Def Let $D$ be a domain in $\mathbb{R}^2$ with boundary $\partial D$.

(1) $\partial D$ is **simple** if it has no self-intersections.

![Simple vs. not simple boundaries](image)

(2) $\partial D$ is positively oriented if it travels in a way that the interior of $D$ lies on the left side.

![Orientation of boundary](image)

**Thm (Green's theorem)**

Let $\mathbf{F} = (P, Q)$ be a differentiable vector field on a domain $D$.

If the boundary $\partial D$ is simple and positively oriented, then

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

On the formula sheet.

**Note** Green's theorem is useful for computing $\int_C \mathbf{F} \cdot d\mathbf{r}$ when

- $C$ is (almost) a loop in $\mathbb{R}^2$
- $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is easy to integrate
Consider the vector field \( \mathbf{F}(x,y) = (x^4 + 4xy^2, 3x^2 - 7y^5) \).

Find \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is the triangular curve with vertices at \((0,0)\), \((1,0)\), and \((1,1)\), oriented counter-clockwise.

**Solution**

\[ D : \text{the region enclosed by } C \]
\[ \Rightarrow \partial D = C \text{ is positively oriented.} \]
\[ D \text{ is given by } 0 \leq x \leq 1, \quad 0 \leq y \leq x. \]

\[ P = x^4 + 4xy^2, \quad Q = 3x^2 - 7y^5 \]
\[ \Rightarrow \frac{\partial P}{\partial y} = 8xy, \quad \frac{\partial Q}{\partial x} = 6x \]

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA \]

Green's thm

\[ = \int_0^1 \int_0^x 6x - 8xy \, dy \, dx = \int_0^1 6xy - 4xy^2 \bigg|_{y=0}^{y=x} \, dx \]

\[ = \int_0^1 6x^2 - 4x^3 \, dx = 2x^3 - x^4 \bigg|_{x=0}^{x=1} = 1 \]

**Note** This solution is very simple compared to a direct computation of the integral over each line segment using parametrizations.
Ex Consider the vector field \( \mathbf{F}(x,y) = (2x^3+y, 2x-3y^4) \)

Find \( \int_{c} \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is the upper half of the circle \((x-1)^2 + y^2 = 4\) with counterclockwise orientation.

**Sol**

\( C' \): the line segment from \((-1,0)\) to \((3,0)\)

\( D \): the region enclosed by \( C \) and \( C' \).

\( \Rightarrow \partial D = C + C' \) is positively oriented.

\[ P = 2x^3 + y, \quad Q = 2x - 3y^4 \Rightarrow \frac{\partial P}{\partial y} = 1, \quad \frac{\partial Q}{\partial x} = 2 \]

\[ \int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{c} \mathbf{F} \cdot d\mathbf{r} + \int_{c'} \mathbf{F} \cdot d\mathbf{r} \]

\( \Rightarrow \int_{c} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r} - \int_{c'} \mathbf{F} \cdot d\mathbf{r} \)  \((*)\)

\[ \int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_{D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \iint_{D} 1 \, dA = \text{Area}(D) = \frac{1}{2} \pi \cdot 2^2 = 2\pi \]

**Green's thm**

\( C' \) is parametrized by \( \mathbf{r}(t) = (t,0) \) with \(-1 \leq t \leq 3\).

\[ \mathbf{F}(\mathbf{r}(t)) = (2t^3 + 0, 2t - 3 \cdot 0^4) = (2t^3, 2t) \]

\[ \mathbf{r}'(t) = (1,0) \]

\( \Rightarrow \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 2t^3 \cdot 1 + 2t \cdot 0 = 2t^3. \)

\( \Rightarrow \int_{c'} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{3} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_{-1}^{3} 2t^3 \, dt = \frac{t^4}{2} \bigg|_{t=-1}^{3} = 40 \]

\[ \int_{c} \mathbf{F} \cdot d\mathbf{r} = 2\pi - 40 \]  \((*)\)
Ex Consider the vortex field \( \mathbf{V}(x, y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \).

Let \( C \) be a simple loop in \( \mathbb{R}^2 \), oriented counterclockwise.

(1) Find \( \int_C \mathbf{V} \cdot d\mathbf{r} \) when \( C \) does not enclose the origin.

Sol

\[ D : \text{the region enclosed by } C \]

\( \Rightarrow \) \( \partial D = C \) is positively oriented.

\( \mathbf{V} \) is defined on \( D \).

(\( D \) does not contain the origin)

\[ \begin{align*}
P &= -\frac{y}{x^2+y^2}, & Q &= \frac{x}{x^2+y^2} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \\
\text{Lecture 32} & \quad \text{(d)} \quad \text{Greens thm} \\
\int_C \mathbf{V} \cdot d\mathbf{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA &= 0
\end{align*} \]

Note You can get the same answer using the fundamental theorem for line integrals. In fact, since \( D \) is simply connected, the vortex field \( \mathbf{V} \) is conservative on \( D \) by the relation \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \).
(2) Find \( \int_C \mathbf{V} \cdot d\mathbf{r} \) when \( C \) encloses the origin.

**Sol**

*We can’t consider the region enclosed by \( C \) since \( \mathbf{V} \) is not defined at the origin.*

\[ C' \]: a circle centered at the origin which encloses \( C \) with counterclockwise orientation.

\( D \): the region bounded by \( C \) and \( C' \)

\( \Rightarrow \partial D = -C + C' \) is positively oriented (\( C \) is negatively oriented)

\( \mathbf{V} \) is defined on \( D \). (\( D \) does not contain the origin)

\[ \int_{\partial D} \mathbf{V} \cdot d\mathbf{r} = -\int_C \mathbf{V} \cdot d\mathbf{r} + \int_{C'} \mathbf{V} \cdot d\mathbf{r} \]

\( \Rightarrow \int_C \mathbf{V} \cdot d\mathbf{r} = -\int_{\partial D} \mathbf{V} \cdot d\mathbf{r} + \int_{C'} \mathbf{V} \cdot d\mathbf{r} \) (\( \ast \))

\[ \int_{\partial D} \mathbf{V} \cdot d\mathbf{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = 0 \]

Green’s thm

\( \Rightarrow \int_C \mathbf{V} \cdot d\mathbf{r} = \int_{C'} \mathbf{V} \cdot d\mathbf{r} = 2\pi \) (\( \ast \))

Lecture 31