ON NONEMPTINESS OF NEWTON STRATA IN THE B_{dR}^+-GRASSMANNIAN FOR GL_n

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ABSTRACT. We study the Newton stratification in the B_{dR}^+-Grassmannian for GL_n associated to an arbitrary (possibly nonbasic) element of $B(\mathrm{GL}_n)$. Our main result classifies all nonempty Newton strata in an arbitrary minuscule Schubert cell. For a large class of elements in $B(\mathrm{GL}_n)$, our classification is given by some explicit conditions in terms of Newton polygons. For the proof, we proceed by induction on n using a previous result of the author that classifies all extensions of two given vector bundles on the Fargues-Fontaine curve.

CONTENTS

1. INTRODUCTION

1.1. Statement of the main result.

The B_{dR}^+-Grassmannian is an analogue of the affine Grassmannian in p-adic geometry. It was introduced by Caraiani-Scholze [CS17] to study the cohomology of certain Shimura varieties, and also used by Scholze-Weinstein [SW20] as a crucial tool for the construction of local Shimura varieties. In addition, it played a fundamental role in the work of Fargues-Scholze [FS21] on the geometrization of the local Langlands correspondence via the geometric Satake equivalence for p-adic groups.

The main objective of this paper is to study a natural stratification of the B_{dR}^+-Grassmannian known as the *Newton stratification*, which we briefly describe now. Let us fix a connected reductive group G over a finite extension E of $\mathbb{Q}_p$. We write Gr_G for the B_{dR}^+-Grassmannian for G, and $\mathrm{Gr}_{G,\mu}$ for the Schubert cell associated to a dominant cocharacter μ of G. For a complete algebraically closed extension C of E, we have

$$\mathrm{Gr}_G(C) = G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+) \quad \text{and} \quad \mathrm{Gr}_{G,\mu}(C) = G(B_{\mathrm{dR}}^+)\mu(t)^{-1}G(B_{\mathrm{dR}}^+)/G(B_{\mathrm{dR}}^+)$$

where B_{dR} is the p-adic de Rham period ring with valuation ring B_{dR}^+, residue field C and a fixed uniformizer t. The Cartan decomposition for G induces a decomposition

$$\mathrm{Gr}_G = \bigsqcup_{\mu \in X_*(T)^+} \mathrm{Gr}_{G,\mu}$$

where $X_*(T)^+$ denotes the set of all dominant cocharacters of G. Moreover, each Schubert cell $\mathrm{Gr}_{G,\mu}$ is related to the (diamond of the) p-adic flag variety $\mathscr{F}\ell(G,\mu)$ via a natural Bialynicki-Birula map

$$\mathrm{BB}_\mu : \mathrm{Gr}_{G,\mu} \longrightarrow \mathscr{F}\ell(G,\mu),$$

which is an isomorphism if μ is minuscule. In order to define the Newton stratification on Gr_G and its Schubert cells, we consider the stack Bun_G of G-bundles on the Fargues-Fontaine curve X. By the result of Fargues [Far20], the topological space $|\mathrm{Bun}_G|$ of Bun_G is in natural bijection with the set $B(G)$ of Frobenius-conjugacy classes of elements of $G(\breve{E})$, where $\breve{E}$ as usual denotes the p-adic completion of the maximal unramified extension of E. We fix an element $b \in B(G)$ and write $\mathcal{E}_b$ for the corresponding G-bundle on X. The theorem of Beauville-Laszlo [BL95] implies that a G-bundle on the Fargues-Fontaine curve is specified by the gluing data of the trivial G-bundles on $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$ and $X - \infty$, where ∞ is a fixed closed point on X with residue field C and completed local ring B_{dR}^+. Hence for every point $x \in \mathrm{Gr}_G(C)$ we can modify the gluing data for $\mathcal{E}_b$ by x to obtain a new G-bundle $\mathcal{E}_{b,x}$. We thus obtain a map

$$\mathrm{Newt}_b : \mathrm{Gr}_G(C) \longrightarrow B(G)$$

which maps each $x \in \mathrm{Gr}_G(C)$ to the element $b' \in B(G)$ corresponding to $\mathcal{E}_{b,x}$. For each Schubert cell $\mathrm{Gr}_{G,\mu}$, the Newton stratification associated to b is a decomposition into subdiamonds

$$\mathrm{Gr}_{G,\mu} = \bigsqcup_{b' \in B(G)} \mathrm{Gr}_{G,\mu,b}^{b'}$$

where $\mathrm{Gr}_{G,\mu,b}^{b'}(C)$ is the preimage of b' in $\mathrm{Gr}_{G,\mu}(C)$ under the map Newt_b.

The Newton stratification of minuscule Schubert cells was originally introduced in the aforementioned work of Caraiani-Scholze [CS17] as a key tool for studying the fibers of the Hodge-Tate period map. It has also been used as a pivotal tool for studying the p-adic period domain by many authors, such as Chen-Fargues-Shen [CFS21], Shen [She19], Chen [Che20], Viehmann [Vie21], Nguyen-Viehmann [NV21], and Chen-Tong [CT22].

For the trivial element $b = 1$, a result of Rapoport [Rap18] shows that the Newton stratum $\mathrm{Gr}_{G,\mu,b}^{b'}$ is nonempty if and only if b' is an element of the set $B(G, -\mu)$ defined by Kottwitz [Kot85]. When b is basic, meaning that $\mathcal{E}_b$ is semistable, Chen-Fargues-Shen [CFS21] and Viehmann [Vie21] extends the result of Rapoport to parametrize all nonempty Newton strata by a generalized Kottwitz set. However, for a general element $b \in B(G)$, no explicit parametrization is known for nonempty Newton strata in an arbitrary Schubert cell.

For $G = \mathrm{GL}_n$ and a minuscule cocharacter μ, our main result classifies all nonempty Newton strata in the Schubert cell $\mathrm{Gr}_{G,\mu}$. Let us recall that, as observed by Kottwitz [Kot85], the set $B(\mathrm{GL}_n)$ is naturally identified with the set of concave polygons on the interval $[0, n]$ with rational slopes and integer breakpoints, where a polygon refers to a continuous piecewise linear function whose graph passes through the origin. Given an element $b \in B(\mathrm{GL}_n)$, we write $\nu(b)$ for the corresponding polygon and denote by $\partial_-\nu(b)$ and $\partial_+\nu(b)$ respectively for its left and right derivatives. We may also represent every dominant cocharacters of GL_n as an n-tuple of descending integers, which we regard as a concave polygon on $[0, n]$ whose slope on $[i-1, i]$ is given by the i-th entry of the tuple.

Theorem 1.1.1. *Let μ be a minuscule dominant cocharacter of $G = \mathrm{GL}_n$ represented by an n-tuple with entries 0 and 1. Given two arbitrary elements $b, b' \in B(\mathrm{GL}_n)$ such that the difference between any two distinct slopes in $\nu(b)$ are greater than 1, the Newton stratum $\mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}$ is nonempty if and only if the following conditions are satisfied:*

(i) The polygon $\nu(b')$ lies below the polygon $\nu(b) + \mu^*$ with the same endpoints, where μ^* denotes the unique dominant cocharacter of GL_n in the conjugacy class of μ^{-1}.

(ii) On the interval $[0, n]$, we have inequalities

$$\partial_-\nu(b') \leq \partial_-\nu(b) \leq \partial_-\nu(b') + 1 \quad \text{and} \quad \partial_+\nu(b') \leq \partial_+\nu(b) \leq \partial_+\nu(b') + 1.$$

(iii) For each breakpoint of $\nu(b)$, there exists a breakpoint of $\nu(b')$ with the same x-coordinate.

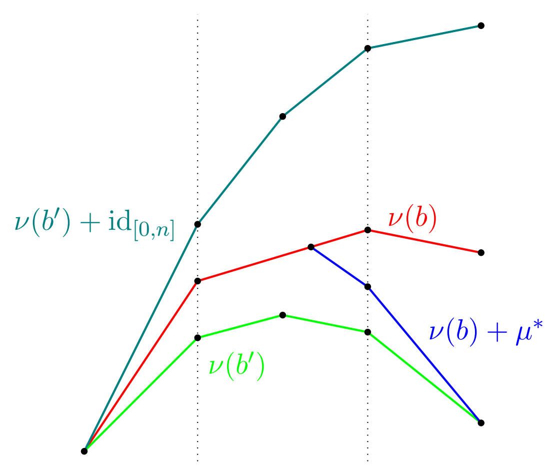

FIGURE 1. Illustration of the conditions in Theorem 1.1.1

The condition (i) is in fact equivalent to having b' in the generalized Kottwitz set considered by Chen-Fargues-Shen [CFS21] and Viehmann [Vie21]. When b is basic, the condition (i) also implies the conditions (ii) and (iii). Hence when b is basic Theorem 1.1.1 agrees with the aforementioned result of Chen-Fargues-Shen [CFS21] and Viehmann [Vie21].

The hypothesis on the cocharacter μ having entries 0 and 1 is insignificant. In fact, without this assumption we still get a similar classification of nonempty Newton strata in $\mathrm{Gr}_{\mathrm{GL}_n,\mu}$ by a simple reduction technique as stated in Proposition 3.1.6. On the other hand, the hypothesis on the slopes in $\nu(b)$ is crucial for Theorem 1.1.1. Without this hypothesis, we can only give an inductive criterion for nonemptiness of a Newton stratum in an arbitrary Schubert cell, as stated in Theorem 3.1.13. However, we can still show that the condition (i) and (ii) are always necessary.

1.2. Outline of the proof.

Given a vector bundle $\mathcal{E}$ on the Fargues-Fontaine curve X, its *minuscule effective modification at* ∞ of degree d refers to an injective bundle map $\mathcal{E}' \hookrightarrow \mathcal{E}$ whose cokernel is the skyscraper sheaf at ∞ with value $C^{\oplus d}$. If we take d to be the number of nonzero entries in μ regarded as an n-tuple of integers, the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is not empty if and only if there exists a minuscule effective modification $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞ of degree d. We thus wish to classify all minuscule effective modifications of $\mathcal{E}_b$ at ∞. If b is basic, the desired classification is given by the aforementioned results of Chen-Fargues-Shen [CFS21] and Viehmann [Vie21]. Let us henceforth assume that b is not basic. We can find a direct sum decomposition

$$\mathcal{E}_b \simeq \mathcal{E}_a \oplus \mathcal{E}_c \quad \text{with} \quad a \in B(\mathrm{GL}_m) \text{ and } c \in B(\mathrm{GL}_{n-m})$$

where a is basic such that $\nu(a)$ equals the line segment in $\nu(b)$ of maximum slope. For every minuscule effective modification $\iota : \mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞, the above decomposition extends to a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}_a & \longrightarrow & \mathcal{E}_b & \longrightarrow & \mathcal{E}_c & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\big\uparrow & & {\scriptstyle\beta}\big\uparrow & & {\scriptstyle\gamma}\big\uparrow & & \\ 0 & \longrightarrow & \mathcal{E}_{a'} & \longrightarrow & \mathcal{E}_{b'} & \longrightarrow & \mathcal{E}_{c'} & \longrightarrow & 0 \end{array}$$

where α and γ are also minuscule effective modifications at ∞. Conversely, given such a commutative diagram we apply a result of Chen-Tong [CT22] to observe that α and γ can be adjusted so that β is a minuscule effective modification at ∞. Then we use a previous result of the author [Hon22] to classify all vector bundles $\mathcal{E}_{a'}$ and $\mathcal{E}_{c'}$ that fit into such a commutative diagram, and consequently proceed by induction to obtain the desired classification.

1.3. Notations and conventions.

Throughout the paper, we fix the following data:

- E is a finite extension of $\mathbb{Q}_p$.
- C is a complete and algebraically closed extension of E.
- G is a reductive group over E with Borel subgroup B and maximal torus $T \subseteq B$.

We also retain the following notations:

- $\breve{E}$ is the p-adic completion of the maximal unramified extension of E.
- $B(G)$ is the set of Frobenius-conjugacy classes of elements of $G(\breve{E})$.
- $X_*(T)^+$ is the set of all dominant cocharacters of G.

In addition, we use the following standard notations:

- Given a valued field K, we write $\mathcal{O}_K$ for its valuation ring.
- Given a ringed space S, we write $\mathcal{O}_S$ for its structure sheaf.
- Given a perfectoid ring R, we write $R^\flat$ for its tilt and R° for its subring of power bounded elements.
- Given a perfect $\mathbb{F}_p$-algebra A, we write $W(A)$ for the ring of Witt vectors over A.

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2. Preliminaries

In this section, we review some basic facts about the B_{dR}^+-Grassmannian and G-bundles on the Fargues-Fontaine curve.

2.1. The B_{dR}^+-Grassmannian.

Proposition 2.1.1 ([Fon82, Proposition 2.4], [KL15, Lemma 3.6.3]). *Let R be a perfectoid algebra over C. There exists a natural surjective homomorphism $W(R^{\circ\flat}) \twoheadrightarrow R^\circ$ whose kernel is a principal ideal of $W(R^{\circ\flat})$.*

Definition 2.1.2. Let R be a perfectoid algebra over C. Choose a generator t of the kernel of the map $W(R^{\circ\flat}) \twoheadrightarrow R^\circ$ in Proposition 2.1.1. We write $B_{\mathrm{dR}}^+(R)$ for the t-adic completion of $W(R^{\circ\flat})[1/p]$, and define the *de Rham period ring* associated to R by $B_{\mathrm{dR}}(R) := B_{\mathrm{dR}}^+(R)[1/t]$.

Proposition 2.1.3 ([Fon82, Proposition 2.17]). *The ring $B_{\mathrm{dR}}(C)$ is a discretely valued field with valuation ring $B_{\mathrm{dR}}^+(C)$ and residue field C.*

We will henceforth write $B_{\mathrm{dR}} := B_{\mathrm{dR}}(C)$ and $B_{\mathrm{dR}}^+ := B_{\mathrm{dR}}^+(C)$. We also fix a uniformizer t of B_{dR} in light of Proposition 2.1.3.

Definition 2.1.4. The B_{dR}^+*-Grassmannian* is the functor Gr_G that associates to each perfectoid affinoid algebra (R, R^+) over C the set of pairs $(\mathcal{E}, \beta)$ consisting of a G-torsor $\mathcal{E}$ over $\mathrm{Spec}\,(B_{\mathrm{dR}}^+(R))$ and a trivialization β of $\mathcal{E}$ over $\mathrm{Spec}\,(B_{\mathrm{dR}}(R))$.

Proposition 2.1.5 ([SW20, Proposition 19.1.2]). *There exists a natural identification*

$$\mathrm{Gr}_G(C) \cong G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+).$$

Remark. In fact, we can naturally identify Gr_G as the étale sheafification of the functor that associates to each perfectoid affinoid algebra (R, R^+) over C the coset $G(B_{\mathrm{dR}}(R))/G(B_{\mathrm{dR}}^+(R))$.

Proposition 2.1.6 ([SW20, Corollary 19.3.4]). *Given $\mu \in X_*(T)^+$, there exists a locally spatial diamond $\mathrm{Gr}_{G,\mu}$ with*

$$\mathrm{Gr}_{G,\mu}(C) = G(B_{\mathrm{dR}}^+)\mu(t)^{-1}G(B_{\mathrm{dR}}^+)/G(B_{\mathrm{dR}}^+).$$

Remark. In this paper, we won't use the language of diamonds in an essential way because we are only interested in the C-valued points of Gr_G and $\mathrm{Gr}_{G,\mu}$.

Definition 2.1.7. Let μ be a dominant cocharacter of G.

(1) We refer to the locally spatial diamond $\mathrm{Gr}_{G,\mu}$ in Proposition 2.1.6 as the *Schubert cell* of Gr_G associated to μ.

(2) We define the *parabolic subgroup of G associated to μ* by

$$P_\mu := \{g \in G : \lim_{t\to 0} \mu(t)g\mu(t)^{-1} \text{ exists}\}.$$

(3) We define the *flag variety* associated to the pair (G, μ) by

$$\mathscr{F}\ell(G, \mu) := G/P_\mu.$$

(4) We define the *Bialynicki-Birula map* associated to μ as the map

$$\mathrm{BB}_\mu : \mathrm{Gr}_{G,\mu}(C) \longrightarrow \mathscr{F}\ell(G, \mu)(C)$$

which associates to $g\mu(t)^{-1}G(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{G,\mu}(C)$ the parabolic subgroup $\overline{g}P_\mu\overline{g}^{-1}$, where $\overline{g}$ denotes the image of g under the natural map $G(B_{\mathrm{dR}}^+) \to G(C)$.

Proposition 2.1.8 ([CS17, Theorem 3.4.5]). *If μ is a minuscule cocharacter of G, the Bialynicki-Birula map BB_μ is bijective.*

2.2. G-bundles on the Fargues-Fontaine curve.

Definition 2.2.1. Fix a uniformizer π of E and a pseudouniformizer ϖ of $C^\flat$. Let q be the number of elements in the residue field of E.

(1) We set
$$\mathcal{Y} := \mathrm{Spa}(W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat}) \setminus \{|\pi[\varpi]| = 0\},$$
where we write $W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat}) := W(\mathcal{O}_{C^\flat}) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ for the ring of ramified Witt vectors over $\mathcal{O}_{C^\flat}$ with coefficients in $\mathcal{O}_E$ and the Teichmuller lift $[\varpi]$ of ϖ, and define the *adic Fargues-Fontaine curve* associated to the pair $(E, C^\flat)$ by
$$\mathcal{X} := \mathcal{Y}/\phi^{\mathbb{Z}}$$
where ϕ denotes the automorphism of $\mathcal{Y}$ induced by the q-Frobenius automorphism on $W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat})$.

(2) We define the *schematic Fargues-Fontaine curve* associated to the pair $(E, C^\flat)$ by
$$X := \mathrm{Proj}\left(\bigoplus_{n\geq 0} H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})^{\phi=\pi^n}\right).$$

Remark. The definition of the adic Fargues-Fontaine curve relies on the fact that the action of ϕ on $\mathcal{Y}$ is properly discontinuous.

Theorem 2.2.2 ([Ked16, Theorem 4.10], [FF18, Théorème 6.5.2], [KL15, Theorem 8.7.7])**.** *For the adic Fargues-Fontaine curve $\mathcal{X}$ and the schematic Fargues-Fontaine curve X, we have the following facts:*

(1) $\mathcal{X}$ is a Noetherian adic space over E.
(2) X is a Dedekind scheme over E.
(3) There exists an equivalence of the categories of vector bundles on $\mathcal{X}$ and X, induced by pullback along a natural map of locally ringed spaces $\mathcal{X} \longrightarrow X$.

Remark. Despite its name, the scheme X is not a curve in the usual sense as it is not of finite type over the base field E.

In light of the statement (3) in Theorem 2.2.2, we will henceforth identify G-bundles on $\mathcal{X}$ with G-bundles on X.

Definition 2.2.3. Given an element $b \in B(G)$, we define the associated G-bundle $\mathcal{E}_b$ on $\mathcal{X}$ (or on X) by descending along the map $\mathcal{Y} \longrightarrow \mathcal{Y}/\phi^{\mathbb{Z}} = \mathcal{X}$ the trivial G-bundle on $\mathcal{Y}$ equipped with the ϕ-linear automorphism given by b.

Theorem 2.2.4 ([Far20, Théorème 5.1])**.** *The map $B(G) \longrightarrow H^1_{\acute{e}t}(X, G)$ sending b to the isomorphism class of $\mathcal{E}_b$ is a bijection.*

Proposition 2.2.5. *The set of isomorphism classes of isocrystals over $\breve{E}$ and the set of isomorphism classes of vector bundles on X admit a natural bijection which is compatible with direct sums, duals, and ranks.*

Proof. Consider an arbitrary integer $n > 0$. Given $b \in B(\mathrm{GL}_n)$, we write N_b for the isocrystal over $\breve{E}$ with underlying vector space $\breve{E}^{\oplus n}$ and the Frobenius-semilinear automorphism given by b. As observed by Kottwitz [Kot85], there exists a natural bijection between $B(\mathrm{GL}_n)$ and the set of isomorphism classes of isocrystals over $\breve{E}$ of rank n where $b \in B(\mathrm{GL}_n)$ maps to the isomorphism class of N_b. Moreover, Theorem 2.2.4 yields a bijection between $B(\mathrm{GL}_n)$ and the set of isomorphism classes of vector bundles over X of rank n where $b \in B(\mathrm{GL}_n)$ maps to the isomorphism class of $\mathcal{E}_b$. We thus obtain a bijection between the set of isomorphism classes of isocrystals over $\breve{E}$ and the set of isomorphism classes of vector bundles on X. It is straight forward to check that this bijection is compatible with diret sums, duals, and ranks. □

Definition 2.2.6. Let $\mathcal{E}$ be a vector bundle on X. We denote by $N(\mathcal{E})$ the isomorphism class of isocrystals over $\breve{E}$ that corresponds to $\mathcal{E}$ under the bijection in Proposition 2.2.5.

(1) We write $\mathrm{rk}(\mathcal{E})$ for the rank of $\mathcal{E}$, and define the *degree* of $\mathcal{E}$, denoted by $\deg(\mathcal{E})$, to be the degree of $N(\mathcal{E})$.
(2) We define the *Harder-Narasimhan (HN) polygon* of $\mathcal{E}$ by $\mathrm{HN}(\mathcal{E}) := -\mathrm{Newt}(N(\mathcal{E})^\vee)$, where $\mathrm{Newt}(N(\mathcal{E})^\vee)$ refers to the Newton polygon of the dual of $N(\mathcal{E})$.
(3) We say that $\mathcal{E}$ is *semistable* of slope λ if $\mathrm{HN}(\mathcal{E})$ is a line segment of slope λ.

Remark. The definition of $\mathrm{HN}(\mathcal{E})$ is in line with the convention that Newton polygons are convex while Harder-Narasimhan polygons are concave. It is also worthwhile to mention that the correct (or usual) definition of semistability should be given in terms of the Harder-Narasimhan formalism for vector bundles on X; in fact, the equivalence of our definition and the correct definition is due to a highly nontrivial result of Fargues-Fontaine [FF18].

Proposition 2.2.7. *Let $\mathcal{E}$ be a vector bundle on X.*

(1) $\mathcal{E}$ admits a direct sum decomposition $\mathcal{E} \simeq \oplus \mathcal{E}_i$ where the $\mathcal{E}_i$'s are semistable vector bundles on X of distinct slopes.
(2) If the $\mathcal{E}_i$'s are arranged in order of descending slope, $\mathrm{HN}(\mathcal{E})$ is given by the concatenation of the polygons $\mathrm{HN}(\mathcal{E}_i)$.

Proof. The assertion is evident by Proposition 2.2.5 and the semisimplicity of isocrystals. □

Remark. The statement (2) implies that the direct summands $\mathcal{E}_i$ are uniquely determined up to permutations.

Definition 2.2.8. Let $\mathcal{E}$ be a vector bundle on X. We refer to the direct sum decomposition $\mathcal{E} \simeq \oplus \mathcal{E}_i$ in Proposition 2.2.7 as the *Harder-Narasimhan (HN) decomposition* of $\mathcal{E}$.

2.3. The Newton stratification of Schubert cells and flag varieties.

For the rest of this paper, we fix a closed point ∞ on X given by the following proposition:

Proposition 2.3.1 ([FF18, Théorèmes 6.5.2 and 7.3.3])**.** *There exists a closed point ∞ on X with the following properties:*

(i) $X - \infty$ is the spectrum of a principal domain $B_e \subseteq B_{\mathrm{dR}}$.
(ii) The completed local ring at ∞ is canonically isomorphic to B_{dR}^+.
(iii) Every G-bundle on X becomes trivial after the pullback along the map $\mathrm{Spec}\,(B_{\mathrm{dR}}^+) \to X$ induced by ∞.

Proposition 2.3.2. *The set $H^1_{\acute{e}t}(X, G)$ is naturally in bijection with the set of isomorphism classes of triples $(\widetilde{\mathcal{E}}, \widehat{\mathcal{E}}, \beta)$ where*

- *$\widetilde{\mathcal{E}}$ is a G-bundle on $X - \infty$,*
- *$\widehat{\mathcal{E}}$ is a trivial G-bundle on $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$, and*
- *β is a gluing map of $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$ over $\mathrm{Spec}\,(B_{\mathrm{dR}})$.*

Proof. By Proposition 2.3.1, the desired assertion immediately follows from the theorem of Beauville-Laszlo [BL95]. □

Definition 2.3.3. Let $\mathcal{E}$ be a G-bundle on X. A *modification* of $\mathcal{E}$ at ∞ is a G-bundle $\mathcal{E}'$ on X together with an isomorphism between $\mathcal{E}$ and $\mathcal{E}'$ on $X - \infty$.

Example 2.3.4. Consider an element $b \in B(G)$ and a point $x \in \mathrm{Gr}_G(C)$. We may write $x = gG(B_{\mathrm{dR}}^+)$ for some $g \in G(B_{\mathrm{dR}})$ under the identification $\mathrm{Gr}_G(C) \cong G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$ noted in Proposition 2.1.5. Now, in light of Proposition 2.3.2 we take a triple $(\widetilde{\mathcal{E}}, \widehat{\mathcal{E}}, \beta)$ corresponding to $\mathcal{E}_b$ and a G-bundle $\mathcal{E}_{b,x}$ on X corresponding to $(\widetilde{\mathcal{E}}, \widehat{\mathcal{E}}, g\beta)$. By construction, $\mathcal{E}_{b,x}$ is naturally a modification of $\mathcal{E}_b$ at ∞.

Definition 2.3.5. Consider an element $b \in B(G)$ and a dominant cocharacter μ of G.

(1) For each $x \in \mathrm{Gr}_G(C)$, we refer to the G-bundle $\mathcal{E}_{b,x}$ constructed in Example 2.3.4 as the *modification of $\mathcal{E}_b$ at ∞ induced by x.*

(2) For each $b' \in B(G)$, we define the associated *Newton stratum with respect to b* in $\mathrm{Gr}_{G,\mu}$ as the subdiamond $\mathrm{Gr}_{G,\mu,b}^{b'}$ of $\mathrm{Gr}_{G,\mu}$ with

$$\mathrm{Gr}_{G,\mu,b}^{b'}(C) = \{x \in \mathrm{Gr}_{G,\mu}(C) : \mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}\}.$$

(3) For each $b' \in B(G)$, we define the associated *Newton stratum with respect to b* in $\mathscr{F}\ell(G,\mu)$ as the subvariety $\mathscr{F}\ell(G,\mu,b)^{b'}$ of $\mathscr{F}\ell(G,\mu)$ such that $\mathscr{F}\ell(G,\mu,b)^{b'}(C)$ is the image of $\mathrm{Gr}_{G,\mu,b}^{b'}(C)$ under the map BB_μ.

Remark. The subdiamond $\mathrm{Gr}_{G,\mu,b}^{b'}$ of $\mathrm{Gr}_{G,\mu}$ is uniquely determined by its set of C-points since $\mathrm{Gr}_{G,\mu}$ is a locally spatial diamond.

2.4. **Subsheaves and extensions of vector bundles on the Fargues-Fontaine curve.**

Definition 2.4.1. Given two integers n and d with $n > 0$, a *rationally tuplar polygon* of rank n and degree d is the graph $\mathscr{P}$ of a continuous function f with the following properties:

(i) f is defined on $[0, n]$ with $f(0) = 0$ and $f(n) = d$.

(ii) f is linear on $[i-1, i]$ for each $i = 1, \cdots, n$ with a rational slope denoted by $\lambda_i(\mathscr{P})$.

Example 2.4.2. We are particularly interested in the following rationally tuplar polygons:

(1) For every vector bundle $\mathcal{E}$ on X of rank n and degree d, its HN polygon $\mathrm{HN}(\mathcal{E})$ is a rationally tuplar polygon of rank n and degree d.

(2) For $G = \mathrm{GL}_n$ with Borel subgroup B of upper triangular matrices and maximal torus T of diagonal matrices, we regard all dominant cocharacters as rationally tuplar polygons of rank n under the natural identification

$$X_*(T)^+ \cong \{(a_i) \in \mathbb{Z}^n : a_1 \geq a_2 \geq \cdots \geq a_n\}.$$

(3) We write $\underline{d/n}^{(n)}$ for the line segment connecting $(0,0)$ and (n,d), which is a rationally tuplar polygon of rank n and degree d.

Definition 2.4.3. Let $\mathbb{P}_n$ denote the set of rationally tuplar polygons of rank n.

(1) We define the *Bruhat order* $\geq$ on $\mathbb{P}_n$ by writing $\mathscr{P} \geq \mathscr{Q}$ if we have

$$\sum_{i=1}^{j} \lambda_i(\mathscr{P}) \geq \sum_{i=1}^{j} \lambda_i(\mathscr{Q}) \qquad \text{for each } j = 1, \cdots, n$$

with equality for $j = n$.

(2) We define the *slopewise dominance order* $\succeq$ on $\mathbb{P}_n$ by writing $\mathscr{P} \succeq \mathscr{Q}$ if we have $\lambda_i(\mathscr{P}) \geq \lambda_i(\mathscr{Q})$ for each $i = 1, \cdots, n$.

Remark. Intuitively, we have $\mathscr{P} \geq \mathscr{Q}$ if and only if $\mathscr{P}$ lies on or above $\mathscr{Q}$ with the same endpoints, as illustrated by Figure 2.

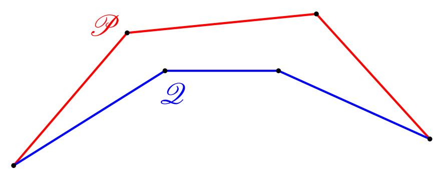

FIGURE 2. Illustration of the Bruhat order

Proposition 2.4.4 ([Hon21, Theorem 1.2.1]). *Let $\mathcal{D}$ and $\mathcal{E}$ be vector bundles on X of rank n. Then $\mathcal{D}$ is a subsheaf of $\mathcal{E}$ if and only if we have $\mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{D})$.*

Definition 2.4.5. Given vector bundles $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ on X, we define an *$\mathcal{E}$-permutation* of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ to be a rationally tuplar polygon $\mathscr{P} \geq \mathrm{HN}(\mathcal{E})$ with the following properties:

(i) The tuple $(\lambda_i(\mathscr{P}))$ is a permutation of the tuple $(\lambda_i(\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})))$.

(ii) For each $i = 1, \cdots, \mathrm{rk}(\mathcal{E})$, we have
- $\lambda_i(\mathscr{P}) < \lambda_i(\mathrm{HN}(\mathcal{E}))$ only if $\lambda_i(\mathscr{P})$ occurs as a slope in $\mathrm{HN}(\mathcal{D})$, and
- $\lambda_i(\mathscr{P}) > \lambda_i(\mathrm{HN}(\mathcal{E}))$ only if $\lambda_i(\mathscr{P})$ occurs as a slope in $\mathrm{HN}(\mathcal{F})$.

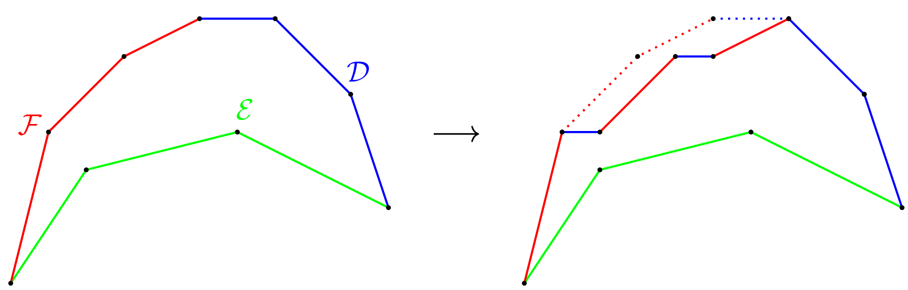

FIGURE 3. Illustration of the conditions in Definition 2.4.5

Proposition 2.4.6 ([FF18, Proposition 5.6.23]). *Given vector bundles $\mathcal{D}$ and $\mathcal{F}$ on X such that the minimum slope in $\mathrm{HN}(\mathcal{D})$ is greater than or equal to the maximum slope in $\mathrm{HN}(\mathcal{F})$, every extension of $\mathcal{F}$ by $\mathcal{D}$ splits.*

Proposition 2.4.7 ([Hon22, Theorem 3.12], [CT22, Proposition 5.3]). *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on X such that there exists a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0.$$

There exists an $\mathcal{E}$-permutation of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$.

Proposition 2.4.8 ([Hon22, Theorem 4.4], [CT22, Proposition 5.9]). *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on X. We write the HN decomposition of $\mathcal{F}$ as*

$$\mathcal{F} \simeq \bigoplus_{i=1}^{m} \mathcal{F}_i$$

where $\mathcal{F}_i$ are arranged in order of descending slope. There exists a short exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

if and only if there exists a sequence of vector bundles $\mathcal{D} = \mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_m = \mathcal{E}$ on X such that the polygon $\mathrm{HN}(\mathcal{E}_{i-1} \oplus \mathcal{F}_i)$ has an $\mathcal{E}_i$-permutation for each $i = 1, \cdots, m$.

3. Nonempty Newton strata in minuscule Schubert cells for GL_n

In this section, we classify all nonempty Newton strata in an arbitrary minuscule Schubert cell for GL_n by studying modifications of vector bundles on the Fargues-Fontaine curve. We first establish in §3.1 an inductive classification for nonempty Newton strata associated to an arbitrary element of $B(\mathrm{GL}_n)$. We then prove in §3.2 some combinatorial lemmas about rationally tuplay polygons and use them in §3.3 to give an explicit classification of all nonempty Newton strata associated to a large class of element of $B(\mathrm{GL}_n)$. Throughout this section, we take dominant cocharacters of GL_n with respect to the standard Borel subgroup of upper triangular matrices and the standard maximal torus of diagonal matrices.

3.1. An inductive classification of nonempty Newton strata.

Definition 3.1.1. Given a rationally tuplar polygon $\mathscr{P}$ of rank n, we define its *dual* to be the rationally tuplar polygon $\mathscr{P}^*$ with $\lambda_i(\mathscr{P}^*) = -\lambda_{n+1-i}(\mathscr{P})$ for each $i = 1, \cdots, n$.

Example 3.1.2. We illustrate the notion of duality for the polygons in Example 2.4.2.

1. For a vector bundle $\mathcal{E}$ on X of rank n, we have $\mathrm{HN}(\mathcal{E})^* = \mathrm{HN}(\mathcal{E}^\vee)$ where $\mathcal{E}^\vee$ denotes the dual bundle of $\mathcal{E}$.
2. For a dominant cocharacter μ of GL_n, the polygon μ^* represents the unique dominant cocharacter in the conjugacy class of μ^{-1}.
3. For arbitrary integers d and n, we have $\underline{d/n}^{(n)*} = \underline{-d/n}^{(n)}$.

Proposition 3.1.3 ([CFS21, Proposition 5.2], [Vie21, Corollary 5.4])**.** *Let b and b' be elements of $B(\mathrm{GL}_n)$ such that $\mathcal{E}_b$ is semistable. Given a dominant cocharacter μ of GL_n, the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is nonempty if and only if we have*

$$\nu(b) + \mu^* \geq \nu(b') \tag{3.1}$$

where $\nu(b)$ and $\nu(b')$ respectively denote $\mathrm{HN}(\mathcal{E}_b)$ and $\mathrm{HN}(\mathcal{E}_{b'})$.

Remark. For a reductive group G and a basic element $b \in B(G)$, the results of Chen-Fargues-Shen [CFS21, Proposition 5.2] and Viehmann [Vie21, Corollary 5.4] classify all nonempty newton strata with respect to b in an arbitrary Schubert cell in terms of the Kottwitz map and the Newton map defined by Kottwitz [Kot85]. In our context, their results are translated to Proposition 3.1.3 by the following facts:

(a) An element $b \in B(\mathrm{GL}_n)$ is basic if and only if $\mathcal{E}_b$ is semistable.
(b) The condition involving the Kottwitz map holds for all elements in $B(\mathrm{GL}_n)$.
(c) The condition involving the Newton map is equivalent to the inequality (3.1) as $\nu(b)$ and $\nu(b')$ are identified with the (concave) Newton polygons of b and b'.

Lemma 3.1.4. *Let b be an element of $B(\mathrm{GL}_n)$. For $x = \underline{1}^{(n)}(t)\,\mathrm{GL}_n(B^+_{\mathrm{dR}}) \in \mathrm{Gr}_{\mathrm{GL}_n,\underline{1}^{(n)}}(C)$, we have $\mathrm{HN}(\mathcal{E}_{b,x}) = \mathrm{HN}(\mathcal{E}_b) - \underline{1}^{(n)}$.*

Proof. Let us write the HN decomposition of $\mathcal{E}_b$ as

$$\mathcal{E}_b \simeq \bigoplus_{i=1}^{m} \mathcal{E}_{b_i} \qquad \text{with } b_i \in B(\mathrm{GL}_{n_i}).$$

In addition, we take $x_i := \underline{1}^{(n_i)}(t)\,\mathrm{GL}_{n_i}(B^+_{\mathrm{dR}}) \in \mathrm{Gr}_{\mathrm{GL}_{n_i},\underline{1}^{(n_i)}}(C)$ for each $i = 1, \cdots, m$. By Proposition 3.1.3, we find $\mathrm{HN}(\mathcal{E}_{b_i,x_i}) = \mathrm{HN}(\mathcal{E}_{b_i}) - \underline{1}^{(n_i)}$ for $i = 1, \cdots, m$. Now the desired assertion follows by the fact that $\mathcal{E}_{b,x}$ is a direct sum of the vector bundles $\mathcal{E}_{b_i,x_i}$. □

Proposition 3.1.5. *Let μ be a dominant cocharacter of GL_n. For elements $b, b' \in B(\mathrm{GL}_n)$, the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is not empty if and only if it contains a C-point.*

Proof. The assertion is evident by definition. □

Proposition 3.1.6. *Let μ be a dominant cocharacter of GL_n with nonnegative slopes. For two elements $b, b' \in B(\mathrm{GL}_n)$, we have the following equivalent conditions:*

(i) $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ *is nonempty.*

(ii) $\mathrm{Gr}^{b}_{\mathrm{GL}_n,\mu^*,b'}$ *is nonempty.*

(iii) $\mathrm{Gr}^{\widetilde{b'}}_{\mathrm{GL}_n,\mu+\underline{1}^{(n)},b}$ *is nonempty for* $\widetilde{b'} \in B(\mathrm{GL}_n)$ *with* $\mathrm{HN}(\mathcal{E}_{\widetilde{b'}}) = \mathrm{HN}(\mathcal{E}_{b'}) - \underline{1}^{(n)}$.

Proof. For $x = g\mu(t)G(B_{\mathrm{dR}}^{+}) \in \mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}(C)$, we take $x^* := g^{-1}\mu^*(t)G(B_{\mathrm{dR}}^{+}) \in \mathrm{Gr}_{\mathrm{GL}_n,\mu^*}(C)$ and find $\mathcal{E}_{b',x^*} \simeq \mathcal{E}_b$, thereby deducing that x^* lies in $\mathrm{Gr}^{b}_{\mathrm{GL}_n,\mu^*,b'}(C)$. Similarly, every point in $\mathrm{Gr}^{b}_{\mathrm{GL}_n,\mu^*,b'}(C)$ gives rise to a point in $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}(C)$. Hence by Proposition 3.1.5 we establish the equivalence of the conditions (i) and (ii).

Now it remains to verify the equivalence of the conditions (i) and (iii). For every $x = g\mu(t)G(B_{\mathrm{dR}}^{+}) \in \mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}(C)$, we take $\widetilde{x} := g\mu(t)\underline{1}^{(n)}(t)G(B_{\mathrm{dR}}^{+}) \in \mathrm{Gr}_{\mathrm{GL}_n,\mu+\underline{1}^{(n)}}(C)$ and find $\mathcal{E}_{b,\widetilde{x}} \simeq \mathcal{E}_{\widetilde{b'}}$ by Lemma 3.1.4, thereby deducing that $\widetilde{x}$ lies in $\mathrm{Gr}^{\widetilde{b'}}_{\mathrm{GL}_n,\mu+\underline{1}^{(n)},b}(C)$. Conversely, for every $\widetilde{x} := g\mu(t)\underline{1}^{(n)}(t)G(B_{\mathrm{dR}}^{+}) \in \mathrm{Gr}_{\mathrm{GL}_n,\mu+\underline{1}^{(n)}}(C)$ we take $x := g\mu(t)G(B_{\mathrm{dR}}^{+}) \in \mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}(C)$ and find $\mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}$ by Lemma 3.1.4, thereby deducing that x lies in $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}(C)$. Hence we complete the proof by Proposition 3.1.5. □

Remark. In light of Proposition 3.1.6, for our desired classification it suffices to consider minuscule cocharacters with slopes 0 and 1.

Definition 3.1.7. Let $\mathcal{E}$ be a vector bundle on X of rank n.

(1) Given a dominant cocharacter μ of GL_n, we define an *effective modification* of $\mathcal{E}$ at ∞ of *type* μ to be an injective $\mathcal{O}_X$-module map $\mathcal{E}' \hookrightarrow \mathcal{E}$ whose cokernel is the skyscraper sheaf at ∞ with value $\displaystyle\bigoplus_{i=1}^{n} B_{\mathrm{dR}}^{+}/t^{\lambda_i(\mu)}B_{\mathrm{dR}}^{+}$.

(2) We say that an effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ is *minuscule of degree d* if its type is minuscule of degree d with slopes 0 and 1.

Proposition 3.1.8. *Let μ be a dominant cocharacter of GL_n with nonnegative slopes. Given two elements $b, b' \in B(\mathrm{GL}_n)$, the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is nonempty if and only if there exists an effective modification $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞ of type μ.*

Proof. If $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is not empty, then Proposition 3.1.5 yields a point $x \in \mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}(C)$, which gives rise to an effective modification $\mathcal{E}_{b,x} \hookrightarrow \mathcal{E}_b$ at ∞ of type μ. Let us now assume for the converse that there exists an effective modification $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞ of type μ. Take triples $(\widetilde{\mathcal{E}_b}, \widehat{\mathcal{E}_b}, \beta_b)$ and $(\widetilde{\mathcal{E}_{b'}}, \widehat{\mathcal{E}_{b'}}, \beta_{b'})$ which respectively correspond to $\mathcal{E}_b$ and $\mathcal{E}_{b'}$ under the bijection in Proposition 2.3.2. We may set $\widetilde{\mathcal{E}_b} = \widetilde{\mathcal{E}_{b'}}$ since the map $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ is an isomorphism on $X - \infty$. Then we conjugate β_b by a suitable element in $\mathrm{GL}_n(B_{\mathrm{dR}}^{+})$ to write $\beta_{b'} = g\mu(t)\beta_b$ for some $g \in \mathrm{GL}_n(B_{\mathrm{dR}})$, and in turn find $g\mu(t)\,\mathrm{GL}_n(B_{\mathrm{dR}}^{+}) \in \mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}(C)$ to complete the proof. □

Corollary 3.1.9. *Let μ be a minuscule dominant cocharacter of GL_n of degree d with slopes 0 and 1. Given two elements $b, b' \in B(\mathrm{GL}_n)$, the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is nonempty if and only if there exists a minuscule effective modification $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞ of degree d.*

Proposition 3.1.10 ([CT22, Proposition 4.6]). *Let $\mathcal{E}$ and $\mathcal{E}'$ be vector bundles on X of rank n. Take a direct sum decomposition*

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F} \tag{3.2}$$

such that $\mathrm{HN}(\mathcal{D})$ *coincides with the line segment of maximal slope in* $\mathrm{HN}(\mathcal{E})$. *There exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ if and only if there exist minuscule effective modifications $\mathcal{D}' \hookrightarrow \mathcal{D}$ and $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ with a short exact sequence*

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Remark. The result of Chen-Tong [CT22, Proposition 4.6] only considers basic elements $b \in B(G)$. However, its proof remains valid for all $b \in B(G)$ after replacing the generalized Kottwitz set $B(M, \kappa(b_M) - \mu^{w,\#}, \nu_{b_M} - \mu^{w,\diamond})$ with the set $\mathcal{N}(M, b_M, \mu^w) := \{b'_M \in B(M) : \mathscr{F}\ell(M, \mu^w, b_M)^{b'_M} \neq \emptyset\}$. If b is basic, these two sets are identical by the result of Chen-Fargues-Shen [CFS21, Proposition 5.2].

For convenience of the readers, let us briefly explain how the result of Chen-Tong [CT22, Proposition 4.6] is translated into Proposition 3.1.10 in our context. We take $b, b' \in B(\mathrm{GL}_n)$ with $\mathcal{E} \simeq \mathcal{E}_b$ and $\mathcal{E}' \simeq \mathcal{E}_{b'}$. We also set μ to be the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$. The direct sum decomposition (3.2) corresponds to a reduction (b_M, h) of b to the Levi subgroup

$$M := \mathrm{GL}_r \times \mathrm{GL}_{n-r} \subseteq \mathrm{GL}_n$$

where r denotes the rank of $\mathcal{D}$. We denote by P the parabolic subgroup of GL_n corresponding to M. We also take an element w in the Weyl group of GL_n and write μ^w for the corresponding conjugate of μ. Then μ^w is minuscule (as a cocharacter of GL_n and also of M), and consequently gives rise to a bijection

$$\mathrm{BB}_{\mu^w} : \mathrm{Gr}_{M,\mu^w}(C) \xrightarrow{\sim} \mathscr{F}\ell(M, \mu^w)(C)$$

by Proposition 2.1.8. Hence for each $b'_M \in B(M)$ with $\{b'_M \in B(M) : \mathscr{F}\ell(M, \mu^w, b_M)^{b'_M} \neq \emptyset\}$, every point $x \in \mathscr{F}\ell(M, \mu^w, b_M)^{b'_M}(C)$ induces minuscule effective modifications $\alpha : \mathcal{D}' \hookrightarrow \mathcal{D}$ and $\beta : \mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ whose degrees d_1 and d_2 are specified by μ^w. Now with $\mathcal{N}(M, b_M, \mu^w)$ in place of $B(M, \kappa(b_M) - \mu^{w,\#}, \nu_{b_M} - \mu^{w,\diamond})$ we can identify the two sets in (4.1.2) of Chen-Tong [CT22] as follows:

- The left set parametrizes minuscule effective modifications $\alpha : \mathcal{D}' \hookrightarrow \mathcal{D}$ and $\beta : \mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ of degrees d_1 and d_2 which fit into a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \alpha \big\uparrow & & \big\uparrow & & \beta \big\uparrow & & \\ 0 & \longrightarrow & \mathcal{D}' & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F}' & \longrightarrow & 0 \end{array} \tag{3.3}$$

  with the top row being split and the middle vertical arrow being a minuscule effective modification at ∞ (of degree d).
- The right set parametrizes minuscule effective modifications $\alpha : \mathcal{D}' \hookrightarrow \mathcal{D}$ and $\beta : \mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ of degrees d_1 and d_2 which fit into a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Hence the result of Chen-Tong [CT22, Proposition 4.6] implies the sufficiency part of Proposition 3.1.10 in our context. The necessity part is evident as every minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ gives rise to a commutative diagram (3.3).

Proposition 3.1.11 ([FF18, §5.5.2.1])**.** *Let $\mathcal{E}$ be a vector bundle on X. For every minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞, its degree is equal to $\deg(\mathcal{E}) - \deg(\mathcal{E}')$.*

Lemma 3.1.12. *Let $\mathcal{E}$ and $\mathcal{E}'$ be vector bundles on X of rank n such that $\mathcal{E}$ is semistable. Take μ to be the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$ with slopes 0 and 1. There exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ if and only if $\mathcal{E}$ and $\mathcal{E}'$ satisfy the following equivalent inequalities:*

$$\mathrm{HN}(\mathcal{E}) + \mu^* \geq \mathrm{HN}(\mathcal{E}') \quad \text{and} \quad \mathrm{HN}(\mathcal{E}') + \underline{1}^{(n)} \succeq \mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{E}'). \tag{3.4}$$

Proof. By Proposition 3.1.3, Corollary 3.1.9 and Proposition 3.1.11, there exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ if and only if $\mathcal{E}$ and $\mathcal{E}'$ satisfy the first inequality in (3.4). If we write λ for the slope of the line segment $\mathrm{HN}(\mathcal{E})$, the polygon $\mathrm{HN}(\mathcal{E}) + \mu^*$ has two distinct slopes λ and $\lambda - 1$. Hence it is not hard to verify the equivalence of the two inequalities in (3.4) by the concavity of HN polygons, thereby deducing the desired assertion. □

Theorem 3.1.13. *Let μ be a minuscule dominant cocharacter of GL_n with slopes 0 and 1. Consider two arbitrary elements $b, b' \in B(\mathrm{GL}_n)$. Take a direct sum decomposition*

$$\mathcal{E}_b \simeq \mathcal{E}_a \oplus \mathcal{E}_c \quad \text{with} \quad a \in B(\mathrm{GL}_r) \text{ and } c \in B(\mathrm{GL}_{n-r})$$

such that $\mathrm{HN}(\mathcal{E}_a)$ coincides with the line segment of maximal slope in $\mathrm{HN}(\mathcal{E}_b)$.

1. *If the degree of μ is not equal to $\deg(\mathcal{E}_b) - \deg(\mathcal{E}_{b'})$, the Newton stratum $\mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}$ is empty.*
2. *If the degree of μ is equal to $\deg(\mathcal{E}_b) - \deg(\mathcal{E}_{b'})$ the Newton stratum $\mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}$ is nonempty if and only if there exist $a' \in B(\mathrm{GL}_r)$ and $c' \in B(\mathrm{GL}_{n-r})$ with the following properties:*
   - *(i) We have $\mathrm{HN}(\mathcal{E}_{a'}) + \underline{1}^{(r)} \succeq \mathrm{HN}(\mathcal{E}_a) \succeq \mathrm{HN}(\mathcal{E}_{a'})$.*
   - *(ii) If we write the HN decomposition of $\mathcal{E}_{c'}$ as*
   $$\mathcal{E}_{c'} \simeq \bigoplus_{i=1}^{m} \mathcal{F}_i$$
   where $\mathcal{F}_i$ are arranged in order of descending slope, there exists a sequence of vector bundles $\mathcal{E}_{a'} = \mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_m = \mathcal{E}_b$ on X such that $\mathrm{HN}(\mathcal{E}_{i-1} \oplus \mathcal{F}_i)$ has an $\mathcal{E}_i$-permutation for each $i = 1, \cdots, r$.
   - *(iii) The Newton stratum $\mathrm{Gr}_{\mathrm{GL}_n,\overline{\mu},c}^{c'}$ is nonempty where $\overline{\mu}$ is a minuscule dominant cocharacter of GL_{n-r} of degree $\overline{d} := \deg(\mathcal{E}_c) - \deg(\mathcal{E}_{c'})$ with slopes 0 and 1.*

Proof. The assertion is straightforward to verify by Proposition 2.4.8, Corollary 3.1.9, Proposition 3.1.10, Proposition 3.1.11, and Lemma 3.1.12. □

Remark. The elements $a \in B(\mathrm{GL}_r)$ and $c \in B(\mathrm{GL}_{n-r})$ are uniquely determined by the HN decomposition of $\mathcal{E}_b$. In addition, the Schubert cell $\mathrm{Gr}_{\mathrm{GL}_n,\overline{\mu}}$ contains finitely many nonemtpy Newton strata, as easily seen by Proposition 2.4.4 and Corollary 3.1.9. Hence the conditions (i) and (iii) together yield finitely many possibilities for $a' \in B(\mathrm{GL}_r)$ and $c' \in B(\mathrm{GL}_{n-r})$. We can thus use Theorem 3.1.13 to inductively classify all nonempty Newton strata in an arbitrary minuscule Schubert cell of $\mathrm{Gr}_{\mathrm{GL}_n}$.

3.2. Concave rationally tuplar polygons.

Definition 3.2.1. Given a rationally tuplar polygon $\mathscr{P}$, we define its *concave rearrangement* to be the rationally tuplar polygon $\widehat{\mathscr{P}}$ such that the tuple $(\lambda_i(\widehat{\mathscr{P}}))$ is the rearrangement of $(\lambda_i(\mathscr{P}))$ in descending order.

Lemma 3.2.2. *For every rationally tuplar polygon $\mathscr{P}$, we have $\widehat{\mathscr{P}} \geq \mathscr{P}$.*

Proof. The assertion is evident by definition. □

Remark. In fact, $\widehat{\mathscr{P}}$ is the maximal rearrangement of $\mathscr{P}$ with respect to the Bruhat order.

Definition 3.2.3. Given two rationally tuplar polygon $\mathscr{P}$ and $\mathscr{Q}$, we define their direct sum $\mathscr{P} \oplus \mathscr{Q}$ to be the concave rearrangement of the concatenation of $\mathscr{P}$ and $\mathscr{Q}$.

Example 3.2.4. Let us record some important examples of direct sums for our purpose.

(1) For two vector bundles $\mathcal{E}$ and $\mathcal{F}$ on X, we have $\mathrm{HN}(\mathcal{E} \oplus \mathcal{F}) = \mathrm{HN}(\mathcal{E}) \oplus \mathrm{HN}(\mathcal{F})$.

(2) For two minuscule dominant cocharacters μ_1 and μ_2 of GL_n with slopes 0 and 1, their direct sum is a minuscule dominant cocharacter of GL_n with slopes 0 and 1.

Lemma 3.2.5. *Given concave rationally tuplar polygons $\mathscr{P}, \mathscr{P}', \mathscr{Q}$ and $\mathscr{Q}'$ with $\mathscr{P} \geq \mathscr{P}'$ and $\mathscr{Q} \geq \mathscr{Q}'$, we have $\mathscr{P} \oplus \mathscr{Q} \geq \mathscr{P}' \oplus \mathscr{Q}'$.*

Proof. Let m and n respectively denote the ranks of $\mathscr{P}$ and $\mathscr{Q}$. Take two sets A and B which form a partition of the set $\{1, \cdots, m+n\}$ with

$$(\lambda_i(\mathscr{P}' \oplus \mathscr{Q}'))_{i \in A} = (\lambda_i(\mathscr{P}')) \quad \text{and} \quad (\lambda_i(\mathscr{P}' \oplus \mathscr{Q}'))_{i \in B} = (\lambda_i(\mathscr{Q}')).$$

Let $\mathscr{R}$ to be the rationally tuplar polygon of rank $m+n$ with

$$(\lambda_i(\mathscr{R}))_{i \in A} = (\lambda_i(\mathscr{P})) \quad \text{and} \quad (\lambda_i(\mathscr{R}))_{i \in B} = (\lambda_i(\mathscr{Q})).$$

Since $\mathscr{P}, \mathscr{P}', \mathscr{Q}$ and $\mathscr{Q}'$ are all concave, the inequalities $\mathscr{P} \geq \mathscr{P}'$ and $\mathscr{Q} \geq \mathscr{Q}'$ together imply $\mathscr{R} \geq \mathscr{P}' \oplus \mathscr{Q}'$. In addition, we have $\mathscr{P} \oplus \mathscr{Q} = \widehat{\mathscr{R}} \geq \mathscr{R}$ by Lemma 3.2.2. Therefore we find $\mathscr{P} \oplus \mathscr{Q} \geq \mathscr{P}' \oplus \mathscr{Q}'$ as desired. □

Remark. Lemma 3.2.5 does not hold without the concavity assumption. For example, if we take $\mathscr{P} = \mathscr{Q} = d/r^{(r)}$ for some integers r and d with $r > 0$, then for arbitrary nonlinear convex polygons $\overline{\mathscr{P}'}$ and $\mathscr{Q}'$ of rank r and degree d we do not have $\mathscr{P} \oplus \mathscr{Q} \geq \mathscr{P}' \oplus \mathscr{Q}'$ despite having $\mathscr{P} \geq \mathscr{P}'$ and $\mathscr{Q} \geq \mathscr{Q}'$, as illustrated in Figure 4.

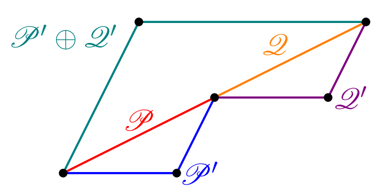

FIGURE 4. A counter example for Lemma 3.2.5 without the concavity assumption

Lemma 3.2.6. *Let $\mathscr{P}$ and $\mathscr{Q}$ be rationally tuplar polygons of rank m and n. For arbitrary rationally tuplar polygons $\mathscr{P}'$ of rank m and $\mathscr{Q}'$ of rank n, we have*

$$(\mathscr{P} \oplus \mathscr{Q}) + (\mathscr{P}' \oplus \mathscr{Q}') \geq (\mathscr{P} + \mathscr{P}') \oplus (\mathscr{Q} + \mathscr{Q}').$$

Proof. We observe that there exist permutations σ and σ' of the set $\{1, \cdots, m+n\}$ with

$$\lambda_i((\mathscr{P} + \mathscr{P}') \oplus (\mathscr{Q} + \mathscr{Q}')) = \lambda_{\sigma(i)}(\mathscr{P} \oplus \mathscr{Q}) + \lambda_{\sigma'(i)}(\mathscr{P}' \oplus \mathscr{Q}') \quad \text{for each } i = 1, \cdots, m+n,$$

and consequently deduce the desired assertion by the concavity of $\mathscr{P} \oplus \mathscr{Q}$ and $\mathscr{P}' \oplus \mathscr{Q}'$. □

3.3. An explicit classification of nonempty Newton strata.

Lemma 3.3.1. *Let $\mathcal{E}$ be a vector bundle on X of rank n. Every minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ gives rise to a minuscule effective modification $\widetilde{\mathcal{E}} \hookrightarrow \mathcal{E}'$ at ∞ with $\mathrm{HN}(\widetilde{\mathcal{E}}) = \mathrm{HN}(\mathcal{E}) - \underline{1}^{(n)}$.*

Proof. Let μ be the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$ with slopes 0 and 1. Take elements b, b' and $\widetilde{b}$ in $B(\mathrm{GL}_n)$ with $\mathcal{E} \simeq \mathcal{E}_b$, $\mathcal{E}' \simeq \mathcal{E}_{b'}$ and $\widetilde{\mathcal{E}} \simeq \mathcal{E}_{\widetilde{b}}$. The effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ yields a point in $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ by Corollary 3.1.9 and Proposition 3.1.11, and in turn yields a point in $\mathrm{Gr}^{\widetilde{b}}_{\mathrm{GL}_n,\mu^*+\underline{1}^{(n)},b'}$ by Proposition 3.1.6. Hence we obtain a minuscule effective modification $\widetilde{\mathcal{E}} \hookrightarrow \mathcal{E}'$ at ∞ by Corollary 3.1.9 as desired. □

Proposition 3.3.2. *Let $\mathcal{E}$ be a vector bundle on X of rank n. For every minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞, we have*

$$\mathrm{HN}(\mathcal{E}) + \mu^* \geq \mathrm{HN}(\mathcal{E}') \quad \text{and} \quad \mathrm{HN}(\mathcal{E}') + \underline{1}^{(n)} \succeq \mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{E}')$$

where μ is the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$ with slopes 0 and 1.

Proof. The second inequality is an immediate consequence of Proposition 2.4.4 and Lemma 3.3.1. Hence it remains to establish the first inequality. Let us write m for the number of distinct slopes in $\mathrm{HN}(\mathcal{E})$ and proceed by induction on m. If $\mathcal{E}$ is semistable, the assertion is evident by Lemma 3.1.12. We henceforth assume that $\mathcal{E}$ is not semistable, so that we have $m > 1$. Take a direct sum decomposition

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F}$$

such that $\mathrm{HN}(\mathcal{D})$ coincides with the line segment of maximal slope in $\mathrm{HN}(\mathcal{E})$. The numbers of distinct slopes in $\mathrm{HN}(\mathcal{D})$ and $\mathrm{HN}(\mathcal{F})$ are respectively 1 and $m-1$. Now Proposition 3.1.10 yields minuscule effective modifications $\alpha : \mathcal{D}' \hookrightarrow \mathcal{D}$ and $\beta : \mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ with a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0. \tag{3.5}$$

Let us denote the types of α and β respectively by μ_1 and μ_2. In a concrete form, we have

$$\mu_1 = \underline{1}^{(d_1)} \oplus \underline{0}^{(n_1-d_1)} \quad \text{and} \quad \mu_2 = \underline{1}^{(d_2)} \oplus \underline{0}^{(n_2-d_2)}$$

where we set $n_1 := \mathrm{rk}(\mathcal{D}) = \mathrm{rk}(\mathcal{D}')$, $n_2 := \mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{F}')$, $d_1 := \deg(\mathcal{D}) - \deg(\mathcal{D}')$ and $d_2 := \deg(\mathcal{F}) - \deg(\mathcal{F}')$. By the induction hypothesis, the minuscule effective modifications α and β at ∞ respectively yield the inequalities

$$\mathrm{HN}(\mathcal{D}) + \mu_1^* \geq \mathrm{HN}(\mathcal{D}') \quad \text{and} \quad \mathrm{HN}(\mathcal{F}) + \mu_2^* \geq \mathrm{HN}(\mathcal{F}').$$

Then by Example 3.2.4, Lemma 3.2.5 and Lemma 3.2.6 we find

$$\mathrm{HN}(\mathcal{E}) + \mu^* = (\mathrm{HN}(\mathcal{D}) \oplus \mathrm{HN}(\mathcal{F})) + (\mu_1^* \oplus \mu_2^*) \geq (\mathrm{HN}(\mathcal{D}) + \mu_1^*) \oplus (\mathrm{HN}(\mathcal{F}) + \mu_2^*) \geq \mathrm{HN}(\mathcal{D}' \oplus \mathcal{F}').$$

In addition, by Proposition 2.4.7 the short exact sequence (3.5) yields the inequality

$$\mathrm{HN}(\mathcal{D}' \oplus \mathcal{F}') \geq \mathrm{HN}(\mathcal{E}').$$

We thus obtain the first inequality, thereby completing the proof. □

Remark. The two inequalities in Proposition 3.3.2 are not equivalent in general, although they are equivalent if $\mathcal{E}$ is semistable as shown in Lemma 3.1.12.

Example 3.3.3. Let us present an example showing that the converse of Proposition 3.3.2 does not hold. Take $\mathcal{E}$ and $\mathcal{E}'$ to be vector bundles on X with

$$\mathrm{HN}(\mathcal{E}) = \underline{4/3}^{(3)} \oplus \underline{3/4}^{(4)} \quad \text{and} \quad \mathrm{HN}(\mathcal{E}') = \underline{1}^{(2)} \oplus \underline{1/3}^{(3)} \oplus \underline{0}^{(2)}.$$

By construction, we have $\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{E}') = 7$, $\deg(\mathcal{E}) = 7$ and $\deg(\mathcal{E}') = 3$. Now for the minuscule dominant cocharacter μ of GL_7 of degree 4 with slopes 0 and 1, we find

$$\mathrm{HN}(\mathcal{E}) + \mu^* \geq \mathrm{HN}(\mathcal{E}') \quad \text{and} \quad \mathrm{HN}(\mathcal{E}') + \underline{1}^{(7)} \succeq \mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{E}').$$

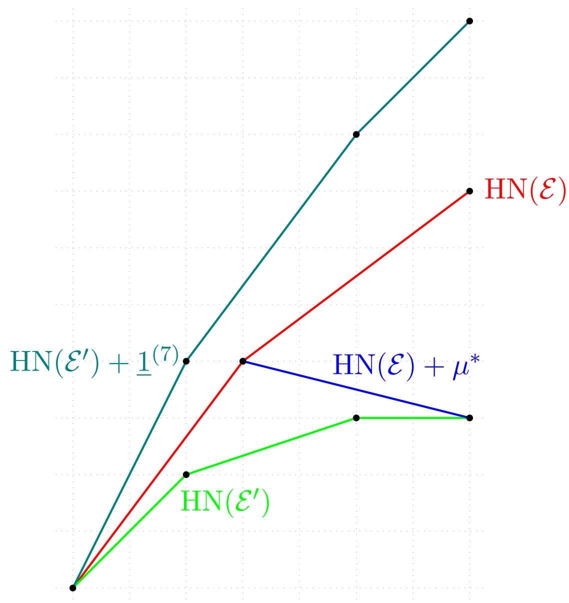

FIGURE 5. A counter example for the converse of Proposition 3.3.2

We wish to show that there does not exist a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞. Suppose for contradiction that such a modification exists. Take a direct sum decomposition

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F}$$

with $\mathrm{HN}(\mathcal{D}) = \underline{4/3}^{(3)}$ and $\mathrm{HN}(\mathcal{F}) = \underline{3/4}^{(4)}$. Proposition 3.1.10 yields minuscule effective modifications $\mathcal{D}' \hookrightarrow \mathcal{D}$ and $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ with a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Then by Proposition 2.4.7 we obtain an $\mathcal{E}'$-permutation $\mathscr{P}$ of $\mathrm{HN}(\mathcal{D}' \oplus \mathcal{F}')$. Since we have $\mathscr{P} \geq \mathrm{HN}(\mathcal{E}')$ by construction, we find

$$\lambda_1(\mathscr{P}) \geq \lambda_1(\mathrm{HN}(\mathcal{E}')) = 1 \quad \text{and} \quad \lambda_2(\mathscr{P}) \geq \lambda_2(\mathrm{HN}(\mathcal{E}')) = 1. \tag{3.6}$$

Moreover, as $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$ by construction, Proposition 2.4.4 implies that all slopes in $\mathrm{HN}(\mathcal{F}')$ are less than or equal to $3/4$. We then deduce by (3.6) that $\lambda_1(\mathscr{P})$ and $\lambda_2(\mathscr{P})$ should occur as a slope of $\mathcal{D}'$, and in turn find that the inequalities in (3.6) are in fact equalities. Therefore $\mathrm{HN}(\mathcal{D}')$ must contain the line segment $\underline{1}^{(2)}$, and consequently is given by $\underline{1}^{(2)} \oplus \underline{d}^{(1)}$ for some integer d. Then we have $d = \lambda_i(\mathscr{P})$ for some $i > 2$ and thus find $d \leq \lambda_i(\mathrm{HN}(\mathcal{E}')) \leq 1/3$. On the other hand, since $\mathcal{D}'$ occurs as a minuscule effective modification of $\mathcal{D}$ at C, Proposition 3.3.2 implies $d \geq 1/3$. Now we have a desired contradiction as d is an integer with $d \leq 1/3$ and $d \geq 1/3$.

Proposition 3.3.4. *Let $\mathcal{E}$ and $\mathcal{E}'$ be vector bundles on X of rank n such that the difference between any two distinct slopes in $\mathrm{HN}(\mathcal{E})$ is greater than 1. Denote by μ the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$ with slopes 0 and 1. There exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ if and only if $\mathcal{E}$ and $\mathcal{E}'$ satisfy the following conditions:*

(i) We have $\mathrm{HN}(\mathcal{E}) + \mu^* \geq \mathrm{HN}(\mathcal{E}')$ and $\mathrm{HN}(\mathcal{E}') + \underline{1}^{(n)} \succeq \mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{E}')$.

(ii) For each breakpoint of $\mathrm{HN}(\mathcal{E})$, there exists a breakpoint of $\mathrm{HN}(\mathcal{E}')$ with the same x-coordinate.

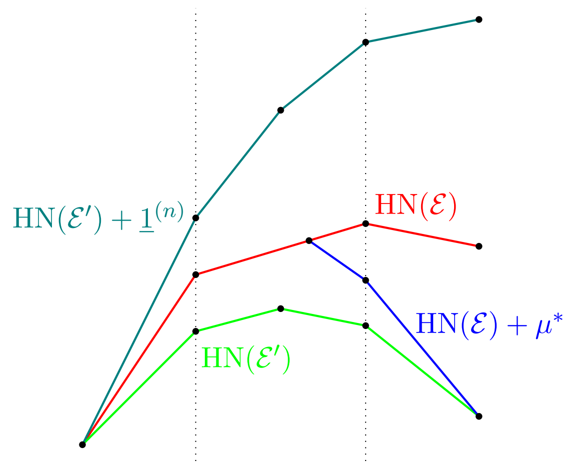

FIGURE 6. Illustration of the conditions in Proposition 3.3.4

Proof. Let us first assume that $\mathcal{E}$ and $\mathcal{E}'$ satisfy the conditions (i) and (ii). We write the HN decomposition of $\mathcal{E}$ as

$$\mathcal{E} \simeq \bigoplus_{i=1}^{m} \mathcal{E}_i \tag{3.7}$$

where the direct summands $\mathcal{E}_i$ are arranged in order of descending slope, and set

$$x_i := \sum_{j=1}^{i} \mathrm{rk}(\mathcal{E}_j) \quad \text{ for } i = 0, \cdots, m.$$

By the condition (ii), we get a direct sum decomposition

$$\mathcal{E}' \simeq \bigoplus_{i=1}^{m} \mathcal{E}'_i \tag{3.8}$$

where each $\mathrm{HN}(\mathcal{E}'_i)$ coincides with the restriction of $\mathrm{HN}(\mathcal{E}')$ on the interval $[x_{i-1}, x_i]$. Then by the condition (i) we find

$$\mathrm{HN}(\mathcal{E}'_i) + \underline{1}^{(x_i - x_{i-1})} \succeq \mathrm{HN}(\mathcal{E}_i) \succeq \mathrm{HN}(\mathcal{E}'_i) \quad \text{ for } i = 1, \cdots, m.$$

Now for each $i = 1, \cdots, m$, Lemma 3.1.12 yields a minuscule effective modification $\mathcal{E}'_i \hookrightarrow \mathcal{E}_i$ at ∞ as $\mathcal{E}_i$ is semistable. Hence we obtain a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ from the direct sum decompositions (3.7) and (3.8).

For the converse, we now assume that there exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞. Since $\mathcal{E}$ and $\mathcal{E}'$ satisfy the condition (i) by Proposition 3.3.2, it remains to establish the condition (ii). We proceed by induction on the number m of distinct slopes in $\mathrm{HN}(\mathcal{E})$.

If $\mathcal{E}$ is semistable, the assertion is vacuously true as $\text{HN}(\mathcal{E})$ does not have a breakpoint. We henceforth assume that $\mathcal{E}$ is not semistable, so that we have $m > 1$. Take a direct sum decomposition

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F} \tag{3.9}$$

such that $\text{HN}(\mathcal{D})$ coincides with the line segment of maximal slope in $\text{HN}(\mathcal{E})$. Let us denote the slope of $\text{HN}(\mathcal{D})$ by λ. By construction, $\text{HN}(\mathcal{F})$ has $m-1$ distinct slopes which are all less than $\lambda - 1$ by our hypothesis on the slopes of $\text{HN}(\mathcal{E})$. In addition, we have $\text{HN}(\mathcal{E}') + \underline{1}^{(n)} \succeq \text{HN}(\mathcal{E})$ by Proposition 3.3.2 and thus find

$$\lambda_i(\text{HN}(\mathcal{E}')) \geq \lambda - 1 \quad \text{ for } i = 1, \cdots, \text{rk}(\mathcal{D}). \tag{3.10}$$

Now we note by Proposition 3.1.10 that there exist minuscule effective modifications $\mathcal{D}' \hookrightarrow \mathcal{D}$ and $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ with a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0. \tag{3.11}$$

Then we find

$$\lambda_i(\text{HN}(\mathcal{F}')) \leq \lambda_i(\text{HN}(\mathcal{F})) < \lambda - 1 \quad \text{ for } i = 1, \cdots, \text{rk}(\mathcal{F}') \tag{3.12}$$

by Proposition 2.4.4, and also obtain an $\mathcal{E}'$-permutation $\mathscr{P}$ of $\text{HN}(\mathcal{D}' \oplus \mathcal{F}')$ by Proposition 2.4.7. For each $i = 1, \cdots, \text{rk}(\mathcal{D})$, the inequalities (3.10) and (3.12) together imply that $\lambda_i(\mathscr{P})$ should occur as a slope in $\text{HN}(\mathcal{D}')$. Since we have $\mathscr{P} \geq \text{HN}(\mathcal{E}')$ by construction, we find

$$\lambda_i(\mathscr{P}) = \lambda_i(\text{HN}(\mathcal{D}')) = \lambda_i(\text{HN}(\mathcal{E}')) \quad \text{ for } i = 1, \cdots, \text{rk}(\mathcal{D})$$

and consequently deduce from the inequalities (3.10) and (3.12) that all slopes in $\text{HN}(\mathcal{D}')$ are greater than all slopes in $\text{HN}(\mathcal{F}')$. Hence the short exact sequence (3.11) induces a direct sum

$$\mathcal{E}' \simeq \mathcal{D}' \oplus \mathcal{F}' \tag{3.13}$$

by Proposition 2.4.6, and consequently yields a breakpoint of $\text{HN}(\mathcal{E}')$ with x-coordinate $\text{rk}(\mathcal{D}') = \text{rk}(\mathcal{D})$. In addition, since we have a minuscule effective modification $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞, we find by the induction hypothesis that for every breakpoint of $\text{HN}(\mathcal{F})$ there exists a breakpoint of $\text{HN}(\mathcal{F}')$ with the same x-coordinate. We thus establish the condition (ii) by the direct sum decompositions (3.9) and (3.13), thereby completing the proof. □

Example 3.3.5. Let us provide an example to show that Proposition 3.3.4 does not hold without the assumption on the slopes in $\text{HN}(\mathcal{E})$. Take $\mathcal{E}$ and $\mathcal{E}'$ to be vector bundles on X with

$$\text{HN}(\mathcal{E}) = \underline{5/4}^{(4)} \oplus \underline{3/4}^{(4)} \quad \text{ and } \quad \text{HN}(\mathcal{E}') = \underline{3/5}^{(5)} \oplus \underline{1/3}^{(3)}.$$

Then $\text{HN}(\mathcal{E})$ and $\text{HN}(\mathcal{E}')$ do not have breakpoints with the same x-coordinates. We wish to show that there exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞. Take vector bundles $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{F}$ and $\mathcal{F}'$ on X with

$$\text{HN}(\mathcal{D}) = \underline{5/4}^{(4)}, \quad \text{HN}(\mathcal{D}') = \underline{1/4}^{(4)}, \quad \text{HN}(\mathcal{F}) = \text{HN}(\mathcal{F}') = \underline{3/4}^{(4)}.$$

By construction, we have a direct sum decomposition

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F}.$$

In addition, we obtain minuscule effective modifications $\mathcal{D}' \hookrightarrow \mathcal{D}$ and $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ by Lemma 3.1.12, and find a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0$$

by Proposition 2.4.8. Therefore Proposition 3.1.10 yields a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ as desired.

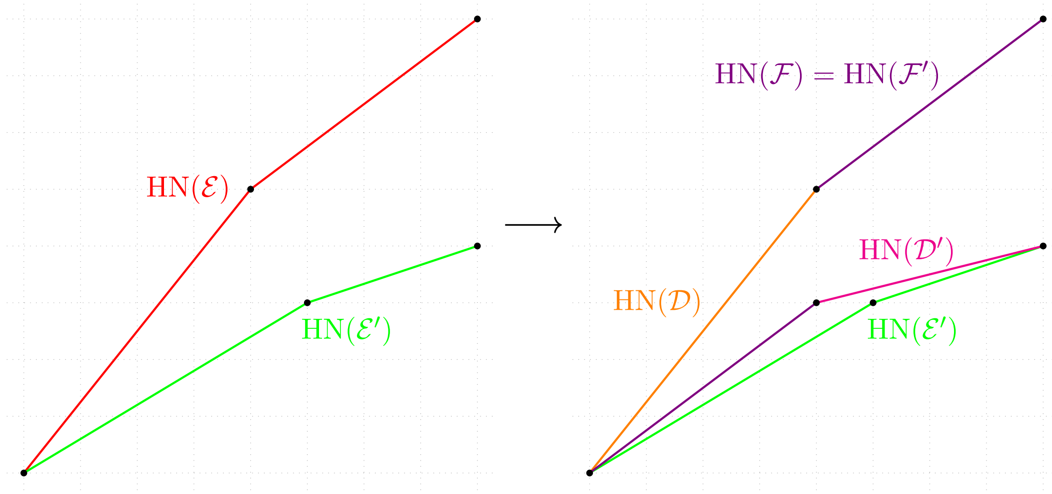

FIGURE 7. Illustration of Example 3.3.5

Theorem 3.3.6. *Let μ be a minuscule dominant cocharacter of GL_n with slopes 0 and 1. Take two arbitrary elements $b, b' \in B(\mathrm{GL}_n)$ and write $\nu(b) := \mathrm{HN}(\mathcal{E}_b)$ and $\nu(b') := \mathrm{HN}(\mathcal{E}_{b'})$. Assume that the difference between any two distinct slopes in $\nu(b)$ is greater than 1. The Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is nonempty if and only if $\nu(b)$ and $\nu(b')$ satisfy the following conditions:*

(i) We have $\nu(b) + \mu^* \geq \nu(b')$ and $\nu(b') + \underline{1}^{(n)} \succeq \nu(b) \succeq \nu(b')$.

(ii) For each breakpoint of $\nu(b)$, there exists a breakpoint of $\nu(b')$ with the same x-coordinate.

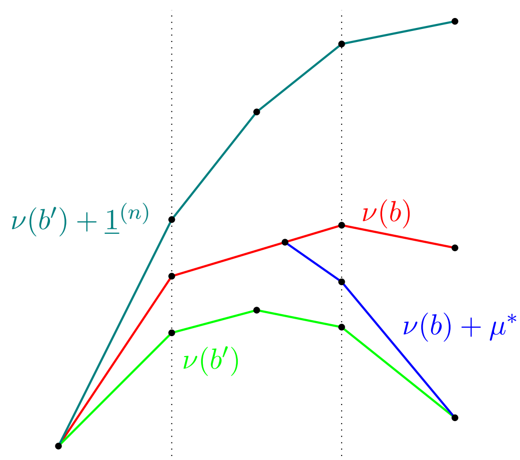

FIGURE 8. Illustration of the conditions in Theorem 3.3.6

Proof. The assertion is an immediate consequence of Corollary 3.1.9, Proposition 3.1.11 and Proposition 3.3.4. □

Remark. For a non-minuscule cocharacter μ of GL_n with slopes in $[0, d]$, we should be able to get a similar classification theorem with d in place of 1 using the Demazure resolution.

References

[BL95] Arnaud Beauville and Yves Laszlo, *Un lemme de descente*, C. R. Acad. Sci. Paris Sr. I Math. **320** (1995), no. 1, 335–340.

[CFS21] Miaofen Chen, Laurent Fargues, and Xu Shen, *On the structure of some p-adic period domains*, Camb. J. Math. **9** (2021), no. 1, 213–267.

[Che20] Miaofen Chen, *Fargues-Rapoport conjecture for p-adic period domains in the non-basic case*, J. Eur. Math. Soc., to appear.

[CS17] Ana Caraiani and Peter Scholze, *On the generic part of the cohomology of compact unitary Shimura varieties*, Annals of Math. **186** (2017), no. 3, 649–766.

[CT22] Miaofen Chen and Jilong Tong, *Weakly admissible locus and Newton stratification in p-adic Hodge theory*, preprint.

[Far20] Laurent Fargues, *G-torseurs en théorie de Hodge p-adique*, Comp. Math. **156** (2020), no. 10, 2076–2110.

[FF18] Laurent Fargues and Jean-Marc Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Astérisque **406** (2018).

[Fon82] Jean-Marc Fontaine, *Sur certains types de representations p-adiques du groupe de Galois d'un corps local; Construction d'un anneau de Barsotti-Tate*, Annals of Mathematics **115** (1982), no. 3, 529–577.

[FS21] Laurent Fargues and Peter Scholze, *Geometrization of the local Langlands correspondence*, arXiv:2102.13459.

[Hon21] Serin Hong, *Classification of subbundles on the Fargues-Fontaine curve*, Algebra & Number Theory **15** (2021), no. 5, 1127–1156.

[Hon22] ______, *Extensions of vector bundles on the Fargues-Fontaine curve II*, arXiv:2203.12838.

[Ked16] Kiran S. Kedlaya, *Noetherian properties of Fargues-Fontaine curves*, Int. Math. Res. Not. IMRN (2016), no. 8, 2544–2567.

[KL15] Kiran S. Kedlaya and Ruochuan Liu, *Relative p-adic Hodge theory: foundations*, Astérisque **371** (2015).

[Kot85] Robert E. Kottwitz, *Isocrystals with additional structure*, Comp. Math. **56** (1985), 201–220.

[NV21] Kieu Hieu Nguyen and Eva Viehmann, *A Harder-Narasimhan stratification of the B^+_{dR}-Grassmannian*, arXiv:2111.01764.

[Rap18] Michael Rapoport, *Accessible and weakly accessible period domains*, Annales de l'ENS **51** (2018), 856–863, appendix to *On the p-adic cohomology of the Lubin-Tate tower* by P. Scholze.

[She19] Xu Shen, *Harder-Narasimhan strata and p -adic period domains*, arXiv:1909.02230.

[SW20] Peter Scholze and Jared Weinstein, *Berkeley lectures on p-adic geometry*, Annals of Math. Studies, vol. 207, 2020.

[Vie21] Eva Viehmann, *On Newton strata in the B^+_{dR}-Grassmannian*, arXiv:2101.07510.

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