parameterizations that we consider in Appendix C, in which we double and halve the size of the shock and double and halve the distribution of risk tolerance. It may be possible, however, to find counterexamples.

Although it need not always be that everyone's shares move in the same direction, we consider this the natural case since the expected return is the main determinant of the optimal shares for people to hold. While intertemporal hedging also matters in determining optimal holdings, the variation in the expected return is much larger than the variation in hedging so its effect dominates. We discuss the magnitude of hedging effects below.

Even though all agents move their portfolio shares in the same direction, some must sell while others buy in order to attain their new positions. The changes in the underlying wealth distribution make such portfolio behavior consistent with general equilibrium.

5.6 Rebalancing by Cohort

So far we have been focusing on the portfolio dynamics of each type of agent at a point in time. Now we track the portfolio rebalancing of each cohort over time. When young, the agents choose an optimal portfolio share after the dividend shock. At the beginning of middle age, before any rebalancing, the value of their tree holdings as a share of their middle-aged wealth changes passively in response to the realized dividend shock and the realized tree return. We refer to this portfolio share as the realized portfolio share.¹⁰ The agents then choose the optimal portfolio share for their middle age (governing wealth accumulation between then and old age). We present the sequence of the three portfolio shares in Table 4. Because we are tracking cohorts (or individuals), we group equilibrium observations by the history of shocks rather than reporting an impulse response as we have done previously. G_Y and G_M denote the shocks that the agents experience when young and when middle-aged, respectively.

Let's first examine the passive movements of the portfolio share of each cohort. Since the cautious young agents are not levered, the realized middle-age portfolio share rises substantially after a good shock at their middle age, and declines slightly after a bad shock. By contrast, since the daring young agents are highly levered, their realized middle-age portfolio share rises after a bad shock, but declines after a good shock. In addition, the magnitude of

¹⁰This realized portfolio share implicitly assumes that consumption is financed by asset sales in proportion to asset shares, including proportionate paying off of debt for agents who have negative bond holdings.

Table 4: Rebalancing: Portfolio Share, by Cohort (Percent)

G_Y, G_M	Cautious Agents			Daring Agents		
	Young	Realized	Middle-Aged	Young	Realized	Middle-Aged
Bad, Bad	53.1	52.0	53.4	706	988	700
Good, Bad	41.8	39.9	53.4	433	594	702
Bad, Good	53.1	71.0	42.7	706	166	395
Good, Good	41.8	59.0	45.9	433	161	467

Note: G_Y and G_M denote the shocks that agents experience when young and when middle-aged, respectively.

the changes in their portfolio share is quite large. For instance, the realized portfolio share rises from 706% to 988% after a history of two bad shocks, and declines from 706% to 166% after a bad shock when young followed by a good shock at middle age.

If the constant share rule were more or less correct, the agents would undo the movements in the realized middle-aged portfolio shares and return to the portfolio share when young. In terms of the direction of the adjustment at middle age, the constant share rule is consistent with the behavior of the cautious agents. The cautious agents do not, however, return to the same portfolio share when young. In fact, the new optimal level is typically quite different from the optimal share when young. The difference in the optimal shares between the two periods is particularly large when the shock they experience when young is different from the shock at middle age.

In addition to the quantitative differences, there are two types of qualitative differences between the optimal rebalancing shown in Table 4 and the constant share rule. First, even returning to the same tree share requires opposite adjustments for people who seek to be more levered than the economy as a whole as compared to agents who seek to be less levered than the economy as a whole. Second, for the daring agents, the direction of their adjustments at middle age is sometimes opposite to the direction that would restore their tree share to what they had earlier. In particular, after a bad shock when young followed by a good shock at middle age, the daring middle-aged agents further increase their portfolio share above the realized portfolio share.

5.7 Intertemporal Hedging

Intertemporal hedging in our model arises when the portfolio shares shift consistently by age to take into account any predictability in returns. Referring back to Table 2, there is little difference in the portfolio choice between the young and the middle-aged. Given that our generational model produces only limited predictability in returns, it is not surprising that we see little intertemporal hedging. On the other hand, the qualitative implications of intertemporal hedging are still interesting and deserve some discussion.

To understand these patterns of intertemporal hedging, recall the portfolio choice problem of the young in equation (17) and of the middle-aged in equation (8). Unlike the middle-aged, who care about simple real returns, the young care about returns relative to the returns of the variable annuity, as indicated by (19). Table 5 presents the annuity price conditional the current and the past shock.

Table 5: Period- t Variable Annuity Price

G_t	Cautious		Daring	
	Bad	Good	Bad	Good
Mean	0.894	0.882	0.785	0.790
G_{t-1}				
Bad	0.891	0.889	0.762	0.802
Good	0.897	0.876	0.767	0.779

For the cautious agents, the variable annuity price is higher after a bad shock than after a good shock. The combination of a low risk-free rate and lower expected tree returns late in life after a bad shock makes it harder to finance a comfortable retirement (see Table 3). This unfavorable movement in the variable annuity price after a bad shock increases the effective risk of holding tree shares when young. By contrast, this variable annuity price is irrelevant for the middle-aged cautious agents; they care only about one-period returns. Thus, the young cautious agents invest less in the tree than the middle-aged cautious agents. The annuity price variation in period t is larger when the dividend shock is good in period $t - 1$. Thus, this finding is reinforced when the cautious agents experienced a good shock when young.

For the daring agents, however, the variable annuity price is lower after a bad shock than after a good shock, since they tend to be so highly levered that the increase in the equity premium after a bad shock makes it easier for them to finance a comfortable retirement. This variable annuity price pattern helps the daring agents smooth consumption at their middle age, and thus the young daring agents invest more in the risky asset than the middle-aged daring agents. The variation in the period- t annuity price is larger when the dividend shock is bad (which yields a greater covariance of the equity premium with the later shock) than when the dividend growth shock is good. Thus, quantitatively, the young daring agents invest more in risky assets than the middle-aged daring agents especially when they experience a bad shock in their youth.

6 Conclusion

We develop a model to illustrate how optimizing agents of different ages and risk preferences rebalance in equilibrium. The dynamics of portfolios differ starkly from benchmark models where agents either hold constant portfolio shares or buy and hold the market portfolio. The model includes agents with different risk tolerance, so there is meaningful trade, and three periods of life, so there is meaningful rebalancing over time. The model has a partial analytic solution. In particular, the state of the economy can be summarized by the distribution of wealth of middle-aged investors. This analytic solution both helps make the findings of the paper transparent and simplifies the solution of the full dynamic equilibrium model.

There are several effects that are key to understanding rebalancing in equilibrium. First, there is a leverage effect. In the model, leverage arises endogenously because of the heterogeneity in risk preferences. Shocks to the valuation of risky assets cause levered agents to rebalance in the opposite direction from unlevered agents. Second, there is an aggregate risk tolerance effect. Shocks to the valuation of risky assets affect agents' wealth differently depending on their initial position in the risky asset. Such shifts in the distribution of wealth change aggregate risk tolerance and therefore the demand for assets going forward. Note that the leverage and aggregate risk tolerance effects are connected because leverage magni-

fies how valuation shocks change the distribution of wealth. For example, after a bad shock leverage increases. The levered agents sell to reduce their leverage and the unlevered agents buy. But the increase in leverage is so great that even after rebalancing levered agents have higher exposure to risky assets. Therefore, all agents become more exposed to risk after a bad shock, and the opposite is true after a good shock

Third, there is an intertemporal hedging effect whereby equilibrium time variation in expected returns affects the portfolio choices of young investors as they exploit mean reversion in asset returns. This intertemporal hedging effect arising from mean reversion is what is often highlighted in analyses emphasizing time- or age-variation in portfolio shares. While it is present in our model, it is quantitatively weak by design. The model is solved under a specification with i.i.d. growth rates, so mean reversion in returns is not built into the specification. Other analyses of intertemporal hedging in the literature are based on empirical properties of actual returns, which might include mean reversion arising from mispricing. Our model has all agents frictionlessly optimizing, hence many sources of mean reversion that may be present in the data are intentionally not present in the model in order to highlight the implications of equilibrium for rebalancing.

Some key implications of the model are broadly consistent with recent rebalancing experience by age in data from the Survey of Consumer Finances. Determining the implications of our model for standard rebalancing data requires some thought. Our model features the underlying fundamentals: one risky asset and risk-free borrowing and lending. Data from markets feature stocks, bonds and other types of risky assets. Of course, these financial assets are claims on the fundamentals that also depend on the corporate capital structure. In particular, leverage in the corporate sector has important implications for the risk exposure of stockholders. For example, consider an investor with 100 percent stocks when the capital structure in the economy is half equity and half debt. In terms of our model, that investor will have a risky asset share of two. Because of the leverage effect described in the model, as stock prices rise a 100 percent stock investor will have a lower risky tree share before rebalancing because her wealth grows faster than the value of her equity. What is

going on is that when the value of all capital in the economy increases, stocks become less levered relative to the economy as a whole. Thus, to maintain the same leverage relative to the economy as a whole, an investor who began with 100 percent in stocks would need to buy more stocks after this market uptick. However, a finding of the paper is that because of general equilibrium forces, an investor will not rebalance to the same tree share as before the market uptick, but to a somewhat lower tree share, tempering somewhat the need to buy more stock. This example highlights how intuition about portfolios in terms of stocks must be interpreted with caution because of the underlying leverage of the corporate sector.

Looking back at the implied rebalancing from Survey of Consumer Finances data reported in Figure 1, several features of the recent history are consistent with the model but inconsistent with both the constant share and buy and hold rules. In the SCF data some of the older investors behave like the daring agents in the model, perhaps because of cohort effects. In Panel A, which reflects recent positive returns, older investors are actively buying to increase their stock share above the passive implied level for 2007. The reason they need to buy is that, because they are levered, when there is a good shock their exposure to risk decreases as wealth increases more than the value of their equity holdings. Younger investors are either actively selling or simply not buying as much. Younger investors act like the cautious agents in our model, avoiding large risky asset holdings after a positive shock. In Panel B, which reflects recent negative returns, the patterns are reversed. Older investors sell more than the passive implied quantity of assets, and younger investors buy more than what would leave them at their passive level. These patterns are broadly consistent with the model, but inconsistent with a constant share or buy and hold rule.

Our framework is useful for understanding how portfolios would respond to a stock market crash in an economy where all investors were behaving optimally. Contrary to both the constant share and buy and hold rules, after a market crash our investors increase their share in risky assets relative to the pre-crash levels, though by different amounts depending on their risk tolerance.

Finally, our analysis points to a principle that must hold much more generally than the

particular model we study here: funds that serve a nontrivial fraction of investors will have to take into account general equilibrium constraints to be feasible and optimal. In particular, many existing target-date funds, which follow simple linear allocation rules regardless of economic conditions, will not be optimal for most people. Moreover, as target-date funds become ubiquitous, they will become infeasible since they will not find counterparties to trade with as they rebalance. Our particular model points to how “universal target-date funds,” or funds that are optimal for all investors given risk tolerance and age, should adjust their target asset allocation in response to economic shocks.

Appendix A: Solution Technique

Consider discrete grid points for the state variable, even though we allow the choice variables to be continuous. We assume that the world ends at the end of period T , and solve the model backward from the last period. Behavior at time T is very simple: all remaining agents consume their wealth and the price-dividend ratio is zero. Thus, we can derive the value functions $v_{YT}^j(w_{MT}^H, w_{MT}^L)$ and $v_{MT}^j(w_{MT}^H, w_{MT}^L)$.

Consider now time $T - 1$ for a state (w_{MT-1}^H, w_{MT-1}^L) . The price-dividend ratio is given by equation (23).¹¹ The optimal consumption allocation can be easily derived. To solve for the portfolio allocation, we need to know $R_{T-1}(w_{MT-1}^H, w_{MT-1}^L)$. For any guess of R_{T-1} , the optimal portfolio share $\theta_{YT-1}^j = \theta_{MT-1}^j$ can be computed as follows:

$$\theta_{MT-1}^j(w_{MT-1}^H, w_{MT-1}^L) = \arg \max_{\theta} \sum_{n'=1}^N \pi_{n'} \left[(1 - \theta)R_{T-1} + \theta \frac{G^{n'}}{p_{T-1}} \right]^{1-\gamma_j}.$$

We check the bond market clearing condition and update R_{T-1} until the market clears. Given the optimal portfolio shares, we compute the certainty equivalent total return functions for the young $\phi_{YT-1}^j(w_{MT-1}^H, w_{MT-1}^L)$ and for the middle-aged $\phi_{MT-1}^j(w_{MT-1}^H, w_{MT-1}^L)$. Thus, we can derive the value function of the young and the middle-aged.

We now study period $T - 2$ for any state (w_{MT-2}^H, w_{MT-2}^L) , which has the same fundamental structure as a generic period. The price-dividend ratio is a function of the sum of the state variables, and the optimal consumption allocation can be easily derived using the first-order conditions. To solve for the model, we need to know the risk-free rate and the portfolio choice of the young. Given any guess of $(R_{T-2}, \theta_{YT-2}^H, \theta_{YT-2}^L)$, the portfolio choice of the young implies the next-period state variable $(w_{MT-1}^{Hn'}, w_{MT-1}^{Ln'})$ and the price-dividend ratio $p_{T-1}^{n'}$, for any dividend growth realization $G^{n'}$. Thus, we can compute the middle-aged optimal portfolio and certainty equivalent total return $\hat{\phi}_{MT-1}^j(w_{MT-1}^{Hn'}, w_{MT-1}^{Ln'})$. We then can compute the optimal portfolio choice of the young θ_{YT-2}^j , which is used to update our initial guess. We also update the interest rate using the bond market clearing condition. We repeat the above procedures until our guesses converge. We then compute the certainty equivalent total return function and the value functions of the middle-aged $v_{MT-2}^j(w_{MT-2}^H, w_{MT-2}^L)$ and the young $v_{YT-2}^j(w_{MT-2}^H, w_{MT-2}^L)$.

We repeat the above procedure for period $T - 3$, $T - 4$, etc, until the value functions of the young and the middle-aged across the two periods converge.

Our Equilibrium as the Fundamental Equilibrium

To be more formal about the tree price discussed in section 4.5.1, begin at the moment in period $t - 1$ when prices are determined in general equilibrium. At that moment, the set of possible equilibria going forward is completely determined by the distribution of wealth

¹¹The asset-market and goods-market equilibrium conditions are modified: the marginal propensity to consume and the optimal risky asset share for the young become the same as for the middle aged.

relative to D_t . Indeed, it is enough to know the normalized wealth per person of each type j of middle-aged agent: w_{Mt}^j : the wealth of the two groups of young agents is always fixed in relation to D_t . By equation (24), $w_{Mt}^H + w_{Mt}^L$ determines p_t and therefore total nonhuman wealth, so that the wealth of the old can be determined as a residual. It is unnecessary to know the division of wealth between the two old types, since their decision rules are identical: consume all wealth and invest nothing.

The decisions at time $t - 1$ on bondholding versus treeholding are made in the light of the overall state of the economy, which in equilibrium determines the probability distribution of future returns over the agents' lifetimes. We solve for the "fundamental equilibrium" in the sense of the limit of the equilibria in a sequence of economies that end in finite time, as the time T remaining before the death of all trees gets longer and longer.

Consider the finite-horizon version of the model. In the final period whose end marks the death of all trees, all living agents consume all of their wealth, with the young consuming all of their labor income. (The final harvest still takes work to harvest.) In the following period all remaining agents starve. In the next to last period, it is known that the value of a tree at the beginning of the next period will be equal to α times the value of the final harvest alone. Therefore, given the price of the tree in the next to last period, the tree returns given good and bad shocks between the next to last and the final period are clear. This anchors the recursive solution of the model from the final period backwards in time.

More generally, in the finite-horizon version of the model, given a length of time $T - t$ before the death of all trees, one can recursively (that is, by backward induction) compute the functions $\Theta_a^j(w_{Mt}^H, w_{Mt}^L, T - t)$ for $j = H, L$ and $a = Y, M, O$. As explained in the text, the aggregate normalized wealth of the daring and cautious middle-aged agents is enough to determine the normalized aggregate wealth of each type to the extent it matters for market prices, since the normalized wealth of young agents is constant and the normalized wealth of old agents can be treated as a residual given that both types of old agents are identical in their market behavior (consuming everything they have) and the total amount of after-consumption wealth is known as the price of the tree. Then the fundamental equilibrium of the infinite horizon model is defined as the limit as $T \rightarrow \infty$, or equivalently, it is defined as the limit of these functions as $t \rightarrow -\infty$.

Appendix B: Price and Return Dynamics

In this appendix we describe the price and return dynamics implied by the model in more detail.

The dependence of the risky asset share of the young cohort as a whole on the shock right before they begin investing makes it possible to explain the behavior of the price-dividend ratio. Recall that the price-dividend ratio is determined exactly by the total middle-aged wealth in each period. See Equation (24). Because of the Cobb-Douglas production function and the constant average propensity to consume, the invested wealth of the young cohort is a fixed fraction of output, or equivalently, in a fixed ratio to the dividend. Their later wealth in middle age in comparison with the dividend at that time depends on their overall portfolio return compared to the growth rate of the dividend. Let us explain the general level of the price-dividend ratio after each of the four possible histories of the last two shocks.

A good shock leading into the youth of the current middle-aged cohort depressed their risky asset share when young, and they were significantly underlevered relative to the economy as a whole. Then a second good shock will actually cause the growth rate of their wealth to fall behind the growth rate of the dividend (which equals the growth rate of economic production as a whole). Middle-aged wealth that is smaller relative to the expanded size of the economy will then lead to a low price-dividend ratio (in part because dividends are large). In contrast, a bad shock as they go into middle age causes the economy to grow more slowly than their relatively safer portfolio, leading to a high middle-aged wealth to dividend ratio, which in turn implies a high price-dividend ratio.

The bad shock leading into the youth of the current middle-aged cohort led that cohort to have slightly elevated risky asset holdings and thus to be slightly overlevered relative to the economy as a whole. A good shock at middle age then leads them to have a high level of wealth relative to the economy as a whole, leading in turn to a high price-dividend ratio. In contrast, a bad shock at middle age causes them to have a low level of wealth relative to the economy as a whole, leading in turn to a low price-dividend ratio.

Note that the last shock makes a bigger difference for the price-dividend ratio between the (Good, Good) and (Good, Bad) histories than it does between the (Bad, Good) and (Bad, Bad) histories. The reason is that the daring middle-aged agents in the previous period pulled more dramatically ahead of the economy after a good shock leading into their middle age than they fell behind the economy after a bad shock leading into their middle age. This led the current middle-aged cohort to have a significantly underlevered portfolio after a good shock leading into their youth, but only slightly overlevered portfolio after a bad shock leading into their youth.

To summarize the history dependence of prices, we present the key price statistics conditional on the current shock, the past shock, and the two past shocks in Table A1. The kind of reasoning above can be extended to interpret the consequences of a longer history of shocks, but the most recent shocks have the most powerful effect on the model economy. One reason we include the effects of a three-shock history is the identity relating the realized

tree return to the history of the price-dividend ratio:

$$R_t^{\text{Tree}} = \frac{P_t + D_t}{P_{t-1}} = \frac{D_t}{D_{t-1}} \left(\frac{1 + (P_t/D_t)}{P_{t-1}/D_{t-1}} \right) = G_t \left(\frac{1 + p_t}{p_{t-1}} \right).$$

Thus, the realized tree return depends on the values of the price-dividend ratio in two different periods. Among three-shock histories, the realized tree return is highest after (Good, Good, Good) since the two good shocks at $t - 2$ and $t - 1$ depress p_{t-1} , while the good shock at t gives a high G_t .¹² The realized tree return is lowest after the three-shock history (Good, Bad, Bad) because the combination of a good shock at $t - 2$ followed by a bad shock at $t - 1$ elevates p_{t-1} , while the bad shock at t gives a low G_t .

Table A1: History Dependence of Key Variables

G_t	P_t/D_t		$\log R_t^{\text{Tree}}$		$\log R_t$		$E(\log R_t^{\text{Tree}})$		$E(\log Z_t)$	
	Bad	Good	Bad	Good	Bad	Good	Bad	Good	Bad	Good
Mean	19.20	18.69	1.61	5.60	1.75	2.10	3.91	4.08	2.16	1.98
G_{t-1}										
Bad	18.99	19.23	1.51	5.60	1.81	1.96	3.96	3.95	2.15	1.98
Good	19.41	19.17	1.71	5.60	1.68	2.23	3.86	4.20	2.18	1.97
G_{t-2}, G_{t-1}										
Bad, Bad	19.07	19.32	1.58	5.67	1.79	1.94	3.94	3.93	2.15	1.99
Good, Bad	18.90	19.15	1.45	5.54	1.84	1.99	3.98	3.97	2.14	1.98
Bad, Good	19.21	17.81	1.53	5.41	1.74	2.33	3.91	4.29	2.17	1.95
Good, Good	19.59	18.52	1.87	5.79	1.62	2.12	3.81	4.11	2.19	1.99

Note: P_t/D_t denotes the price-dividend ratio, $\log R_t$ denotes the log risk free interest rate, $\log R_t^{\text{Tree}}$ denotes the log realized tree return, given by $\log(D_t + P_t) - \log P_{t-1}$, $E(\log R_t^{\text{Tree}})$ denotes the expected log tree return, and $E(\log Z_t)$ denotes the expected log excess return. All prices are annualized. All prices are in percentage, except the price-dividend ratio. G_t , G_{t-1} and G_{t-2} denote the dividend growth rate shock in period t , $t - 1$ and $t - 2$, respectively.

Table A1 also shows that the risk free rate is higher after a good shock than after a bad shock in period t . This pattern becomes particularly pronounced after a good shock in period $t - 1$. The expected tree return is higher after a good shock in period t . However, the pattern does not hold when we consider a two-period history of shocks. After a good shock in period $t - 1$, the expected tree return is higher after a good shock. In contrast, after a bad shock in period $t - 1$, the expected tree return is higher after a bad shock. The expected excess return is lower after a good shock in period t , given the high aggregate risk tolerance after a good shock in period t . The expected excess return is especially low after a history of the bad and good shocks.

¹²The realized tree return is second-highest after the three-shock history (Bad, Bad, Good) not only because two bad shocks at $t - 2$ and $t - 1$ depress p_{t-1} somewhat, but also because the tail end of a bad shock at $t - 1$ followed by a good shock at t puts p_t at a reasonably high level as well as giving a high G_t .

Appendix C: Alternative Parameterizations

We solve the model with different sets of numerical assumptions to determine how robust our primary findings are. First, we recalculate the risk tolerances of cautious and daring agents setting the standard deviation of risk tolerance equal to 0.085, which is half of the standard deviation of the risk tolerances listed in Table 1. We then solve the model with the other assumptions in Table 1 and report results analogous to Table 2 below. Next we double the standard deviation of risk tolerances. Finally, we both halve and double the standard deviations of economic shocks and report the results below. In each case we find results that are qualitatively similar to those obtained with our main parameterization in the text.

Table A2: Low Dispersion in Tolerance

G_t	Portfolio Share θ			Savings Weight $\frac{W_i}{W}$		Tree Amount $\frac{\theta W_i}{W}$	
	Average	Bad	Good	Bad	Good	Bad	Good
Cautious	72	76	69	92	88	70	61
Young	72	76	69	60	61	46	42
Middle	73	76	70	32	27	24	19
Daring	359	388	330	8	12	30	39
Young	360	389	330	5	5	20	18
Middle	359	388	330	2	7	10	22
Total	95	101	90	100.0	100.0	100	100
Young	95	101	90	66	66	66	59
Middle	96	101	90	34	34	34	41

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. G_t denotes the dividend growth rate shock in period t .

Table A3: High Dispersion in Tolerance

G_t	Portfolio Share θ			Savings Weight $\frac{W_i}{W}$		Tree Amount $\frac{\theta W_i}{W}$	
	Average	Bad	Good	Bad	Good	Bad	Good
Cautious	23	27	18	94	81	25	14
Young	21	27	15	60	62	16	10
Middle	24	28	21	34	19	9	4
Daring	860	1226	495	6	19	75	86
Young	853	1213	492	5	5	63	27
Middle	868	1238	499	1	14	12	59
Total	90	123	56	100.0	100.0	100	100
Young	88	122	54	65	67	79	37
Middle	92	124	59	35	33	22	63

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. G_t denotes the dividend growth rate shock in period t .

Table A4: Low Dispersion in Growth

G_t	Portfolio Share θ			Savings Weight $\frac{W_i}{W}$		Tree Amount $\frac{\theta W_i}{W}$	
	Average	Bad	Good	Bad	Good	Bad	Good
Cautious	66	69	63	93	89	64	56
Young	66	69	62	60	61	42	38
Middle	66	69	63	32	29	22	18
Daring	454	489	418	7	11	36	44
Young	454	490	418	5	5	26	22
Middle	453	489	418	2	5	10	22
Total	97	103	91	100.0	100.0	100	100
Young	97	103	90	66	66	68	60
Middle	97	103	91	34	34	32	40

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. G_t denotes the dividend growth rate shock in period t .

Table A5: High Dispersion in Growth

G_t	Portfolio Share θ			Savings Weight $\frac{W_i}{W}$		Tree Amount $\frac{\theta W_i}{W}$	
	Average	Bad	Good	Bad	Good	Bad	Good
Cautious	19	22	17	93	75	20	12
Young	19	22	15	59	65	13	10
Middle	20	22	18	34	10	7	2
Daring	746	1091	401	7	25	80	88
Young	750	1100	401	5	6	57	24
Middle	742	1083	401	2	19	23	64
Total	77	108	47	100.0	100.0	100	100
Young	77	108	46	64	70	70	34
Middle	78	107	48	36	30	31	66

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. G_t denotes the dividend growth rate shock in period t .

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