16.6. Parametric surfaces

Def (1) A **parametric surface** is an object parametrized by a vector function of two variables.

(2) A **grid curve** of a vector function \( \mathbf{r}(u,v) \) is given by setting either \( u \) or \( v \) to be constant.

E.g. The cylinder \( x^2 + y^2 = 1 \) is parametrized by

\[
\mathbf{r}(\theta, z) = (\cos \theta, \sin \theta, z)
\]

- \( \theta \) constant \( \Rightarrow \) vertical lines
- \( z \) constant \( \Rightarrow \) circles.

\[\star\] **Note** The graph \( z = f(x,y) \) is parametrized by
\[
\mathbf{r}(x,y) = (x, y, f(x,y)) \quad "xy\text{-parametrization}".
\]

**Prop** Consider a vector function
\[
\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)),
\]

(1) The partial derivatives of \( \mathbf{r}(u,v) \) are
\[
\mathbf{r}_u = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \quad \text{and} \quad \mathbf{r}_v = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right).
\]

\[\star\] (2) If a surface \( S \) is parametrized by \( \mathbf{r}(u,v) \), then the tangent plane to \( S \) has a normal vector \( \mathbf{r}_u \times \mathbf{r}_v \).

\( \mathbf{r}_u \) and \( \mathbf{r}_v \) are tangent vectors of the grid curves.
Sketch the surface parametrized by
\[ \mathbf{r}(u,v) = (2u \cos v, 2u \sin v, v) \] with \( 1 \leq u \leq 3, \ 0 \leq v \leq \pi \).

**Sol:**

Idea: Sketch grid curves.

\[ u=1 \Rightarrow \mathbf{r}(1,v) = (2 \cos v, 2 \sin v, v) \]
\[ \sim \text{a helix from } (-2,0,0) \text{ to } (2,0,\pi) \]

\[ u=2 \Rightarrow \mathbf{r}(2,v) = (4 \cos v, 4 \sin v, v) \]
\[ \sim \text{a helix from } (-4,0,0) \text{ to } (4,0,\pi) \]

\[ u=3 \Rightarrow \mathbf{r}(3,v) = (6 \cos v, 6 \sin v, v) \]
\[ \sim \text{a helix from } (-6,0,0) \text{ to } (6,0,\pi) \]

\[ v=0 \Rightarrow \mathbf{r}(u,0) = (-u,0,0) \]
\[ \sim \text{a line segment from } (-1,0,0) \text{ to } (-3,0,0) \]

\[ v=\frac{\pi}{3} \Rightarrow \mathbf{r}(u,\frac{\pi}{3}) = \left( \frac{u}{2}, \frac{\sqrt{3}}{2} u, \frac{\pi}{3} \right) \]
\[ \sim \text{a line segment from } \left( \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3} \right) \text{ to } \left( \frac{3}{2}, \frac{3\sqrt{3}}{2}, \frac{\pi}{3} \right) \]

\[ v=\frac{2\pi}{3} \Rightarrow \mathbf{r}(u,\frac{2\pi}{3}) = \left( -\frac{u}{2}, \frac{\sqrt{3}}{2} u, \frac{2\pi}{3} \right) \]
\[ \sim \text{a line segment from } \left( \frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{5\pi}{3} \right) \text{ to } \left( \frac{3}{2}, -\frac{3\sqrt{3}}{2}, \frac{5\pi}{3} \right) \]

\[ v=\pi \Rightarrow \mathbf{r}(u,\pi) = (-u,0,\pi) \]
\[ \sim \text{a line segment from } (-1,0,\pi) \text{ to } (-3,0,\pi) \]
Ex: Find a parametrization of the hemisphere \( x^2 + y^2 + z^2 = 4 \) with \( z \geq 0 \).

**Sol 1**

\[
x^2 + y^2 + z^2 = 4 \implies z = \sqrt{4 - x^2 - y^2} \quad (z \geq 0)
\]

The shadow on the xy-plane is given by \( x^2 + y^2 \leq 4 \).

\[
\mathbf{r}(x,y) = (x, y, \sqrt{4-x^2-y^2}) \quad \text{with} \quad x^2 + y^2 \leq 4.
\]

**Sol 2**

In cylindrical coordinates:

\[
x^2 + y^2 + z^2 = 4 \implies r^2 + z^2 = 4 \implies z = \sqrt{4-r^2} \quad (z \geq 0)
\]

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad z = \sqrt{4-r^2}
\]

The shadow on the xy-plane: \( 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2 \)

\[
\mathbf{s}(r, \theta) = (r \cos \theta, r \sin \theta, \sqrt{4-r^2}) \quad \text{with} \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2
\]

**Sol 3**

In spherical coordinates:

\[
x^2 + y^2 + z^2 = 4 \implies \rho^2 = 4 \implies \rho = 2.
\]

\[
x = 2 \sin \phi \cos \theta, \quad y = 2 \sin \phi \sin \theta, \quad z = 2 \cos \phi
\]

\[
\phi \geq 0 \implies 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{2}
\]

\[
\mathbf{t}(\theta, \phi) = (2 \sin \phi \cos \theta, 2 \sin \phi \sin \theta, 2 \cos \phi)
\]

with \( 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{2} \)
Ex Find an equation of the tangent plane to the paraboloid 
\[ z = x^2 + y^2 \] at \((1, 1, 2)\).

\(\textbf{Sol 1} \) (Using the gradient)

\[ z = x^2 + y^2 \Rightarrow x^2 + y^2 - z = 0 \]

is a level surface of \(f(x, y, z) = x^2 + y^2 - z\).

\[ \nabla f = (f_x, f_y, f_z) = (2x, 2y, -1) \]

A normal vector is \(\nabla f(1, 1, 2) = (2, 2, -1)\)

The tangent plane at \((1, 1, 2)\) is given by

\[ 2(x-1) + 2(y-1) - (z-2) = 0 \]

\(\textbf{Sol 2} \) (Using a parametrization)

The paraboloid \(z = x^2 + y^2\) is parametrized by

\[ \vec{r}(x, y) = (x, y, x^2 + y^2) \]

\[ \Rightarrow \vec{r}_x = (1, 0, 2x), \quad \vec{r}_y = (0, 1, 2y) \]

\[ \Rightarrow \vec{r}_x \times \vec{r}_y = (-2x, -2y, 1) \]

At \((1, 1, 2)\):

\[ \vec{r}_x \times \vec{r}_y = (-2, -2, 1) \]

The tangent plane at \((1, 1, 2)\) is given by

\[ -2(x-1) - 2(y-1) + (z-2) = 0 \]

\(\text{Note} \) You can also use a cylindrical parametrization

\[ \vec{S}(r, \theta) = (r \cos \theta, r \sin \theta, r^2) \] with \(r = \sqrt{2}, \theta = \frac{\pi}{4}\) at \((1, 1, 2)\).

However, the computation is quite tedious.
Ex. Let $S$ be the surface parametrized by

$$F(u,v) = (u^3+1, v^2+1, u+v)$$ with $u,v > 0$.

(1) Find an equation of the tangent plane to $S$ at $(2,5,3)$

**Sol.** $F_u = (3u^2, 0, 1)$ and $F_v = (0, 2v, 1)$

$$\Rightarrow F_u \times F_v = (-2v, -3u^2, 6uv)$$

Find $u$ and $v$ at $(2,5,3)$.

$$F(u,v) = (2,5,3)$$

$$\Rightarrow u^3 + 1 = 2, v^2 + 1 = 5, u + v = 3 \Rightarrow u = 1, v = 2$$

$v = -2$ works for the second equation, but not for the last equation.

A normal vector is $F_u \times F_v = (-4, -3, 12)$.

The tangent plane at $(2,5,3)$ is given by

$$-4(x-2)-3(y-5)+12(z-3) = 0$$

(2) Find all points on $S$ where the tangent plane is parallel to the $xy$-plane.

**Sol.** If the tangent plane is parallel to the $xy$-plane, the normal vector $F_u \times F_v = (-2v, -3u^2, 6uv)$ must be parallel to $k = (0,0,1)$

$$\Rightarrow -2v = 0, -3u^2 = 0 \Rightarrow u,v > 0 \Rightarrow \text{no solutions}$$