Note: There are two ways to set up a triple integral.

1) Projection method
   Step 1. Choose the innermost variable.
   Step 2. Find the bounds for the outer double integral by looking at the shadow.
   Step 3. Find the bounds for the innermost integral.

2) Cross section method
   Step 1. Choose the outermost variable.
   Step 2. Find the bounds for the outermost integral by looking at extreme values.
   Step 3. Find the bounds for the inner double integral by looking at the cross section.

Remark: The projection method requires a 3-dimensional sketch, but with minimum algebra.
The cross section method only requires a 2-dimensional sketch, but with heavy algebra.
For the exam, you can use the method of your preference.
Ex. Let \( E \) be the solid bounded by the planes \( x=0, \ y=0, \ x+y+z=2 \). Express the volume of \( E \) as a triple integral in the order \( dy \, dx \, dz \).

**Sol.** (Projection method)

\[
\begin{align*}
\text{Shadow} & \quad \text{x+y+z=2 : a plane} \\
\text{E} & \quad \begin{cases}
\text{x-intercept} = 2 \\
\text{y-intercept} = 2 \\
\text{z-intercept} = 2.
\end{cases}
\end{align*}
\]

For the outer double integral, look at the shadow \( D \) on the \( xz \)-plane:

\[
\begin{align*}
\text{The bounds are given by} & \\
0 \leq z \leq 2, & \quad 0 \leq x \leq 1 - \frac{z}{2}.
\end{align*}
\]

For each point on \( D \): \( x \leq y \leq 2 - x - z \).

\[
\Rightarrow \text{Vol}(E) = \iiint_E 1 \, dV = \int_0^2 \int_0^{1 - \frac{z}{2}} \int_x^{2 - x - z} 1 \, dy \, dx \, dz.
\]
Sol 2 (Cross section method)

The outermost integral is with respect to $dz$.
The boundary equations: $x=0$, $z=0$, $x+y+z=2$.
The extreme values occur at the intersections:
$x=0$, $x=y$, $x+y+z=2$ \Rightarrow $z=2$.
The bounds for $z$ are $0 \leq z \leq 2$.
For the inner double integral, look at the cross section with constant $z$.
The boundary equations: $x=0$, $x=y$, $x+y=2-z$.

\[ \text{Intersection: } x=y \text{ and } x+y=2-z. \]
\[ \Rightarrow x=y=1-\frac{z}{2}. \]
The bounds are given by
$0 \leq x \leq 1-\frac{z}{2}$, $x \leq y \leq 2-x-z$.

\[ \Rightarrow \text{Vol}(E) = \iiint_E 1 \, dV = \int_0^2 \int_{-\frac{3}{2}}^{2-x-z} \int_x^{2-z} 1 \, dy \, dx \, dz. \]
Ex Rewrite the integral \( \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} 1 \, dz \, dy \, dx \) in the order \( dy \, dx \, dz \).

Sol I (Projection method)

The bounds are \(-1 \leq x \leq 1, \ x^2 \leq y \leq 1, \ 0 \leq z \leq 1-y\).

For the outer double integral, look at the shadow D on the \(xz\)-plane:

The curve is given by \( y = x^2 \) and \( z = 1-y \)

\( \Rightarrow z = 1-x^2 \) or \( x = \pm \sqrt{1-z} \)

The bounds are given by \( 0 \leq z \leq 1, \ -\sqrt{1-z} \leq x \leq \sqrt{1-z} \)

For each point on D: \( x^2 \leq y \leq 1-z \).

\[ \Rightarrow \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} 1 \, dz \, dy \, dx = \int_{0}^{1} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} 1 \, dy \, dx \, dz \]
Sol 2 (Cross section method)

The bounds are \(-1 \leq x \leq 1, x^2 \leq y \leq 1, 0 \leq z \leq 1-y\). The outermost integral is with respect to \(dz\).

\[ \Rightarrow 0 \leq z \leq 1-y \leq 1-x^2 \leq 1 \Rightarrow 0 \leq z \leq 1. \]

For the inner double integral, look at the cross section with constant \(z\).

The boundary conditions: \(-1 \leq x \leq 1, x^2 \leq y \leq 1, y \leq 1-z\).

Intersections: \(y = x^2\) and \(y = 1-z\)

\[ \Rightarrow x = \pm \sqrt{1-z}. \]

The bounds are given by \(-\sqrt{1-z} \leq x \leq \sqrt{1-z}, x^2 \leq y \leq 1-z\).

\[ \Rightarrow \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} 1 \, dzdydx = \int_{0}^{1} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} 1 \, dydxdz \]