Winter 2017, MATH 215 Calculus III, Exam 2

3/23/2017, 6:10-7:40pm (90 minutes)

• Your name: _______________________________________________________________

• Circle your section and write your Lab time:

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Instructions:

• This examination booklet contains 7 problems.

• If you want extra space, write on the back.

• DO NOT remove any sheets or the staple from the exam booklet.

• The formula sheet is not collected back and not graded.

• This is a closed book exam. Electronic devices, calculators, and note-cards are not allowed.

• Show your work and explain clearly.
1. (10 points) Find the volume below the surface \( z = x^4 + y^4 \) and above the square in the \( x-y \) plane with vertices at \((x, y) = (\pm 1, 0), (0, \pm 1)\). The square is shown here:

\[
\text{Volume} = \iiint_\mathcal{D} x^4 + y^4 \, dA
\]

\( x^4 + y^4 \) is even in both \( x \) and \( y \).

\( \mathcal{D} \) is symmetric about the \( x \)-axis and the \( y \)-axis.

By symmetry, \( \text{volume} = 4 \iint_\mathcal{D} x^4 + y^4 \, dA \).

\( D_1 = \left\{ (x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \ 0 \leq y \leq 1-x \right\} \)

\[
D_1 = \left\{ (x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1, \ 0 \leq x \leq 1-y \right\}
\]

\[
\iint_{D_1} x^4 + y^4 \, dA = \int_0^1 \int_0^{1-x} x^4 + y^4 \, dy \, dx
\]

\[
= \int_0^1 \left[ x^4 y + \frac{y^5}{5} \right]_{y=0}^{y=1-x} \, dx
\]

\[
= \int_0^1 x^4 (1-x) + \frac{(1-x)^5}{5} \, dx
\]

\[
= \int_0^1 x^4 (1-x) \, dx + \int_0^1 \frac{(1-x)^5}{5} \, dx
\]
\[
\int_0^1 x^4 (1-x) \, dx = \int_0^1 x^4 - x^5 \, dx = \frac{x^5}{5} - \frac{x^6}{6} \bigg|_{x=0}^{x=1} = \frac{1}{30}.
\]
\[
\int_0^1 \frac{(1-x)^5}{5} \, dx = \int_0^1 -\frac{u^5}{5} \, du = -\frac{u^6}{30} \bigg|_{u=0}^{u=1} = \frac{1}{30}.
\]

\[
\text{Volume} = 4 \int_D x^4 + y^4 \, dA = \frac{4}{15}.
\]

\textbf{Note} The domain D remains unchanged upon swapping the x and y axes.

\[
\Rightarrow \int_D x^4 \, dA = \int_D y^4 \, dA.
\]
\[
\text{Volume} = \int_D x^4 + y^4 \, dA = \int_D x^4 \, dA + \int_D y^4 \, dA
\]
\[
= 2 \int_D x^4 \, dA = 8 \int_D x^4 \, dA.
\]

This integral is simpler to compute.
2. Consider the iterated triple integral

\[ \int_0^1 \int_y^1 f(x, y, z) \, dz \, dx \, dy. \]

In this integral, \( z \) is innermost, \( x \) is in the middle, and \( y \) is outermost.

(a) (5 points) Rewrite the integral with \( x \) innermost, \( y \) in the middle, and \( z \) outermost.

\[ E = \{ (x, y, z) \in \mathbb{R}^3 : 0 \leq y \leq 1, \ y \leq x \leq 1, \ 0 \leq z \leq y \} \]

Outer double integral is for \( dy \, dz \).

For each \( y \) and \( z \) : \( y \leq x \leq 1 \)

\[ \int_0^1 \int_y^1 f(x, y, z) \, dx \, dy \, dz \]
Note: You can also use the cross section method.

\[ E = \{ (x, y, z) \in \mathbb{R}^3 : 0 \leq y \leq 1, \ y \leq x \leq 1, \ 0 \leq z \leq y \} \]

Outermost integral is for \( dz \).

Bounds: \( 0 \leq z \leq y \leq x \leq 1 = 0 \leq z \leq 1 \).

Inner double integral is for \( dx \ dy \) (\( z \) fixed).

Boundary: \( 0 \leq y \leq 1, \ y \leq x \leq 1, \ z \leq y \).

\[ \int_0^1 \int_y^1 \int_z^1 f(x, y, z) \, dx \, dy \, dz \]

\( 0 \leq z \leq 1 \) on the line \( y = z \) lies between the lines \( y = 0 \) and \( y = 1 \).
(b) (5 points) Rewrite the integral with $z$ innermost, $y$ in the middle, and $x$ outermost.

Outer double integral is for $dy \, dx$.

For each $x$ and $y$: $0 \leq z \leq y$.

$$\int_0^1 \int_0^y f(x,y,z) \, dz \, dy \, dx$$

Note: You can also use the cross section method.

Outermost integral is for $dx$.

Bounds: $0 \leq y \leq x \leq 1 \Rightarrow 0 \leq x \leq 1$

Innermost integral is for $dz \, dy$ (x fixed).

Bounds: $0 \leq y \leq 1$, $y \leq x$, $0 \leq z \leq y$.

$$0 \leq y \leq x, \ 0 \leq z \leq y.$$
3. Let \( f(x, y) = x^4 + y^4 + 4xy \).

(a) (5 points) Find three critical points of \( f(x, y) \).

\[
\nabla f = (f_x, f_y) = (4x^3 + 4y, 4y^3 + 4x)
\]

\[
\nabla f = 0 \iff \begin{cases} 4x^3 + 4y = 0 & \Rightarrow y = -x^3 \\ 4y^3 + 4x = 0 & \Rightarrow x = -y^3 = x^9 \end{cases}
\]

\[
x = x^9 \iff x = -1, 0, 1.
\]

\[
y = -x^3 \Rightarrow (x, y) = (-1, 1), (0, 0), (1, -1)
\]

(b) (5 points) Pick one of the three critical points and classify it as local minimum, local maximum, or saddle.

\[
f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{2}{\partial x} (4x^3 + 4y) = 12x^2.
\]

\[
f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} (4x^3 + 4y) = 4
\]

\[
f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} (4y^3 + 4x) = 12y^2
\]

\[
H = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \det \begin{bmatrix} 12x^2 & 4 \\ 4 & 12y^2 \end{bmatrix}
\]

\[
= 144x^2y^2 - 16.
\]
At \((1, -1)\): \[ H = 144 \cdot (-1)^2 \cdot 1^2 - 16 > 0 \]

\[ f_{xx} = 12 > 0 \]

\(\Rightarrow\) A local minimum at \((-1, 1)\)

At \((0, 0)\): \[ H = 144 \cdot 0^2 - 0^2 - 16 < 0 \]

\(\Rightarrow\) A saddle point at \((0, 0)\)

At \((1, -1)\): \[ H = 144 \cdot (-1)^2 \cdot 1^2 - 16 > 0 \]

\[ f_{xx} = 12 > 0 \]

\(\Rightarrow\) A local minimum at \((1, -1)\)
4. (10 points) Find a critical point (you do not need to classify it as a local maximum or minimum) of
\[ f(x, y, z) = -x \log x - 2y \log y - 3z \log z \]
subject to the constraint
\[ g(x, y, z) = x + 2y + 3z - 1 = 0. \]
Evaluate \( f \) at that point. Here \( \log \) is the natural logarithm, as usual, so that \( \frac{d \log x}{dx} = \frac{1}{x} \).

\[ \nabla f = (f_x, f_y, f_z) = (-1 - \log x, -2 - 2 \log y, -3 - 3 \log z) \]

\[ f_x = \frac{d}{dx} (-x \log x) = -1 \cdot \log x - x \cdot \frac{1}{x} = -\log x - 1 \]

↑ product rule

Similar computation for \( f_y \) and \( f_z \)

\[ \nabla g = (g_x, g_y, g_z) = (1, 2, 3) \]

\[ \nabla f = \lambda \nabla g : (-1 - \log x, -2 - 2 \log y, -3 - 3 \log z) = \lambda (1, 2, 3) \]

\[ \begin{aligned}
-1 - \log x &= \lambda & \Rightarrow \log x &= -1 - \lambda & \Rightarrow x &= e^{-1-\lambda} \\
-2 - 2 \log y &= 2 \lambda & \Rightarrow \log y &= -1 - \lambda & \Rightarrow y &= e^{-1-\lambda} \\
-3 - 3 \log z &= 3 \lambda & \Rightarrow \log z &= -1 - \lambda & \Rightarrow z &= e^{-1-\lambda}
\end{aligned} \]

\[ x = y = z. \]

\[ g = 0 : x + 2y + 3z = 1 \Rightarrow x = y = z = \frac{1}{6}. \]

\( \Rightarrow \) A critical point is at \( (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}) \)

\[ f(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}) = -\frac{1}{6} \log (\frac{1}{6}) - \frac{2}{6} \log (\frac{1}{6}) - \frac{3}{6} \log (\frac{1}{6}) = -\log (\frac{1}{6}) = \log (6). \]
5. (10 points) Find the two points at which the parabola \( y = \frac{x^2}{2} \) intersects the circle \( x^2 + y^2 = 8 \). If \( D \) is the region bounded by that parabola and circle (see below), evaluate the double integral:

\[
\int \int_D x^2 y \, dx \, dy.
\]

The region of integration \( D \) looks as follows:

Note: Once you have a numerical answer you do not need to simplify it to a fraction. In the textbook, the area element \( dx \, dy \) in the integral is given as \( dA \).

**Semicircle:** \( x^2 + y^2 = 8 \), \( y \geq 0 \) \( \Rightarrow \) \( y = \sqrt{8-x^2} \).

**Intersection:** \( y = \frac{x^2}{2} \) and \( x^2 + y^2 = 8 \)

\( \Rightarrow \) \( x^2 = 2y \) and \( x^2 = 8-y^2 \)

\( \Rightarrow \) \( 2y = 8-y^2 \) \( \Rightarrow \) \( y = 2, -4 \)

\( \Rightarrow \) \( x = \pm \sqrt{2y} = \pm 2 \).

\( \Rightarrow \) \( (x, y) = (-2, 2), (2, 2) \)

\( D = \{ (x, y) \in \mathbb{R}^2 : -2 \leq x \leq 2, \ \frac{x^2}{2} \leq y \leq \sqrt{8-x^2} \} \).
\[ \iint_D x^2y \, dA = \int_{-2}^{2} \int_{y^2/2}^{\sqrt{8-x^2}} x^2y \, dy \, dx \]

\[ = \int_{-2}^{2} \frac{x^2y^2}{2} \bigg|_{y^2=\sqrt{8-x^2}} \, dx \]

\[ = \int_{-2}^{2} \frac{x^2(8-x^2)}{2} - \frac{x^6}{8} \, dx \]

\[ = \int_{-2}^{2} 4x^2 - \frac{x^4}{2} - \frac{x^6}{8} \, dx \]

\[ = \left[ \frac{4}{3}x^3 - \frac{x^5}{10} - \frac{x^7}{56} \right]_{x=-2}^{x=2} \]

\[ = \frac{1088}{105} \]

Note: \( D \) is symmetric about the \( y \)-axis, while the function \( x^2y \) is even in \( x \).

\[ \Rightarrow \iint_D x^2y \, dA = 2 \iint_{D_1} x^2y \, dA \]

where \( D_1 \) is the part of \( D \) on the first quadrant.
6. Consider the integral

\[ \int \int_D \frac{dx \, dy}{(x^2 + y^2)^{1/2}} \]

where \( D \) is the disc \((x-1)^2 + y^2 \leq 1\).

(a) (4 points) Describe the region of integration in polar coordinates.

\((x-1)^2 + y^2 = 1\): a disk of radius 1, centered at \((1,0)\).

Write the circle equation in polar coordinates:

\((x-1)^2 + y^2 = 1 \Rightarrow x^2 - 2x + 1 + y^2 = 1 \Rightarrow x^2 + y^2 = 2x \Rightarrow r^2 = 2r \cos \theta \Rightarrow r = 2 \cos \theta\)

 Bounds for \( \theta \) are given by the \( y \)-axis

\[ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}. \]

For each \( \theta \): \( 0 \leq r \leq 2 \cos \theta \)

(b) (6 points) Evaluate the integral.

\[ \int \int_D \frac{1}{\sqrt{x^2 + y^2}} \, dA = \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} \frac{1}{r} \cdot r \, dr \, d\theta \]

\[ = \int_{-\pi/2}^{\pi/2} 2 \sin \theta \mid_{\theta=-\pi/2}^{\theta=\pi/2} = 4. \]
7. (10 points) Sketch the sector of the unit disc bounded by the lines \( x = y, \ y = 0, \) and the circle \( x^2 + y^2 = 1 \) in the first quadrant of the \( x-y \) plane. Assuming constant density, find the \( x \) and \( y \) coordinates of the center of mass.

**Bounds for \( \theta \) are given by**

the lines \( y=0 \) and \( y=x \)

\( \theta_{\text{min}} = 0, \ \theta_{\text{max}} = \tan^{-1}(1) = \frac{\pi}{4} \)

\( \Rightarrow 0 \leq \theta \leq \frac{\pi}{4} \)

For each \( \theta : \ 0 \leq r \leq 1 \)

We may assume that the density \( \rho(x,y) = 1 \).

Mass \( m = \iint_D \rho(x,y) \, dA = \iint_D 1 \, dA \)

\( = \text{Area}(D) = \frac{\pi}{8}. \)

\( \bar{x} = \frac{1}{m} \iint_D x \rho(x,y) \, dA = \frac{1}{\pi/8} \iint_D x \, dA \)

\( = -\frac{1}{\pi/8} \int_0^1 \int_0^{\pi/4} r \cos \theta \cdot r \, dr \, d\theta \)

\( = \frac{8}{\pi} \int_0^{\pi/4} \int_0^1 r^2 \cos \theta \, dr \, d\theta \)

\( = \frac{8}{\pi} \int_0^{\pi/4} \frac{r^3}{3} \cos \theta \bigg|_{r=0}^{r=1} \, d\theta \)

\( = \frac{8}{\pi} \int_0^{\pi/4} \frac{1}{3} \cos \theta \, d\theta \)
\[
\bar{y} = \frac{1}{m} \iint_D y p(x, y) \, dA = \frac{1}{\pi^2} \iint_D y \, dA \\
= \frac{8}{\pi} \int_0^{\pi/4} \int_0^1 r \sin \theta \cdot r \, dr \, d\theta \\
= \frac{8}{\pi} \int_0^{\pi/4} \int_0^1 r^2 \sin \theta \, dr \, d\theta \\
= \frac{8}{\pi} \int_0^{\pi/4} \frac{r^3}{3} \sin \theta \bigg|_{r=0}^{r=1} \, d\theta \\
= \frac{8}{\pi} \int_0^{\pi/4} \frac{1}{3} \sin \theta \, d\theta \\
= \frac{8}{3\pi} \left(-\cos \theta \right) \bigg|_0^{\pi/4} \\
= \frac{8}{3\pi} \left(1 - \frac{\sqrt{2}}{2}\right).
\]

Center of mass: \( \left( \frac{4\sqrt{2}}{3\pi}, \frac{8}{3\pi} \left(1 - \frac{\sqrt{2}}{2}\right) \right) \)