1. [10 points] Consider the function $f(x, y) = x - y$ on the region $D$ in $\mathbb{R}^2$ defined by $x + y \geq 0$ and $x^2 + y^2 \leq 4$.
   a. [5 points] Sketch the region $D$.
   b. [5 points] Calculate the integral of $f$ over $D$.

   **Solu 1 (Polar coordinates)**

   \[
   \int_{\frac{3\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{2} (r\cos\theta - r\sin\theta) \cdot r \, dr \, d\theta
   = \int_{\frac{3\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{2} r^2 \cos\theta - r^2 \sin\theta \, dr \, d\theta
   \]

   $\theta_{\text{min}} = -\frac{\pi}{4}$

   $\theta_{\text{max}} = \frac{3\pi}{4}$

   $0 \leq r \leq 2$
\[ = \int_{-\pi/4}^{3\pi/4} \frac{r^3}{3} (\cos \theta - 5 \sin \theta) \, d\theta \]

\[ = \int_{-\pi/4}^{3\pi/4} \frac{8}{3} (\cos \theta - 5 \sin \theta) \, d\theta \]

\[ = \frac{8}{3} \left[ \sin \theta + \cos \theta \right]_{-\pi/4}^{3\pi/4} = 0. \]

**Solution 2 (Symmetry)**

A point \((a,b)\) reflects to \((b,a)\) along this line of symmetry.

\[ f(a,b) = a - b, \quad f(b,a) = b - a = -f(a,b) \]

These values "cancel out" in \( \iint_D f(x,y) \, dA \).

\[ = \iint_D f(x,y) \, dA = 0. \]

\( y = x \) : line of symmetry for \( D \).
2. [12 points] Find and classify the critical points for the function \( h(x, y) = x^4 + y^3 - 6y - 2x^2 \).

\[
h_x = 4x^3 - 4x, \quad h_y = 3y^2 - 6.
\]

\( \nabla h = (4x^3 - 4x, \ 3y^2 - 6) \)

\( \nabla h \rightarrow 0 \Rightarrow 4x^3 - 4x = 0 \ and \ 3y^2 - 6 = 0. \)

\( 4x^3 - 4x = 0 \Rightarrow x(x^2 - 1) = 0 \Rightarrow x = 0, \pm 1. \)

\( 3y^2 - 6 = 0 \Rightarrow y^2 = 2 \Rightarrow y = \pm \sqrt{2}. \)

**Critical points:** \((-1, \pm \sqrt{2}), \ (0, \pm \sqrt{2}), \ (1, \pm \sqrt{2})\)

**Use the 2nd derivative test.**

\[
h_{xx} = \frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x} (4x^3 - 4x) = 12x^2 - 4.
\]

\[
h_{xy} = \frac{\partial^2 h}{\partial y \partial x} = \frac{\partial}{\partial y} (4x^3 - 4x) = 0.
\]

\[
h_{yy} = \frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y} (3y^2 - 6) = 6y.
\]

**Hessian** \( D = h_{xx} \cdot h_{yy} - h_{xy}^2 = (12x^2 - 4) \cdot 6y - 0^2 = 24(3x^2 - 1)y. \)
At \((-1, \sqrt{2})\):

\[ D = 24 \cdot (3\cdot(-1)^2 - 1) \sqrt{2} > 0 \]

\[ h_{xx} = 12 \cdot (-1)^2 - 4 > 0 \]

\[ \text{loc. min at } (-1, \sqrt{2}) \]

At \((-1, -\sqrt{2})\):

\[ D = 24 \cdot (3\cdot(-1)^2 - 1) (-\sqrt{2}) < 0 \]

\[ \text{saddle pt at } (-1, -\sqrt{2}) \]

At \((0, \sqrt{2})\):

\[ D = 24 \cdot (3\cdot0^2 - 1) \sqrt{2} < 0 \]

\[ \text{saddle pt at } (0, \sqrt{2}) \]

At \((0, -\sqrt{2})\):

\[ D = 24 \cdot (3\cdot0^2 - 1) (-\sqrt{2}) > 0 \]

\[ h_{xx} = 12 \cdot 0^2 - 3 < 0 \]

\[ \text{loc. max at } (0, -\sqrt{2}) \]

At \((1, \sqrt{2})\):

\[ D = 24 \cdot (3\cdot1^2 - 1) \sqrt{2} > 0 \]

\[ h_{xx} = 12 \cdot 1^2 - 3 > 0 \]

\[ \text{loc. min at } (1, \sqrt{2}) \]

At \((1, -\sqrt{2})\):

\[ D = 24 \cdot (3\cdot1^2 - 1) (-\sqrt{2}) < 0 \]

\[ \text{saddle pt at } (1, -\sqrt{2}) \]
3. [15 points] The graph below is a plot of some of the level curves of a function $g$ in a rectangular region $R = [0.4, 5.8] \times [0.4, 5.8]$. Assume that as we move between adjacent level curves the value of $g$ increases or decreases by exactly one. The arrows point in the direction of $\nabla g$.

![Graph of level curves](image)

a. [9 points] Identify the approximate coordinates of the critical points of $g$ in $R$. For each critical point, indicate if it is a point where the function has a local maximum, a local minimum, or a saddle.

Recall: crit. pts typically look as follows:

- Local max/min
- Saddle point
For saddle pts, you can also look for "peanut shapes."

There will be an intersection of hidden level curves at (likely) a saddle pt.

For local max/mins, look at arrows to figure out whether it's a max or a min (i.e., if points in direction of increase)

e.g.

\[ \nabla g \text{ increasing toward } P, \text{ decreasing toward } Q \]

\[ \Rightarrow P \text{ is a local max, } Q \text{ is a local min.} \]
a loc. max
(∇g pointing in)

a loc. min
(∇g pointing out)

a saddle
(hidden intersection
(∇g pointing in & out)

(0.9, 0.8) : a loc. max
(3.2, 3) : a loc. min
(3.8, 4.3) : a saddle pt
(4.8, 4.9) : a loc. min.
b. [4 points] Identify the approximate coordinates of the points where the function $g$ attains its global maximum and global minimum over the rectangle $R$.

$R$: closed and bounded

I need to check crit. pts and boundary.

Note: consecutive levels differ by 1.

$k$ arrows point in direction of increase

$\frac{\nabla g}{\nabla f}$

$\Rightarrow$ We can find relative positions of all levels

E.g.

Level $B = \text{Level } A+1$, Level $C = \text{Level } D+1$.

Idea: Set one curve to be at level 0, and find levels for all other curves.

Then compare values at crit. pts and boundary.
$\star$ Value at $A$ is bigger than boundary because arrows point to $A$ from boundary.

$\implies A > \text{boundary} > C > B > D$.

$\implies$ global max at $A = (0.9, 0.8)$

global min at $D = (4.8, 4.9)$
c. [2 points] If the value of $g$ at its global minimum on $R$ is between 23 and 24, then the value of the global maximum of $g$ is between _____ and ______.

Global min at $(4.8, 4.9)$ and value between 23 and 24.

So in our picture from (b), levels must be shifted by 25.

Global max at $(0.9, 0.8)$:

This curve must be at level $g + 25 = 33$.

$\Rightarrow$ Value at $A$ must be between 33 and 34.
4. [10 points] The sides of the cube below have length six. The point \( a \) is at the midpoint of its edge. Let \( P \) be the plane that contains the points \( a, b, \) and \( c \).

![Diagram of a cube with labeled points]

a. [8 points] Set up, but do not evaluate, an integral for the volume of that part of the cube that lies below the plane \( P \).

b. [2 points] Calculate the volume of that part of the cube that lies below the plane \( P \).

\[
\begin{align*}
\text{Find the equation of the plane.} \\
\overrightarrow{ca} &= (6,0,-3), \quad \overrightarrow{cb} = (0,6,0) \\
\text{normal vector} \quad \overrightarrow{n} &= \overrightarrow{ca} \times \overrightarrow{cb} = (18,0,36) \\
The plane contains \quad c &= (0,0,6) \\
18(x-0) + 0(y-0) + 36(z-6) &= 0 \\
x + 2(z-6) &= 0 \\
z &= 6 - \frac{x}{2}.
\end{align*}
\]
We look at the solid under the graph 
\( z = 6 - \frac{x}{2} \) and above the rectangular domain 
\( R = [0, 6] \times [0, 6] \).

(a) Volume \( = \iiint_R 6 - \frac{x}{2} \, dA \)

\[ = \int_0^6 \int_0^6 6 - \frac{x}{2} \, dx \, dy \]

*Note*: Can also use the dydx order.

(b) Volume \( = \iiint_R 6 - \frac{x}{2} \, dx \, dy \)

\[ = \int_0^6 \int_0^6 6x - \frac{x^2}{4} \, dy \]

\[ = \int_0^6 6 \cdot 27 \cdot dy \]

\[ = 27 \cdot 6 = 162 \]
5. [12 points] Suppose $a, b$ are positive constants.
In this problem we will consider rectangles that are inscribed in the ellipse with equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). The sides of the inscribed rectangles are parallel to the coordinate axes, as in the figure below.

![Ellipse with inscribed rectangle](image)

a. [10 points] Using the method of Lagrange multipliers, find the rectangle of largest area that can be inscribed in the ellipse with equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). What is its area?
No credit given for solutions that do not use the method of Lagrange multipliers.

\[(x, y) = \text{vertex on 1st quadrant} \]
\[ \Rightarrow \text{Area} = 2x \cdot 2y = 4xy, \quad x > 0, y > 0 \]
\[\text{Maximize } f(x, y) = 4xy \text{ on } x, y > 0 \]
\[\text{subject to } g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.\]

\[\nabla f = (4y, 4x), \quad \nabla g = (\frac{2x}{a^2}, \frac{2y}{b^2}) \]

We solve the system
\[\nabla f = \lambda \nabla g \text{ and } g = 0\]
\[\Rightarrow (4y, 4x) = \lambda \left( \frac{2x}{a^2}, \frac{2y}{b^2} \right) \quad \text{(1)}\]
\[\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \text{(2)}\]
\((1)\) \quad \begin{align*} 4y &= \frac{2\lambda x}{a^2} \quad \Rightarrow \quad y = \frac{\lambda x}{2a^2} \quad (\text{**}) \\
4x &= \frac{2\lambda y}{b^2} \quad \Rightarrow \quad 4x = \frac{\lambda^2 x/a^2}{b^2} = \frac{\lambda^2 x}{a^2 b^2} \\
\Rightarrow \quad 4 &= \frac{\lambda^2}{a^2 b^2} \quad \Rightarrow \quad \lambda = \pm 2ab \\
x^2 + y^2 = 1 \\
\text{Constraint:} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{x^2}{a^2} + \frac{b^2 y^2 / a^2}{b^2} = 1 \\
\Rightarrow \quad \frac{2x^2}{a^2} = 1 \quad \Rightarrow \quad x = \frac{a}{\sqrt{2}} \quad \text{(c: } x > 0) \\
(\text{**}*) \quad y = \frac{b}{a} x = \frac{b}{a} \cdot \frac{a}{\sqrt{2}} = \frac{b}{\sqrt{2}} \\
\Rightarrow \quad (x, y) = \left( \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right) \text{ is the only solution.} \\
\Rightarrow \quad f_c \left( \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right) = 4 \cdot \frac{a}{\sqrt{2}} \cdot \frac{b}{\sqrt{2}} = 2ab \\
\Rightarrow \quad \boxed{\text{Max area of } 2ab \text{ at } \left( \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right)}
6. [15 points] Suppose \( g: \mathbb{R}^2 \to \mathbb{R} \) is a continuous function, and consider the iterated double integral
\[
\int_0^4 \int_{y^{1/2}}^2 g(x, y) \, dx \, dy.
\]
In this integral, \( x \) is innermost and \( y \) is outermost.

a. [5 points] Sketch the region of integration.

b. [5 points] Rewrite the integral with \( y \) innermost and \( x \) outermost.

c. [5 points] Evaluate the integral for \( g(x, y) = \sin(x^3 - 1) \).

\[(a) \quad \text{Domain } D: \quad 0 \leq y \leq 4, \quad \sqrt{y} \leq x \leq 2\]

(b) Describe \( D \) in the opposite order:
\[
D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2, \quad 0 \leq y \leq x^2\}
\]

\[
\Rightarrow \int_0^{x^2} \int_0^2 g(x, y) \, dy \, dx.
\]
(c) Use the integral from part (b).

\[ \int_0^2 \int_0^{x^2} \sin(x^3 - 1) \, dy \, dx \]

constant integral

\[ = \int_0^2 x^2 \sin(x^3 - 1) \, dx \]

\( u = x^3 - 1 \Rightarrow du = 3x^2 \, dx \)

\[ = \int_{-1}^7 \frac{1}{3} \sin u \, du \]

\[ = -\frac{1}{3} \cos u \bigg|_{-1}^7 = \frac{1}{3} \left( \cos(-1) - \cos(7) \right) \]
7. [15 points] Indicate if each of the following is true or false by circling the correct answer. No partial credit will be given.

a. [3 points] Suppose \( f: \mathbb{R}^2 \to \mathbb{R} \) is a differentiable function and \( (c, d) \in \mathbb{R}^2 \). If \( f_{xx}(c, d)f_{yy}(c, d) < [f_{xy}(c, d)]^2 \) then \( f \) has a saddle point at \( (c, d) \).

The statement is **False**.

This looks true, because of the second derivative test, but it is not. The issue is that the second derivative test applies only to critical points. So for this statement to be true, you need to assume that \( \nabla f(c, d) = \mathbf{0} \).

b. [3 points] Suppose \( h: \mathbb{R}^2 \to \mathbb{R} \) is a differentiable function. Suppose \( (a, b) \in \mathbb{R}^2 \) and \( C \) is the curve in \( \mathbb{R}^2 \) described by the equation \( h(x, y) = h(a, b) \). If \( \ell \) is the tangent line to \( C \) at \( (a, b) \), then \( \nabla h(a, b) \) is perpendicular to \( \ell \).

The statement is **True**.

This is in fact one of key properties of the gradient vector.

c. [3 points] If \( f: [0, 1] \to \mathbb{R} \) is continuous, then

\[
\int_0^1 \int_0^1 f(t)f(s) \, ds \, dt = \left[ \int_0^1 f(u) \, du \right]^2.
\]

The statement is **True**.

You can see this as follows:
\[ \int_0^1 \int_0^1 f(t) f(s) \, ds \, dt = \int_0^1 \int_0^1 f(t) \, ds \, dt \]

\[ \text{if } f(t) \text{ is constant for inner integral} \]

\[ = \int_0^1 \int f(s) ds \cdot \int_0^1 f(t) \, dt \]

\[ \text{if } \int_0^1 f(s) ds \text{ is constant} \]

\[ = \left( \int_0^1 f(u) du \right)^2 \]

\[ \text{two integrals are the same.} \]

d. [3 points] Suppose \( g: \mathbb{R}^3 \to \mathbb{R} \) is a differentiable function and \( k = (0, 0, 1) \). We have

\[ D_k g(x, y, z) = g_z(x, y, z). \]

**This statement is \boxed{True}.**

To see this, you compute

\[ D_k g(x, y, z) = \nabla g(x, y, z) \cdot k \]

\[ = (g_x(x, y, z), g_y(x, y, z), g_z(x, y, z)) \cdot (0, 0, 1) \]

\[ = g_z(x, y, z) \]
\[
\int_{-1}^{1} \int_{0}^{1} e^{x^2+y^2} \sin(x) \, dy \, dx = 0.
\]

The statement is **True**.

Note that the domain \( R = [-1, 1] \times [0, 1] \) is symmetric about the y-axis.

Also, the integrand \( f(x,y) = e^{x^2+y^2} \sin(x) \) is odd in \( x \).

\[
\implies \text{By symmetry, } \int_{R} f(x,y) \, dA = 0
\]

You can also see this as follows:

\[
\int_{-1}^{1} \int_{0}^{1} e^{x^2+y^2} \sin(x) \, dy \, dx = \int_{0}^{1} \int_{-1}^{1} e^{x^2+y^2} \sin(x) \, dx \, dy
\]

\[\text{Fubini} \]

\[
= \int_{0}^{1} e^{y^2} \int_{-1}^{1} e^{x^2} \sin(x) \, dx \, dy
\]

\[\text{For } e^{y^2} \text{ is constant for } dx \]

\[
= \int_{0}^{1} e^{y^2} \cdot 0 \, dy = 0.
\]
8. [11 points] Recall from your homework that the Law of Cosines states that for a triangle with sides of length \(a, b,\) and \(c\) we have \(c^2 = a^2 + b^2 - 2ab \cos(\theta)\) where \(\theta\) is the measure (in radians) of the angle opposite side \(c\). Thus, the Law of Cosines implicitly defines \(\theta\) as a function of the side lengths \(a, b,\) and \(c\).

![Triangle diagram](image)

a. [3 points] Compute \(\frac{\partial \theta}{\partial c}\).

Set \(f(a, b, c, \theta) = a^2 + b^2 - c^2 - 2ab \cos(\theta)\).

The Law of Cosines: \(f(a, b, c, \theta) = 0\).

By the Implicit function theorem,

\[
\frac{\partial \theta}{\partial c} = -\frac{f_c}{f_\theta} = -\frac{-2c}{2ab \sin(\theta)} = \frac{c}{ab \sin(\theta)}
\]

b. [3 points] Compute \(\frac{\partial \theta}{\partial a}\).

By the Implicit function theorem,

\[
\frac{\partial \theta}{\partial a} = -\frac{f_a}{f_\theta} = -\frac{2a - 2bc \cos(\theta)}{2ab \sin(\theta)} = \frac{bc \cos(\theta) - a}{ab \sin(\theta)}
\]

c. [5 points] Suppose the lengths of the sides of the triangle (measured in meters) are changing as a function of time (measured in seconds) according to the rules \(a(t) = 3 + t,\) \(b(t) = 3,\) and \(c(t) = 3 + 2t\). What is \(\frac{d\theta}{dt}\) at time \(t = 1\)?

Consider \(\theta\) as a function of \(a, b, c\).

\[
\left. \frac{d\theta}{dt} \right|_{t=1} = \frac{\partial \theta}{\partial a} \cdot \frac{da}{dt} + \frac{\partial \theta}{\partial b} \cdot \frac{db}{dt} + \frac{\partial \theta}{\partial c} \cdot \frac{dc}{dt}
\]

\(\text{Chain rule}\)
\[
\frac{da}{dt} = \frac{1}{3t} (3+t) = 1, \\
\frac{db}{dt} = \frac{1}{3t} (3) = 0. \\
\frac{dc}{dt} = \frac{1}{3t} (3+2t) = 2. \\
\]

\[
4 \frac{d\theta}{dt} = \frac{d\theta}{da} + 2 \frac{d\theta}{dc} \\
= \frac{b \cos \theta - a}{abs \sin \theta} + \frac{2c}{abs \sin \theta} \\
\]

\[
(a) \quad (b) \\
\]

At \( t = 1 \), \( a(1) = 4, \ b(1) = 3, \ c(1) = 5 \).

\[
\cos \theta = \frac{a^2 + b^2 - c^2}{2ab} = \frac{4^2 + 3^2 - 5^2}{2 \cdot 4 \cdot 3} = 0 \\
\]

\[
\Rightarrow \theta = \frac{\pi}{2} \Rightarrow \sin \theta = 1. \\
\]

\[
(\text{#}) \quad \frac{d\theta}{dt} = \frac{3 \cdot 0 - 4}{4 \cdot 3 \cdot 1} + \frac{2 \cdot 5}{3 \cdot 4 \cdot 1} = \frac{1}{2} \text{ (rad/sec)} \\
\]