12.4. The cross product

Def (1) The determinant of a matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is
\[
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} := ad - bc
\]

(2) The cross product (or vector product) of \( \mathbf{v} = (a_1, b_1, c_1) \) and \( \mathbf{w} = (a_2, b_2, c_2) \) is
\[
\mathbf{v} \times \mathbf{w} := (b_1 c_2 - b_2 c_1, -a_1 c_2 + a_2 c_1, a_1 b_2 - a_2 b_1)
\]
\[
= \left( \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right)
\]
\[
\text{don't forget}
\]

Rmk (1) The output of a cross product is a vector.
(2) The cross product is defined only for 3-dimensional vectors.

Prop (Algebraic properties of the cross product)

(1) \( \mathbf{v} \times \mathbf{v} = \mathbf{0} \)

★(2) \( \mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v} \)

(3) \( \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \)

(4) \( (r \mathbf{v}) \times \mathbf{w} = r (\mathbf{v} \times \mathbf{w}) \) for any number \( r \)

(5) \( \mathbf{v} \times \mathbf{0} = \mathbf{0} \)
Thm (1) \(|\vec{v} \times \vec{w}| = |\vec{v}| |\vec{w}| \sin \theta|

where \(\theta\) is the angle between \(\vec{v}\) and \(\vec{w}\).

\(\star\) (2) The direction of \(\vec{v} \times \vec{w}\) is perpendicular to both \(\vec{v}\) and \(\vec{w}\), given by the right hand rule.

\(\star\) Cor The area of the triangle ABC is

\[ \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} |\vec{BA} \times \vec{BC}| = \frac{1}{2} |\vec{CA} \times \vec{CB}|. \]

\(\star\star\) Thm \(\vec{v}\) and \(\vec{w}\) are perpendicular if and only if \(\vec{v} \cdot \vec{w} = 0\).

(Explanation: \(\vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \theta\)

\[ \vec{v} \cdot \vec{w} = 0 \iff \cos \theta = 0 \iff \theta = \frac{\pi}{2} \)
Ex Find two unit vectors which are orthogonal to \( \vec{v} = (-2, 0, 2) \)

\( \| \vec{v} \| \) perpendicular

Sol 1 (Cross product)

Idea: Choose a random vector \( \vec{w} \) and take the unit vector of \( \vec{v} \times \vec{w} \).

Take \( \vec{w} = (1, 0, 0) \)

\( \Rightarrow \) \( \vec{v} \times \vec{w} = (-2, 0, 2) \times (1, 0, 0) = (0, 2, 0) \)

\( 1 \| \vec{v} \times \vec{w} \| = \sqrt{0^2 + 2^2 + 0^2} = 2 \).

The unit vectors are

\[ \pm \frac{\vec{v} \times \vec{w}}{\| \vec{v} \times \vec{w} \|} = \pm \frac{1}{2} (0, 2, 0) = \pm (0, 1, 0) \]

Note This method works as long as you choose a vector \( \vec{w} \) which is not parallel to \( \vec{v} \).

If \( \vec{w} \) is parallel to \( \vec{v} \), then you get \( \vec{v} \times \vec{w} = \vec{0} \) and thus find no unit vectors.

To see whether two vectors are parallel, you compare the ratios of their coordinates.

e.g. \((2,1,3)\) is parallel to \((4,2,6)\), but not to \((6,3,4)\)
Sol 2 (Dot product)

Idea: Find a vector \( \vec{w} \) with \( \vec{v} \cdot \vec{w} = 0 \), then take the unit vector of \( \vec{w} \).

Set \( \vec{w} = (a, b, c) \).

\[ \Rightarrow \vec{v} \times \vec{w} = (2, 0, -2) \cdot (a, b, c) = 2a - 2c. \]

We want \( \vec{v} \cdot \vec{w} = 0 \)

\[ \Rightarrow 2a - 2c = 0 \quad \Rightarrow \quad a = c. \]

Take \( a = c = 0, \ b = 1 \)

\( \Rightarrow \vec{w} = (0, 1, 0) \)

\[ |\vec{w}| = \sqrt{0^2 + 1^2 + 0^2} = 1. \]

The unit vectors are

\[ \pm \frac{13}{|13|} = \pm (0, 1, 0) \]

Note: This method works for any \( \vec{w} \) with \( \vec{v} \cdot \vec{w} = 0 \). For example, you may set \( a = c = 2, \ b = 1 \) and get \( \vec{w} = (2, 1, 2) \), which yields \[ \pm \frac{13}{|13|} = \pm \frac{1}{3} (2, 1, 2) \]
Ex Consider the points \( P = (1, 1, 1) \), \( Q = (3, 1, 2) \), \( R = (1, 4, 0) \).

(1) Find the two unit vectors which are perpendicular to both \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \).

\[ \text{Sol} \] We take the unit vector of \( \overrightarrow{PQ} \times \overrightarrow{PR} \)
\[ \text{(\because \overrightarrow{PQ} \times \overrightarrow{PR} \text{ is perpendicular to both } \overrightarrow{PQ} \text{ and } \overrightarrow{PR})} \]
\[ \overrightarrow{PQ} = (2, 0, 1), \quad \overrightarrow{PR} = (0, 3, -1) \]
\[ \overrightarrow{PQ} \times \overrightarrow{PR} = (2, 0, 1) \times (0, 3, -1) = (-3, 2, 6) \]
\[ |\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-3)^2 + 2^2 + 6^2} = 7 \]
\[ \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \pm \frac{1}{7} (-3, 2, 6) \]

Note You can also solve this problem using dot products, but with much more work.

(2) Find the area of the triangle \( PQR \).

\[ \text{Sol} \] Area = \( \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{7}{2} \]
Addendum for HW 1

The volume of the parallelepiped given by \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) is \( |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \)