14.6. Directional derivatives and the gradient vector

Def Let \( f(x,y,z) \) be a differentiable function.

1. Its gradient is given by
   \[
   \nabla f := (f_x, f_y, f_z) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).
   \]

2. Its directional derivative at \((a,b,c)\) along a unit vector \( \mathbf{u} = (u_1,u_2,u_3) \) is defined by
   \[
   D_{\mathbf{u}}f(a,b,c) := \lim_{h \to 0} \frac{f(ath_1, b+hu_2, c+hu_3) - f(a,b,c)}{h},
   \]
   "change rate in the direction of \( \mathbf{u} \)."

Note (1) The gradient is a vector, while the directional derivatives are scalars.

(2) The partial derivatives are special cases of the directional derivatives:
   \[
   D_i f = \frac{\partial f}{\partial x}, \quad D_j f = \frac{\partial f}{\partial y}, \quad D_k f = \frac{\partial f}{\partial z}
   \]
   where \( \mathbf{i} = (1,0,0), \quad \mathbf{j} = (0,1,0), \quad \mathbf{k} = (0,0,1) \).
Prop Let \( f(x,y,z) \) be a differentiable function.

1. \( \nabla f(a,b,c) = \nabla f(a,b,c) \cdot \vec{u} \)

2. \( \nabla f(a,b,c) \) points in the direction of fastest increase at \((a,b,c)\), whereas \(-\nabla f(a,b,c)\) points in the direction of fastest decrease at \((a,b,c)\).

3. \(|\nabla f(a,b,c)|\) is equal to the maximum directional derivative at \((a,b,c)\), while \(-|\nabla f(a,b,c)|\) is equal to the minimum directional derivative at \((a,b,c)\).

4. \( \nabla f(a,b,c) \) is a normal vector of the tangent plane to the level surface of \( f \) at \((a,b,c)\).

5. If \( x, y, z \) are given by a vector function \( \vec{r}(t) \),
   then \( \frac{df}{dt} = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \)
Ex  Consider $f(x, y) = x^2 + y^2$.

1) Draw a contour map of $f(x, y)$ with levels at $0, 1, 2, 3$.

$\textbf{Sol}$ Level curve at $k$ is given by $x^2 + y^2 = k$

$\sim$ a circle of radius $\sqrt{k}$ and center $(0,0)$.

(2) Draw $\nabla f(1,0)$ and $\nabla f(-1,1)$ on the contour map.

$\textbf{Sol}$ $\nabla f = (f_x, f_y) = (2x, 2y)$

$\nabla f(1,0) = (2, 0)$, $\nabla f(-1,1) = (-2, 2)$.

Note Our sketch shows that these vectors point in the direction of fastest increase at each point and are perpendicular to the tangent lines to each level curve.
Consider \( g(x, y, z) = x^2 + y^2 + z^2 + xyz \).

1. Find the directional derivative at \( P = (0, 1, 1) \) along the direction of \( \vec{V} = (4, 0, 3) \).

**Sol.** The unit vector of \( \vec{V} \) is

\[
\vec{u} = \frac{\vec{V}}{|\vec{V}|} = \frac{(4, 0, 3)}{\sqrt{4^2 + 0^2 + 3^2}} = \frac{1}{5} (4, 0, 3)
\]

\( \nabla g = (g_x, g_y, g_z) = (2x + yz, 2y + xz, 2z + xy) \)

\( \nabla g(0, 1, 1) = (1, 2, 2) \)

\( \implies D_{\vec{u}} g(0, 1, 1) = \nabla g(0, 1, 1) \cdot \vec{u} \)

\( = (1, 2, 2) \cdot \frac{1}{5} (4, 0, 3) = 2 \)

2. Find the unit vector that points in the direction of fastest increase at \( P \).

**Sol.** This is given by the unit vector of \( \nabla g(0, 1, 1) \).

\[
\frac{\nabla g(0, 1, 1)}{|\nabla g(0, 1, 1)|} = \frac{(1, 2, 2)}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3} (1, 2, 2)
\]

**Note.** The direction of fastest decrease at \( P \) is given by \(- \frac{1}{3} (1, 2, 2)\).
(3) Find the directional derivative at $P$ along the unit vector in (2).

**Sol** This is equal to the maximum directional derivative at $P$, given by $|\nabla g(0,1,1)| = 3$

(4) Find a unit vector $\vec{w}$ along which the directional derivative at $P$ is zero.

**Sol** Set $\vec{w} = (a,b,c)$.

$$
\Rightarrow D_{\vec{w}} g(0,1,1) = \nabla g(0,1,1) \cdot \vec{w} = (1,2,2) \cdot (a,b,c) = a + 2b + 2c = 0
$$

We can take $\vec{w}$ to be any unit vector with $a + 2b + 2c = 0$.

$$
\vec{w} = \frac{(0,1,-1)}{|(0,1,-1)|} = \frac{(0,1,-1)}{\sqrt{0^2+1^2+(-1)^2}} = \frac{1}{\sqrt{2}} (0,1,-1)
$$

Note There are infinitely many possible answers.

e.g. $\vec{w} = \frac{1}{\sqrt{5}} (2,-1,0)$ or $\vec{w} = \frac{1}{3} (2,-2,1)$. 
Ex For each surface, find an equation of the tangent plane at \((3,2,0)\).

(1) The surface \(z = \ln(x-y)\).

**Sol** \[ z = \ln(x-y) \Rightarrow \ln(x-y) - z = 0 \]

\(\nabla f = \left( \frac{1}{x-y}, -\frac{1}{x-y}, -1 \right)\)

A normal vector is \(\nabla f(3,2,0) = (1,-1,-1)\)

The tangent plane is given by

\[ 1 \cdot (x-3) - 1 \cdot (y-2) - 1 \cdot (z-0) = 0 \]

(2) The surface \(x^2 - y^3 + 2yz = e^z\)

**Sol** \[ x^2 - y^3 + 2yz = e^z \Rightarrow x^2 - y^3 + 2yz - e^z = 0 \]

\(\nabla g = (2x, -3y^2 + 2z, 2y - e^z)\)

A normal vector is \(\nabla g(3,2,0) = (6,-12,3)\)

The tangent plane is given by

\[ 6(x-3) - 12(y-2) + 3(z-0) = 0 \]

**Note** The tangent plane formula for graphs can be used for (1), but not for (2).
Ex Suppose that the electric potential at a point is given by $V(x,y,z) = 5x^2 - 3xy + xy^2$.

A particle is moving so that its position at time $t$ is given by $\mathbf{r}(t) = (t^2 - 1, 2 - t, t)$. Find the change rate of the electric potential along the path of particle at $t=1$.

**Sol**
\[
\frac{dV}{dt} \bigg|_{t=1} = \nabla V(\mathbf{r}(1)) \cdot \mathbf{r}'(1).
\]

\[
\nabla V = (V_x, V_y, V_z) = (10x - 3y + y^2, -3x + x^2, y^2)
\]

\[
\mathbf{r}'(1) = (0, 1, 1)
\]

\[
\nabla V(\mathbf{r}(1)) = \nabla V(0, 1, 1) = (-2, 0, 1)
\]

\[
\mathbf{r}'(t) = (2t, -1, 1) \Rightarrow \mathbf{r}'(1) = (2, -1, 1)
\]

\[
\Rightarrow \frac{dV}{dt} \bigg|_{t=1} = (-2, 0, 1) \cdot (2, -1, 1) = -3
\]