14.8. Lagrange multipliers

\*Thm (Method of Lagrange multipliers)

Given a differentiable function \( f(x,y,z) \) with a constraint \( g(x,y,z) = 0 \), the extreme values can be found as follows if they exist.

Step 1. Solve the equations \( \nabla f = \lambda \nabla g \) and \( g = 0 \).

Step 2. Evaluate \( f(x,y,z) \) at all solutions.

Step 3. Compare all values from Step 2.

Maximum = the largest of these values
Minimum = the smallest of these values

\* Explanation:

At extrema, level curves (or surfaces) of \( f \) and \( g \) must be tangent.

\[ \implies \nabla f \text{ and } \nabla g \text{ are parallel} \]

\[ \implies \nabla f = \lambda \nabla g \]

Note (1) In Math 215, you can always assume that extreme values exist. In general, however, there may be no extreme values.

(2) Step 1 often involves heavy algebra.
Ex: Find the minimum value of \( x^2 + y^2 + z \) on the plane \( 2x + 2y + z = 9 \).

\[ \text{Sol 1 (Lagrange multipliers)} \]
Constraint: \( 2x + 2y + z = 9 \) \( \Rightarrow \) \( 2x + 2y + z - 9 = 0 \)
Set \( f(x, y, z) = x^2 + y^2 + z \) and \( g(x, y, z) = 2x + 2y + z - 9 \).
Solve \( \nabla f = \lambda \nabla g \) and \( g = 0 \)
\( \Rightarrow (2x, 2y, 1) = \lambda (2, 2, 1) \) and \( 2x + 2y + z - 9 = 0 \)
\( \Rightarrow 2x = 2\lambda, \ 2y = 2\lambda, \ 1 = \lambda \), \( 2x + 2y + z = 9 \)
\( \Rightarrow \lambda = 1, \ x = 1, \ y = 1, \ z = 9 - 2x - 2y = 5 \)
The minimum value is \( f(2, 2, 5) = 7 \)

\[ \text{Sol 2 (Removing the constraint)} \]
\( 2x + 2y + z = 9 \) \( \Rightarrow \) \( z = 9 - 2x - 2y \)
\( \Rightarrow x^2 + y^2 + z = x^2 + y^2 + 9 - 2x - 2y \).
We find the minimum of \( h(x, y) = x^2 + y^2 + 9 - 2x - 2y \) on \( \mathbb{R}^2 \).
Since \( \mathbb{R}^2 \) is open, the minimum is at a critical point.
\( \nabla h = (0, 0) \) \( \Rightarrow (2x - 2, 2y - 2) = 0 \) \( \Rightarrow x = 1, \ y = 1 \).
\( \Rightarrow \) The minimum is \( h(1, 1) = 7 \)

Note: It's not always possible to remove the constraint in this way.
Ex Find the shortest distance from the origin to the surface 
\[ 2x + 4y + z^2 = 20. \]

Sol Distance from the origin is \( \sqrt{x^2 + y^2 + z^2}. \)

We find the minimum of \( x^2 + y^2 + z^2 \) subject to the constraint 
\[ 2x + 4y + z^2 - 20 = 0. \]

Set \( f(x, y, z) = x^2 + y^2 + z^2 \) and \( g(x, y, z) = 2x + 4y + z^2 - 20 = 0 \)

Solve \( \nabla f = \lambda \nabla g \) and \( g = 0 \)

\[ (2x, 2y, 2z) = \lambda (2, 4, 2z) \]

\[ 2x + 4y + z^2 - 20 = 0 \]

\[ 2x = 2\lambda, \ 2y = 4\lambda, \ 2z = 2\lambda z, \ 2x + 4y + z^2 = 20. \]

Case 1 \( z = 0 \):

\[ x = \lambda, \ y = 2\lambda, \ 2x + 4y = 20 \]

\[ \Rightarrow 2\lambda + 4 \cdot 2\lambda = 20 \Rightarrow 10\lambda = 20 \Rightarrow \lambda = 2 \]

\[ \Rightarrow x = 2, \ y = 4, \ z = 0. \]

Case 2 \( z \neq 0 \):

\[ 2z = 2\lambda z \Rightarrow \lambda = 1. \]

\[ 2x = 2\lambda = 2 \Rightarrow x = 1, \ 2y = 4\lambda = 4 \Rightarrow y = 2 \]

\[ 2x + 4y + z^2 = 20 \Rightarrow 10 + z^2 = 20 \Rightarrow z = \pm \sqrt{10} \]

\[ \Rightarrow x = 1, \ y = 2, \ z = \pm \sqrt{10}. \]

\[ f(2, 4, 0) = 20, \ f(1, 2, \sqrt{10}) = f(1, 2, -\sqrt{10}) = 15 \]

\[ \Rightarrow \text{The minimum of } f(x, y, z) = x^2 + y^2 + z^2 \text{ is } 15. \]

\[ \Rightarrow \text{The shortest distance is } \sqrt{15}. \]
Ex Find the extreme values of $6x + 8y$ on the domain given by $x^2 + y^2 \leq 25$.

\[ D \text{ is closed and bounded.} \]

We apply the Extreme value theorem with $f(x,y) = 6x + 8y$

\[ \nabla f = (6, 8) \neq (0,0) \Rightarrow \text{no critical points.} \]

Step 1. Evaluate $f(x,y)$ at all critical points.

\[ \nabla f = (6, 8) = (0,0) \Rightarrow \text{no critical points.} \]

Step 2. Find the extrema of $f(x,y)$ on the boundary.

Set $g(x,y) = x^2 + y^2 - 25 \Rightarrow g(x,y) = 0$ on the boundary.

Solve $\nabla f = \lambda \nabla g$ and $g = 0$

$\Rightarrow (6, 8) = \lambda (2x, 2y)$ and $x^2 + y^2 - 25 = 0$

$\Rightarrow 6 = 2\lambda x, \quad 8 = 2\lambda y, \quad x^2 + y^2 = 25$

$\Rightarrow x = \frac{3}{\lambda}, \quad y = \frac{4}{\lambda}$ (X)

$x^2 + y^2 = 25 \Rightarrow \frac{9}{\lambda^2} + \frac{16}{\lambda^2} = 25 \Rightarrow 25 = 25\lambda^2 \Rightarrow \lambda = \pm 1$

$\lambda = 1 \Rightarrow x = 3, \quad y = 4, \quad \lambda = -1 \Rightarrow x = -3, \quad y = -4$

$f(3, 4) = 50, \quad f(-3, -4) = -50$

Step 3. Compare all values from steps 1 and 2.

Maximum = 50, minimum = -50
Ex Find the minimum surface area of a rectangular box with volume 1.

Sol

\[
\begin{align*}
\text{Width } x, \text{ length } y, \text{ height } z \\
\Rightarrow \quad \text{Volume} &= xyz = 1 \\
\text{Surface area} &= 2xy + 2yz + 2zx
\end{align*}
\]

Set \( f(x,y,z) = 2xy + 2yz + 2zx \) and \( g(x,y,z) = xyz - 1 \).

Solve \( \nabla f = \lambda \nabla g \) and \( g = 0 \)

\( \Rightarrow (2y + 2z, 2x + 2z, 2x + 2y) = \lambda (yz, zx, xy) \) and \( xyz - 1 = 0 \).

\( \Rightarrow 2y + 2z = \lambda yz, \, 2x + 2z = \lambda zx, \, 2x + 2y = \lambda xy, \, xyz = 1 \).

\( \Rightarrow \lambda yz = x(2y + 2z) = y(2x + 2z) = z(2x + 2y) \)

\( \Rightarrow 2xy + 2x \cdot z = 2xy + 2y \cdot z = 2xz + 2yz \)

\( \Rightarrow xy = yz = zx \Rightarrow \frac{xy}{x} = \frac{yz}{z} = \frac{xz}{x} \Rightarrow x = y = z \).

\( xyz = 1 \Rightarrow x = y = z = 1 \).

The minimum is \( f(1,1,1) = 6 \)

Note Alternatively, you can remove the constraint \( xyz = 1 \) by writing \( z = \frac{1}{xy} \) and considering

\[ 2xy + 2y \cdot \frac{1}{xy} + 2z \cdot x = 2xy + 2y \cdot \frac{1}{xy} \cdot x = 2xy + \frac{2}{x} + \frac{2}{y} \]

on the domain \( x, y > 0 \).