EXTENSIONS OF VECTOR BUNDLES ON THE FARGUES-FONTAINE CURVE II

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Abstract. Given two arbitrary vector bundles on the Fargues-Fontaine curve, we completely classify all vector bundles which arise as their extensions.

Contents

1. Introduction

The *Fargues-Fontaine curve* is a geometric object that plays a fundamental role in many parts of arithmetic geometry. It was originally constructed by Fargues-Fontaine [FF18] as a main tool for a geometric formulation of classical p-adic Hodge theory. The work of Scholze-Weinstein [SW20] then used it to develop new technical frameworks for p-adic geometry, including local Shimura varieties and the B_{dR}^+-Grassmanninans. Recently, the seminal work of Fargues-Scholze [FS21] established a semi-simplified local Langlands correspondence for a general p-adic group G in terms of the stack of G-bundles on the Fargues-Fontaine curve.

In this article, we investigate the question of classifying all vector bundles on the Fargues-Fontaine curve that arise as extensions of two given vector bundles. This question naturally arises in the study of p-adic flag varieties and the B_{dR}^+-Grassmanninans, as in the work of Viehmann [Vie21] and Chen-Tong [CT22]. Our main result gives a complete answer to this question, extending the previous work of the author [Hon20]. In addition, our main result leads to a classification of all nonempty Newton strata in minuscule p-adic flag varieties for GL_n by the subsequent work of the author [Hon22]. We hope that our result will further lead to a concise description of how the Harder-Narasimhan strata and the Newton strata on the B_{dR}^+-Grassmanninans intersect, building upon the work of Shen [She19], Viehmann [Vie21], and Nguyen-Viehmann [NV21].

For a precise statement of our main result, we set up some basic notations and collect some fundamental facts about the Fargues-Fontaine curve. Let E be a nonarchimedean local field with finite residue field $\mathbb{F}_q$, where q is a power of a fixed prime number p, and let F an algebraically closed nonarchimedean complete field of characteristic p. We denote by $X_{E,F}$ the Fargues-Fontaine curve associated to the pair (E, F), which is a regular noetherian scheme over E of Krull dimension 1. The Picard group of $X_{E,F}$ is naturally isomorphic to $\mathbb{Z}$, and thus yields a good Harder-Narasimhan formalism for vector bundles on $X_{E,F}$. The main result of Fargues-Fontaine [FF18] states that the Harder-Narasimhan filtration of an arbitrary vector bundle $\mathcal{V}$ on $X_{E,F}$ splits. In other words, the isomorphism class of every vector bundle $\mathcal{V}$ on $X_{E,F}$ is determined by its Harder-Narasimhan polygon $\mathrm{HN}(\mathcal{V})$. We regard $\mathrm{HN}(\mathcal{V})$ as (the graph of) a concave piecewise linear function with the left endpoint at the origin, and denote by $\mu_i(\mathrm{HN}(\mathcal{V}))$ the slope of $\mathrm{HN}(\mathcal{V})$ on the interval $[i-1, i]$ for each integer $i > 0$.

Theorem 1.1. *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X_{E,F}$ such that $\mathcal{D}$ or $\mathcal{F}$ is semistable. There exists a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

if and only if the line segments of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ can be rearranged so that the resulting (possibly non-concave) polygon $\mathscr{P}$ satisfies the following properties:

(i) $\mathscr{P}$ lies above $\mathrm{HN}(\mathcal{E})$ with the same endpoints.

(ii) For $i = 1, \cdots, rank(\mathcal{E})$, the polygon $\mathscr{P}$ has a constant slope $\mu_i(\mathscr{P})$ on $[i-1, i]$ with

- *$\mu_i(\mathscr{P}) < \mu_i(\mathrm{HN}(\mathcal{E}))$ only if $\mu_i(\mathscr{P})$ occurs as a slope in $\mathrm{HN}(\mathcal{D})$,*
- *$\mu_i(\mathscr{P}) > \mu_i(\mathrm{HN}(\mathcal{E}))$ only if $\mu_i(\mathscr{P})$ occurs as a slope in $\mathrm{HN}(\mathcal{F})$.*

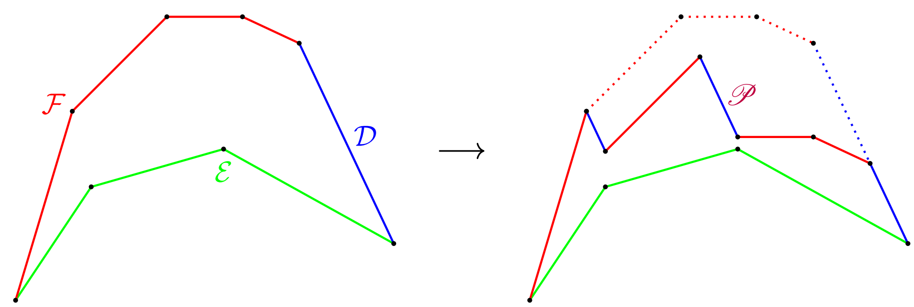

FIGURE 1. Illustration of the conditions in Theorem 1.1

Theorem 1.2. *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X_{E,F}$. There exists a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

if and only if $\mathcal{E}$ admits a filtration

$$\mathcal{D} = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$$

such that the induced filtration

$$0 = \mathcal{E}_0/\mathcal{D} \subset \mathcal{E}_1/\mathcal{D} \subset \cdots \subset \mathcal{E}_r/\mathcal{D} = \mathcal{E}/\mathcal{D}$$

coincides with the Harder-Narasimhan filtration of $\mathcal{F}$.

Let us make some remarks about our main results. By Theorem 1.1, we can formulate Theorem 1.2 purely in terms of Harder-Narasimhan polygons; we refer the readers to Theorem 4.4 for a precise statement. After completing this work, we became aware that Chen-Tong [CT22] independently obtained our main results in a similar way. For vector bundles on $\mathbb{P}^1$, there is an analogue of our main results due to Schlesinger [Sch00].

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2. Vector bundles on the Fargues-Fontaine curve

Throughout this paper, we fix a field F of characteristic $p > 0$ which is complete, nonarchimedean, and algebraically closed. We also let E denote an arbitrary nonarchimedean local field whose residue field is finite of characteristic p.

Definition 2.1. Let $\mathcal{O}_E$ and $\mathcal{O}_F$ respectively denote the valuation rings of E and F. Fix a uniformizer π of E and a pseudouniformizer ϖ of F. Let q be the number of elements in the residue field of E.

(1) If E is of equal characteristic, we set
$$\mathcal{Y}_{E,F} := \mathrm{Spa}(\mathcal{O}_F[\![\pi]\!]) \setminus \{|\pi\varpi| = 0\},$$
and define the *adic Fargues-Fontaine curve* associated to the pair (E, F) by
$$\mathcal{X}_{E,F} := \mathcal{Y}_{E,F}/\phi^{\mathbb{Z}}$$
where ϕ denotes the automorphism of $\mathcal{Y}_{E,F}$ induced by the q-Frobenius automorphism on $\mathcal{O}_F[\![\pi]\!]$.

(2) If E is of mixed characteristic, we set
$$\mathcal{Y}_{E,F} := \mathrm{Spa}(W_{\mathcal{O}_E}(\mathcal{O}_F) \setminus \{|\pi[\varpi]| = 0\},$$
where $W_{\mathcal{O}_E}(\mathcal{O}_F)$ denotes the ring of ramified Witt vectors over $\mathcal{O}_F$ with coefficients in $\mathcal{O}_E$ and the Teichmuller lift $[\varpi]$ of ϖ, and define the *adic Fargues-Fontaine curve* associated to the pair (E, F) by
$$\mathcal{X}_{E,F} := \mathcal{Y}_{E,F}/\phi^{\mathbb{Z}}$$
where ϕ denotes the automorphism of $\mathcal{Y}_{E,F}$ induced by the q-Frobenius automorphism on $W_{\mathcal{O}_E}(\mathcal{O}_F)$.

(3) We define the *schematic Fargues-Fontaine curve* associated to the pair (E, F) by
$$X_{E,F} := \mathrm{Proj}\left(\bigoplus_{n\geq 0} H^0(\mathcal{Y}_{E,F}, \mathcal{O}_{\mathcal{Y}_{E,F}})^{\phi=\pi^n}\right).$$

Remark. The definition of the adic Fargues-Fontaine curve makes sense since the action of ϕ on $\mathcal{Y}_{E,F}$ is properly discontinuous.

Theorem 2.2 ([KL15, Theorems 6.3.12 and 8.7.7])**.** *There is a natural map of locally ringed spaces*
$$\mathcal{X}_{E,F} \longrightarrow X_{E,F}$$
which induces by pullback an equivalence of the categories of vector bundles.

In light of Theorem 2.2, which may be regarded as GAGA for the Fargues-Fontaine curve, we will henceforth identify vector bundles on $\mathcal{X}_{E,F}$ with vector bundles on $X_{E,F}$.

Definition 2.3. Let $\lambda = d/r$ be a rational number written in lowest terms with $r > 0$. We define the vector bundle $\mathcal{O}_{E,F}(\lambda)$ on $\mathcal{X}_{E,F}$ (or on $X_{E,F}$) by descending along the map $\mathcal{Y}_{E,F} \longrightarrow \mathcal{Y}_{E,F}/\phi^{\mathbb{Z}} = \mathcal{X}_{E,F}$ the trivial bundle $\mathcal{O}_{\mathcal{Y}_{E,F}}^{\oplus r}$ equipped with the isomorphism $\phi^*\mathcal{O}_{\mathcal{Y}_{E,F}}^{\oplus r} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}_{E,F}}^{\oplus r}$ represented by the matrix
$$\left(\begin{array}{c|ccc} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \hline \pi^{-d} & & & \end{array}\right).$$

Proposition 2.4 ([FF18, Théorème 6.5.2])**.** *The schematic Fargues-Fontaine curve $X_{E,F}$ is a Dedekind scheme over E, with a natural isomorphism from its Picard group* $\mathrm{Pic}(X_{E,F})$ *to* $\mathbb{Z}$ *which associates each $d \in \mathbb{Z}$ with $\mathcal{O}_{E,F}(d)$.*

Remark. While Proposition 2.4 suggests that $X_{E,F}$ behaves much as algebraic curves do, $X_{E,F}$ itself is not an algebraic curve for not being of finite type over the base field E.

Definition 2.5. Let $\mathcal{V}$ be an arbitrary nonzero vector bundle on $X_{E,F}$.

(1) We write $\mathrm{rk}(\mathcal{V})$ for the rank of $\mathcal{V}$, and define the *degree* of $\mathcal{V}$ to be the integer $\deg(\mathcal{V})$ which corresponds to the isomorphism class of the determinant line bundle $\wedge^{\mathrm{rk}(\mathcal{V})}(\mathcal{V})$ under the natural isomorphism $\mathrm{Pic}(X_{E,F}) \cong \mathbb{Z}$ in Proposition 2.4.

(2) We define the *slope* of $\mathcal{V}$ to be

$$\mu(\mathcal{V}) := \frac{\deg(\mathcal{V})}{\mathrm{rk}(\mathcal{V})}.$$

Proposition 2.6 ([FF18, Proposition 5.6.23],[Ked05, Proposition 4.1.3])**.** *Let $\lambda = d/r$ be a rational number written in lowest terms with $r > 0$.*

(1) The vector bundle $\mathcal{O}_{E,F}(\lambda)$ has rank r, degree d, and slope $\lambda = d/r$.

(2) The dual of $\mathcal{O}_{E,F}(\lambda)$ is isomorphic to $\mathcal{O}_{E,F}(-\lambda)$.

(3) $\mathrm{Hom}(\mathcal{O}_{E,F}(\lambda), \mathcal{O}_{E,F}(\mu))$ *is trivial for all rational numbers $\mu < \lambda$.*

(4) $\mathrm{Ext}^1(\mathcal{O}_{E,F}(\lambda), \mathcal{O}_{E,F}(\mu))$ *is trivial for all rational numbers $\mu \geq \lambda$.*

Proposition 2.7 ([FF18, Proposition 5.6.23])**.** *Let E' be an unramified finite extension of E. Denote by d the degree of E' over E.*

(1) There exists a canonical isomorphism $X_{E',F} \cong X_{E,F} \times_{\mathrm{Spec}\,(E)} \mathrm{Spec}\,(E')$.

(2) The projection map

$$\pi : X_{E',F} \cong X_{E,F} \times_{\mathrm{Spec}\,(E)} \mathrm{Spec}\,(E') \longrightarrow X_{E,F}$$

induces a natural identification

$$\pi^*\mathcal{O}_{E,F}(\lambda) \cong \mathcal{O}_{E',F}(d\lambda)^{\oplus m} \qquad \textit{for each } \lambda \in \mathbb{Q}$$

with $m = \mathrm{rk}(\mathcal{O}_{E,F}(\lambda))/\mathrm{rk}(\mathcal{O}_{E',F}(d\lambda))$.

Definition 2.8. Let $\mathcal{V}$ be a vector bundle on $X_{E,F}$.

(1) We say that $\mathcal{V}$ is *stable* if we have $\mu(\mathcal{W}) < \mu(\mathcal{V})$ for all nonzero subbundles $\mathcal{W} \subsetneq \mathcal{V}$.

(2) We say that $\mathcal{V}$ is *semistable* if we have $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$ for all nonzero subbundles $\mathcal{W} \subsetneq \mathcal{V}$.

Theorem 2.9 ([FF18, Théorème 8.2.10])**.** *Let $\mathcal{V}$ be a vector bundle on $X_{E,F}$.*

(1) $\mathcal{V}$ is stable of slope λ if and only if it is isomorphic to $\mathcal{O}_{E,F}(\lambda)$.

(2) $\mathcal{V}$ is semistable of slope λ if and only if it is isomorphic to $\mathcal{O}_{E,F}(\lambda)^{\oplus m}$ for some m.

(3) $\mathcal{V}$ admits a direct sum decomposition

$$\mathcal{V} \simeq \bigoplus_{i=1}^{l} \mathcal{O}_{E,F}(\lambda_i)^{\oplus m_i} \qquad \textit{with } \lambda_i \in \mathbb{Q},$$

where the λ_i's are all distinct and uniquely determined up to permutations.

Remark. Prior to the work of Fargues-Fontaine [FF18], Theorem 2.9 had been obtained in a different language by Hartl-Pink [HP04, Theorem 11.1] and Kedlaya [Ked05, Theorem 4.5.7].

Definition 2.10. Let $\mathcal{V}$ be a vector bundle on $X_{E,F}$. Fix a direct sum decomposition

$$\mathcal{V} \simeq \bigoplus_{i=1}^{l} \mathcal{O}_{E,F}(\lambda_i)^{\oplus m_i} \qquad \text{with } \lambda_i \in \mathbb{Q} \tag{2.1}$$

given by Theorem 2.9.

(1) We refer to the numbers λ_i as the *Harder-Narasimhan (HN) slopes* of $\mathcal{V}$, or often simply as the *slopes* of $\mathcal{V}$.
(2) We write $\mu_{\max}(\mathcal{V})$ for the maximum HN slope of $\mathcal{V}$, and $\mu_{\min}(\mathcal{V})$ for the minimum HN slope of $\mathcal{V}$.
(3) For every $\mu \in \mathbb{Q}$, we define the direct summands

$$\mathcal{V}^{\geq \mu} := \bigoplus_{\lambda_i \geq \mu} \mathcal{O}_{E,F}(\lambda_i)^{\oplus m_i} \qquad \text{and} \qquad \mathcal{V}^{\leq \mu} := \bigoplus_{\lambda_i \leq \mu} \mathcal{O}_{E,F}(\lambda_i)^{\oplus m_i},$$

and similarly define $\mathcal{V}^{>\mu}$ and $\mathcal{V}^{<\mu}$.
(4) We define the *Harder-Narasimhan (HN) polygon* of $\mathcal{V}$, denoted by $\mathrm{HN}(\mathcal{V})$, as the upper convex hull of the points $(0,0)$ and $\big(\mathrm{rk}(\mathcal{V}^{\geq \lambda_i}), \deg(\mathcal{V}^{\geq \lambda_i})\big)$.

Remark. For our work, the choice of the decomposition (2.1) does not play a significant role. In fact, since $\mathcal{V}$ determines the numbers λ_i and m_i in the decomposition (2.1), it consequently determines the isomorphism classes of the direct summands $\mathcal{V}^{\geq \mu}$, $\mathcal{V}^{\leq \mu}$, $\mathcal{V}^{>\mu}$, and $\mathcal{V}^{<\mu}$ for each $\mu \in \mathbb{Q}$ and also determines the HN polygon $\mathrm{HN}(\mathcal{V})$.

3. Extensions and permutations of HN polygons

Our goal in this section is to state and derive the necessary conditions for the existence of a short exact sequence involving three given vector bundles on the Fargues-Fontaine curve.

Definition 3.1. A *rationally tuplar polygon* is a graph $\mathscr{P}$ of a continuous function on a closed interval which satisfies the following properties:

(i) The endpoints of $\mathscr{P}$ are integer points, with the left endpoint being $(0,0)$.
(ii) For each integer i in the domain, $\mathscr{P}$ is linear on the interval $[i-1, i]$ with a rational slope denoted by $\mu_i(\mathscr{P})$.

Remark. For every vector bundle $\mathcal{V}$ on $X_{E,F}$, its Harder-Narasimhan polygon $\mathrm{HN}(\mathcal{V})$ is a rationally tuplar polygon with the following additional properties:

- All breakpoints of $\mathrm{HN}(\mathcal{V})$ are integer points.
- The slopes of $\mathrm{HN}(\mathcal{V})$ are decreasing.

However, in this article we will consider rationally tuplar polygons which do not necessarily enjoy these properties.

Lemma 3.2. *Let $\mathcal{V}$ and $\mathcal{W}$ be arbitrary vector bundles on $X_{E,F}$.*

(1) If $\mathcal{W}$ is a subsheaf of $\mathcal{V}$, we have

$$\mu_i(\mathrm{HN}(\mathcal{W})) \leq \mu_i(\mathrm{HN}(\mathcal{V})) \qquad \text{for } i = 1, \cdots, \mathrm{rk}(\mathcal{W}).$$

(2) If $\mathcal{W}$ is a quotient of $\mathcal{V}$, we have

$$\mu_i(\mathrm{HN}(\mathcal{W})) \geq \mu_{i+\mathrm{rk}(\mathcal{V})-\mathrm{rk}(\mathcal{W})}(\mathrm{HN}(\mathcal{V})) \qquad \text{for } i = 1, \cdots, \mathrm{rk}(\mathcal{W}).$$

Proof. Let us establish the first statement. Suppose for contradiction that $\mathcal{W}$ is a subsheaf of $\mathcal{V}$ with $\mu_i(\mathrm{HN}(\mathcal{W})) > \mu_i(\mathrm{HN}(\mathcal{V}))$ for some positive integer $i \leq \mathrm{rk}(\mathcal{W})$. Let us fix an injective bundle map $\mathcal{W} \hookrightarrow \mathcal{V}$ and write $\mu := \mu_i(\mathrm{HN}(\mathcal{W}))$. We have direct sum decompositions

$$\mathcal{V} \simeq \mathcal{V}^{\geq \mu} \oplus \mathcal{V}^{<\mu} \qquad \text{and} \qquad \mathcal{W} \simeq \mathcal{W}^{\geq \mu} \oplus \mathcal{W}^{<\mu}.$$

Since every bundle map from $\mathcal{W}^{\geq \mu}$ to $\mathcal{V}^{<\mu}$ is zero by Proposition 2.6, the injective map $\mathcal{W} \hookrightarrow \mathcal{V}$ restricts to an injective map $\mathcal{W}^{\geq \mu} \hookrightarrow \mathcal{V}^{\geq \mu}$. Hence we find $\mathrm{rk}(\mathcal{W}^{\geq \mu}) \leq \mathrm{rk}(\mathcal{V}^{\geq \mu})$. On the other hand, we have $\mathrm{rk}(\mathcal{W}^{\geq \mu}) \geq i > \mathrm{rk}(\mathcal{V}^{\geq \mu})$ by the concavity of HN polygons and the inequality $\mu_i(\mathrm{HN}(\mathcal{W})) > \mu_i(\mathrm{HN}(\mathcal{V}))$. We thus obtain a contradiction as desired.

For the second statement, we now assume that $\mathcal{W}$ is a quotient of $\mathcal{V}$. Let us write $\mathcal{V}^\vee$ and $\mathcal{W}^\vee$ respectively for the duals of $\mathcal{V}$ and $\mathcal{W}$. Since $\mathcal{W}^\vee$ is a subbundle of $\mathcal{V}^\vee$, we use Proposition 2.6 and the first statement to find

$$\mu_i(\mathrm{HN}(\mathcal{W})) = -\mu_{\mathrm{rk}(\mathcal{W})+1-i}(\mathrm{HN}(\mathcal{W}^\vee)) \geq -\mu_{\mathrm{rk}(\mathcal{W})+1-i}(\mathrm{HN}(\mathcal{V}^\vee)) = \mu_{i+\mathrm{rk}(\mathcal{V})-\mathrm{rk}(\mathcal{W})}(\mathrm{HN}(\mathcal{V}))$$

for each $i = 1, \cdots, \mathrm{rk}(\mathcal{W})$, thereby deducing the second statement. □

Remark. The converse of the first statement is true by a previous result of the author [Hon21, Theorem 1.1.2], whereas the converse of the second statement does not by another previous result of the author [Hon19, Theorem 1.1.2]. However, our argument won't need these results.

Definition 3.3. Let $\mathscr{P}$ and $\mathscr{Q}$ be rational tuplar polygons. We say that $\mathscr{P}$ *dominates* $\mathscr{Q}$ and write $\mathscr{P} \geq \mathscr{Q}$ if $\mathscr{P}$ and $\mathscr{Q}$ satisfy the following properties:

(i) $\mathscr{P}$ and $\mathscr{Q}$ have the same endpoints.

(ii) If r denotes the x-coordinate of their common right endpoint, we have

$$\sum_{i=1}^{j} \mu_i(\mathscr{P}) \geq \sum_{i=1}^{j} \mu_i(\mathscr{Q}) \qquad \text{for each } j = 1, \cdots, r.$$

Remark. In other words, we have $\mathscr{P} \geq \mathscr{Q}$ if and only if $\mathscr{P}$ lies on or above $\mathscr{Q}$ with the same endpoints, as in Mazur's inequality.

Lemma 3.4. *The binary relation $\geq$ is a partial order on the set of rational tuplar polygons.*

Proof. This is straightforward to check using Definition 3.3. □

Definition 3.5. Given vector bundles $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ on $X_{E,F}$, we define an *$\mathcal{E}$-permutation* of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ to be a rationally tuplar polygon $\mathscr{P} \geq \mathrm{HN}(\mathcal{E})$ with the following properties:

(i) The tuple $(\mu_i(\mathscr{P}))$ is a permutation of the tuple $(\mu_i(\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})))$.

(ii) For each $i = 1, \cdots, \mathrm{rk}(\mathcal{E})$, we have
- $\mu_i(\mathscr{P}) < \mu_i(\mathrm{HN}(\mathcal{E}))$ only if $\mu_i(\mathscr{P})$ occurs as a slope of $\mathcal{D}$, and
- $\mu_i(\mathscr{P}) > \mu_i(\mathrm{HN}(\mathcal{E}))$ only if $\mu_i(\mathscr{P})$ occurs as a slope of $\mathcal{F}$.

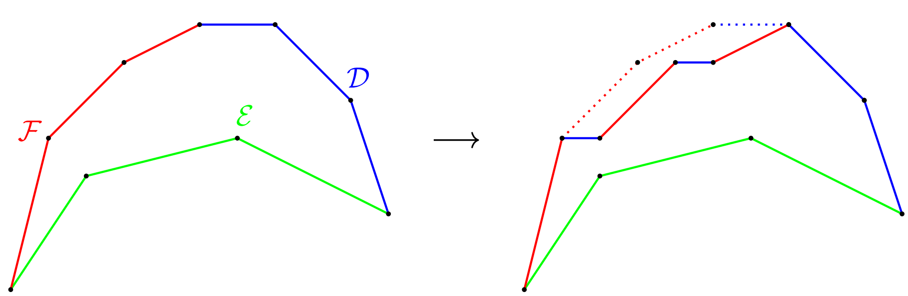

FIGURE 2. Illustration of the conditions in Definition 3.5

Lemma 3.6. *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be arbitrary vector bundles on $X_{E,F}$. A rationally tuplar polygon $\mathscr{P} \geq \mathrm{HN}(\mathcal{E})$ is an $\mathcal{E}$-permutation of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ if and only if there exists an ordered pair $(S_\mathcal{D}, S_\mathcal{F})$ of sets satisfying the following properties:*

(i) The sets $S_\mathcal{D}$ and $S_\mathcal{F}$ form a partition of the index set $\{1, \cdots, \mathrm{rk}(\mathcal{E})\}$.

(ii) The tuple $(\mu_i(\mathscr{P}))_{i \in S_\mathcal{D}}$ permutes the tuple $(\mu_i(\mathrm{HN}(\mathcal{D})))$ with

$$\mu_i(\mathscr{P}) \leq \mu_i(\mathrm{HN}(\mathcal{E})) \text{ for all } i \in S_\mathcal{D}.$$

(iii) The tuple $(\mu_i(\mathscr{P}))_{i \in S_\mathcal{F}}$ permutes the tuple $(\mu_i(\mathrm{HN}(\mathcal{F})))$ with

$$\mu_j(\mathscr{P}) \geq \mu_j(\mathrm{HN}(\mathcal{E})) \text{ for all } j \in S_\mathcal{F}.$$

Proof. This is evident by Definition 3.5. □

Definition 3.7. Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X_{E,F}$ with an $\mathcal{E}$-permutation $\mathscr{P}$ of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$.

(1) We refer to an ordered pair $(S_\mathcal{D}, S_\mathcal{F})$ as in Lemma 3.6 as a *$\mathscr{P}$-partition pair.*

(2) Given a $\mathscr{P}$-partition pair $(S_\mathcal{D}, S_\mathcal{F})$, we say that $\mathscr{P}$ is *$(S_\mathcal{D}, S_\mathcal{F})$-sorted* if we have

$$(\mu_i(\mathscr{P}))_{i \in S_\mathcal{D}} = (\mu_i(\mathrm{HN}(\mathcal{D}))) \quad \text{and} \quad (\mu_j(\mathscr{P}))_{j \in S_\mathcal{F}} = (\mu_j(\mathrm{HN}(\mathcal{F}))).$$

Remark. For our work, we will consider tuples of various sizes and/or index sets, as in the two equalities above. When we write an equality of tuples, we always specify the index set for one side to specify the size of tuples. In addition, when we do not specify the index set for a tuple, it means that the index set is take to be maximal (ranging from 1 to the length of the tuple).

Lemma 3.8. *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X_{E,F}$. Suppose that there exists an $\mathcal{E}$-permutation $\mathscr{P}$ of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ with a $\mathscr{P}$-partition pair $(S_\mathcal{D}, S_\mathcal{F})$. There exists an $\mathcal{E}$-permutation $\mathscr{Q}$ of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ with the following properties:*

(i) $(S_\mathcal{D}, S_\mathcal{F})$ is a $\mathscr{Q}$-partition pair.

(ii) $\mathscr{Q}$ is $(S_\mathcal{D}, S_\mathcal{F})$-sorted.

Proof. Since there are finitely many $\mathcal{E}$-permutations of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$, we can take a permutation $\mathscr{Q}$ of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ which is maximal among those with property (i). We wish to show that $\mathscr{Q}$ is $(S_\mathcal{D}, S_\mathcal{F})$-sorted. By concavity of HN polygons, it suffices to prove that the tuples $(\mu_i(\mathscr{Q}))_{i \in S_\mathcal{D}}$ and $(\mu_j(\mathscr{Q}))_{i \in S_\mathcal{F}}$ are sorted in descending order.

Let us first verify that $(\mu_i(\mathscr{Q}))_{i \in S_\mathcal{D}}$ is sorted in descending order. Suppose for contradiction that there exist integers $a, b \in S_\mathcal{D}$ with $a < b$ and $\mu_a(\mathscr{Q}) < \mu_b(\mathscr{Q})$. We have

$$\mu_b(\mathrm{HN}(\mathcal{E})) \leq \mu_a(\mathrm{HN}(\mathcal{E})) \leq \mu_a(\mathscr{Q}) < \mu_b(\mathscr{Q}) \tag{3.1}$$

where the first inequality follows from the concavity of $\mathrm{HN}(\mathcal{E})$. Take $\mathscr{Q}'$ to be the rationally tuplar polygon such that the tuple $(\mu_i(\mathscr{Q}'))$ swaps the positions of $\mu_a(\mathscr{Q})$ and $\mu_b(\mathscr{Q})$ in the tuple $(\mu_i(\mathscr{Q}))$. It follows from the inequality (3.1) that $\mathscr{Q}'$ is an $\mathcal{E}$-permutation of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ with $\mathscr{Q}' \geq \mathscr{Q}$ and $\mathscr{Q}' \neq \mathscr{Q}$ such that $(S_\mathcal{D}, S_\mathcal{F})$ is a $\mathscr{Q}''$-partition pair. Hence we obtain a contradiction to the maximality of $\mathscr{Q}$ as desired.

It remains to check that $(\mu_j(\mathscr{Q}))_{j \in S_\mathcal{F}}$ is sorted in descending order. Suppose for contradiction that there exist integers $c, d \in S_\mathcal{F}$ with $c < d$ and $\mu_c(\mathscr{Q}) < \mu_d(\mathscr{Q})$. We have

$$\mu_c(\mathscr{Q}) < \mu_d(\mathscr{Q}) \leq \mu_d(\mathrm{HN}(\mathcal{E})) \leq \mu_c(\mathrm{HN}(\mathcal{E})) \tag{3.2}$$

where the last inequality follows from the concavity of $\mathrm{HN}(\mathcal{E})$. Take $\mathscr{Q}''$ to be the rationally tuplar polygon such that the tuple $(\mu_j(\mathscr{Q}''))$ swaps the positions of $\mu_c(\mathscr{Q})$ and $\mu_d(\mathscr{Q})$ in the tuple $(\mu_j(\mathscr{Q}))$. It follows from the inequality (3.2) that $\mathscr{Q}''$ is an $\mathcal{E}$-permutation of $\mathrm{HN}(\mathcal{D}\oplus\mathcal{F})$ with $\mathscr{Q}'' \geq \mathscr{Q}$ and $\mathscr{Q}'' \neq \mathscr{Q}$ such that $(S_\mathcal{D}, S_\mathcal{F})$ is a $\mathscr{Q}''$-partition pair. Hence we obtain a contradiction to the maximality of $\mathscr{Q}$ as desired. $\square$

Definition 3.9. Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X_{E,F}$ with a short exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Let A and B be sets which form a partition of the set $\{1, \cdots, \mathrm{rk}(\mathcal{E})\}$. An (A, B)-*decomposition* of $\mathcal{E}$ is a direct sum decomposition $\mathcal{E} \simeq \mathcal{E}_A \oplus \mathcal{E}_B$ satisfying the following properties:

(i) We have $(\mu_i(\mathrm{HN}(\mathcal{E}_A))) = (\mu_i(\mathrm{HN}(\mathcal{E})))_{i\in A}$ and $(\mu_i(\mathrm{HN}(\mathcal{E}_B))) = (\mu_i(\mathrm{HN}(\mathcal{E})))_{i\in B}$.

(ii) The map $\mathcal{E}_B \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{F}$ is injective.

Proposition 3.10. *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X_{E,F}$ with integer slopes. If there exists a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0, \tag{3.3}$$

then there exists an $\mathcal{E}$-permutation $\mathscr{P}$ of $\mathrm{HN}(\mathcal{D}\oplus\mathcal{F})$ with a $\mathscr{P}$-partition pair $(S_\mathcal{D}, S_\mathcal{F})$ and an $(S_\mathcal{D}, S_\mathcal{F})$-decomposition of $\mathcal{E}$.

Proof. Let us proceed by induction on $\mathrm{rk}(\mathcal{E})$. The assertion is trivial when $\mathcal{E}$ is zero. We henceforth assume that $\mathcal{E}$ is nonzero. The vector bundle $\mathcal{E}$ admits a decomposition

$$\mathcal{E} \simeq \overline{\mathcal{E}} \oplus \mathcal{O}_{E,F}(\mu_{\min}(\mathcal{E}))$$

where $\overline{\mathcal{E}}$ is a vector bundle on $X_{E,F}$ with $\mathrm{rk}(\overline{\mathcal{E}}) = \mathrm{rk}(\mathcal{E}) - 1$; indeed, $\mathrm{HN}(\overline{\mathcal{E}})$ is given by the restriction of $\mathrm{HN}(\mathcal{V})$ on the interval $[0, \mathrm{rk}(\mathcal{E})-1]$. We write $\overline{\mathcal{D}}$ for the preimage of $\overline{\mathcal{E}}$ under the map $\mathcal{D} \longrightarrow \mathcal{E}$, and $\overline{\mathcal{F}}$ for the image of $\overline{\mathcal{E}}$ under the map $\mathcal{E} \longrightarrow \mathcal{F}$. Then we have a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \overline{\mathcal{D}} \arrow[r] \arrow[d, hook] & \overline{\mathcal{E}} \arrow[r] \arrow[d, hook] & \overline{\mathcal{F}} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{D} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd} \tag{3.4}$$

where all vertical maps are injective. By the induction hypothesis, there exists an $\overline{\mathcal{E}}$-permutation $\overline{\mathscr{P}}$ of $\mathrm{HN}(\overline{\mathcal{D}} \oplus \overline{\mathcal{F}})$ with an $\overline{\mathscr{P}}$-partition pair $(S_{\overline{\mathcal{D}}}, S_{\overline{\mathcal{F}}})$ and an $(S_{\overline{\mathcal{D}}}, S_{\overline{\mathcal{F}}})$-decomposition $\overline{\mathcal{E}} \simeq \overline{\mathcal{E}}_{\overline{\mathcal{D}}} \oplus \overline{\mathcal{E}}_{\overline{\mathcal{F}}}$. In light of Lemma 3.8, we may assume that $\overline{\mathscr{P}}$ is $(S_{\overline{\mathcal{D}}}, S_{\overline{\mathcal{F}}})$-sorted. In addition, we find that $\mathrm{rk}(\overline{\mathcal{F}})$ is equal to either $\mathrm{rk}(\mathcal{F}) - 1$ or $\mathrm{rk}(\mathcal{F})$.

We consider the case where $\mathrm{rk}(\overline{\mathcal{F}})$ is equal to $\mathrm{rk}(\mathcal{F}) - 1$. The surjective map $\mathcal{E} \twoheadrightarrow \mathcal{F}$ induces a surjective map $\mathcal{O}_{E,F}(\mu_{\min}(\mathcal{E})) \simeq \mathcal{E}/\overline{\mathcal{E}} \twoheadrightarrow \mathcal{F}/\overline{\mathcal{F}}$, which must be an isomorphism as both $\mathcal{O}_{E,F}(\mu_{\min}(\mathcal{E}))$ and $\mathcal{F}/\overline{\mathcal{F}}$ are of rank 1. This isomorphism yields a map

$$\mathcal{F}/\overline{\mathcal{F}} \simeq \mathcal{O}_{E,F}(\mu_{\min}(\mathcal{E})) \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{F}$$

which splits the short exact sequence

$$0 \longrightarrow \overline{\mathcal{F}} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\overline{\mathcal{F}} \longrightarrow 0$$

and consequently induces a direct sum decomposition

$$\mathcal{F} \simeq \overline{\mathcal{F}} \oplus \mathcal{F}/\overline{\mathcal{F}} \simeq \overline{\mathcal{F}} \oplus \mathcal{O}_{E,F}(\mu_{\min}(\mathcal{E})).$$

In particular, we find $\mu_{\min}(\mathcal{F}) \leq \mu_{\min}(\mathcal{E})$. Meanwhile, Lemma 3.2 implies $\mu_{\min}(\mathcal{F}) \geq \mu_{\min}(\mathcal{E})$ as $\mathcal{F}$ is a quotient of $\mathcal{E}$. Hence we have $\mu_{\min}(\mathcal{E}) = \mu_{\min}(\mathcal{F})$. We also obtain an isomorphism

$\overline{\mathcal{D}} \simeq \mathcal{D}$ by applying the snake lemma to the diagram (3.4). Let us now take $\mathscr{P}$ to be the rationally tuplar polygon with

$$\mu_i(\mathscr{P}) = \begin{cases} \mu_i(\overline{\mathscr{P}}) & \text{for } i < \mathrm{rk}(\mathcal{E}), \\ \mu_{\min}(\mathcal{F}) & \text{for } i = \mathrm{rk}(\mathcal{E}) \end{cases}$$

We also set $S_{\mathcal{D}} := S_{\overline{\mathcal{D}}}$, $S_{\mathcal{F}} := S_{\overline{\mathcal{F}}} \cup \{\mathrm{rk}(\mathcal{E})\}$, $\mathcal{E}_{\mathcal{D}} := \overline{\mathcal{E}}_{\overline{\mathcal{D}}}$, and $\mathcal{E}_{\mathcal{F}} := \overline{\mathcal{E}}_{\overline{\mathcal{F}}} \oplus \mathcal{O}_{E,F}(\mu_{\min}(\mathcal{E}))$. It is then straightforward to verify that $\mathscr{P}$ is an $\mathcal{E}$-permutation of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ with a $\mathscr{P}$-partition pair $(S_{\mathcal{D}}, S_{\mathcal{F}})$ and an $(S_{\mathcal{D}}, S_{\mathcal{F}})$-decomposition $\mathcal{E} \simeq \mathcal{E}_{\mathcal{D}} \oplus \mathcal{E}_{\mathcal{F}}$, as illustrated by Figure 3.

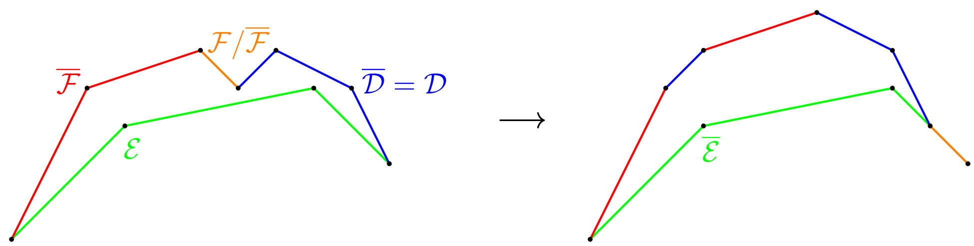

FIGURE 3. Construction of $\mathscr{P}$ in the case $\mathrm{rk}(\overline{\mathcal{F}}) = \mathrm{rk}(\mathcal{F}) - 1$

It remains to consider the case where $\mathrm{rk}(\overline{\mathcal{F}})$ and $\mathrm{rk}(\mathcal{F})$ are equal. Let us set

$$S_{\mathcal{D}} := S_{\overline{\mathcal{D}}} \cup \{\mathrm{rk}(\mathcal{E})\}, \quad S_{\mathcal{F}} := S_{\overline{\mathcal{F}}}, \quad \mathcal{E}_{\mathcal{D}} := \overline{\mathcal{E}}_{\overline{\mathcal{D}}} \oplus \mathcal{O}_{E,F}(\mu_{\min}(\mathcal{E})), \quad \mathcal{E}_{\mathcal{F}} := \overline{\mathcal{E}}_{\overline{\mathcal{F}}}.$$

We also take $\mathscr{P}$ to be the rationally tuplar polygon with

$$(\mu_i(\mathscr{P}))_{i \in S_{\mathcal{D}}} = (\mu_i(\mathrm{HN}(\mathcal{D}))) \quad \text{and} \quad (\mu_i(\mathscr{P}))_{i \in S_{\mathcal{F}}} = (\mu_i(\mathrm{HN}(\mathcal{F}))).$$

The $(S_{\overline{\mathcal{D}}}, S_{\overline{\mathcal{F}}})$-decomposition $\overline{\mathcal{E}} \simeq \overline{\mathcal{E}}_{\overline{\mathcal{D}}} \oplus \overline{\mathcal{E}}_{\overline{\mathcal{F}}}$ and the commutative diagram (3.4) together yield an $(S_{\mathcal{D}}, S_{\mathcal{F}})$-decomposition $\mathcal{E} \simeq \mathcal{E}_{\mathcal{D}} \oplus \mathcal{E}_{\mathcal{F}}$. Moreover, the injective map $\mathcal{E}_{\mathcal{F}} \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{F}$ is an isomorphism at the generic point as we have $\mathrm{rk}(\mathcal{E}_{\mathcal{F}}) = \mathrm{rk}(\overline{\mathcal{E}}_{\overline{\mathcal{F}}}) = \mathrm{rk}(\overline{\mathcal{F}}) = \mathrm{rk}(\mathcal{F})$. We then find from the short exact sequence (3.3) that the map $\mathcal{D} \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{E}_{\mathcal{F}} \cong \mathcal{E}_{\mathcal{D}}$ is also an isomorphism at the generic point. Now Lemma 3.2 yields inequalities

$$\begin{aligned} \mu_i(\mathrm{HN}(\mathcal{D})) &\leq \mu_i(\mathrm{HN}(\mathcal{E}_{\mathcal{D}})) && \text{for } i = 1, \cdots, \mathrm{rk}(\mathcal{D}), \\ \mu_j(\mathrm{HN}(\mathcal{F})) &\geq \mu_j(\mathrm{HN}(\mathcal{E}_{\mathcal{F}})) && \text{for } j = 1, \cdots, \mathrm{rk}(\mathcal{F}). \end{aligned}$$

We can rewrite these inequalities as

$$\begin{aligned} \mu_i(\mathscr{P}) &\leq \mu_i(\mathrm{HN}(\mathcal{E})) && \text{for all } i \in S_{\mathcal{D}}, \\ \mu_j(\mathscr{P}) &\geq \mu_j(\mathrm{HN}(\mathcal{E})) && \text{for all } j \in S_{\mathcal{F}}. \end{aligned}$$

In addition, since $\overline{\mathcal{D}}$ and $\overline{\mathcal{F}}$ are respectively subsheaves of $\mathcal{D}$ and $\mathcal{F}$, Lemma 3.2 yields

$$\begin{aligned} \mu_i(\mathrm{HN}(\overline{\mathcal{D}})) &\leq \mu_i(\mathrm{HN}(\mathcal{D})) && \text{for } i = 1, \cdots, \mathrm{rk}(\mathcal{D}) - 1, \\ \mu_j(\mathrm{HN}(\overline{\mathcal{F}})) &\leq \mu_j(\mathrm{HN}(\mathcal{F})) && \text{for } j = 1, \cdots, \mathrm{rk}(\mathcal{F}). \end{aligned}$$

As $\overline{\mathscr{P}}$ is $(S_{\overline{\mathcal{D}}}, S_{\overline{\mathcal{F}}})$-sorted, we find $\mathscr{P} \geq \mathrm{HN}(\mathcal{E})$. Hence we deduce by Lemma 3.6 that $\mathscr{P}$ is an $\mathcal{E}$-permutation of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ with a $\mathscr{P}$-partition pair $(S_{\mathcal{D}}, S_{\mathcal{F}})$ and an $(S_{\mathcal{D}}, S_{\mathcal{F}})$-decomposition $\mathcal{E} \simeq \mathcal{E}_{\mathcal{D}} \oplus \mathcal{E}_{\mathcal{F}}$, as illustrated by Figure 4. $\square$

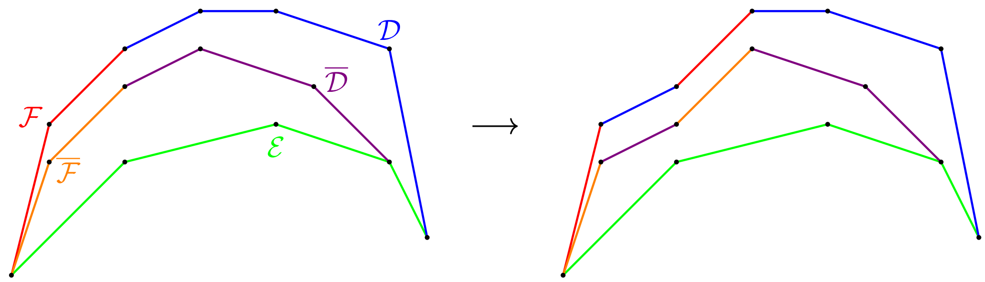

FIGURE 4. Construction of $\mathscr{P}$ in the case $\mathrm{rk}(\overline{\mathcal{F}}) = \mathrm{rk}(\mathcal{F})$

Theorem 3.11. *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X_{E,F}$ with a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0. \tag{3.5}$$

There exists an $\mathcal{E}$-permutation of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$.

Proof. Take an integer d such that $d\lambda$ is an integer for all slopes of $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$. Let E' be the unramified extension of E of degree d. Proposition 2.7 yields a projection map

$$\pi : X_{E',F} \cong X_{E,F} \times_{\mathrm{Spec}\,(E)} \mathrm{Spec}\,(E') \longrightarrow X_{E,F}$$

such that every vector bundle $\mathcal{V}$ on $X_{E,F}$ satisfies

$$\mu_i(\mathrm{HN}(\pi^*\mathcal{V})) = d \cdot \mu_i(\mathrm{HN}(\mathcal{V})) \qquad \text{for } i = 1, \cdots, \mathrm{rk}(\mathcal{V}).$$

In particular, all slopes of $\pi^*\mathcal{D}$, $\pi^*\mathcal{E}$, and $\pi^*\mathcal{F}$ are integers. Moreover, since π is evidently flat, the exact sequence (3.5) gives rise to a short exact sequence

$$0 \longrightarrow \pi^*\mathcal{D} \longrightarrow \pi^*\mathcal{E} \longrightarrow \pi^*\mathcal{F} \longrightarrow 0.$$

By Proposition 3.10, there exists a $\pi^*\mathcal{E}$-permutation $\mathscr{P}'$ of $\mathrm{HN}(\pi^*\mathcal{D} \oplus \pi^*\mathcal{F})$. Hence we get an $\mathcal{E}$-permutation $\mathscr{P}$ of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ with

$$\mu_i(\mathscr{P}) = \mu_i(\mathscr{P}')/d \qquad \text{for } i = 1, \cdots, \mathrm{rk}(\mathcal{E}),$$

thereby completing the proof. $\square$

4. CLASSIFICATION THEOREMS FOR EXTENSIONS

In this section, we establish two classification theorems for extensions of vector bundles on the Fargues-Fontaine curve.

Lemma 4.1. *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X_{E,F}$. Suppose that there exists an $\mathcal{E}$-permutation $\mathscr{P}$ of* $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$.

(1) We have $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F}) \geq \mathscr{P} \geq \mathrm{HN}(\mathcal{E})$

(2) $\mathcal{F}^{<\mu_{min}(\mathcal{D})}$ *is a direct summand of* $\mathcal{E}$.

Proof. The first statement is evident by the concavity of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$. Let us now consider the second statement. Suppose for the sake of contradiction that $\mathcal{F}^{<\mu_{\min}(\mathcal{D})}$ is not a direct summand of $\mathcal{E}$. By the concavity of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$, we find

$$(\mu_i(\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})))_{i > \mathrm{rk}(\mathcal{E}) - \mathrm{rk}(\mathcal{F}^{<\mu_{\min}(\mathcal{D})})} = (\mu_i(\mathrm{HN}(\mathcal{F}^{<\mu_{\min}(\mathcal{D})}))).$$

Take l to be the largest integer with $\mu_l(\mathrm{HN}(\mathcal{E})) \neq \mu_l(\mathrm{HN}(\mathcal{D} \oplus \mathcal{F}))$. Then we must have $l > \mathrm{rk}(\mathcal{E}) - \mathrm{rk}(\mathcal{F}^{<\mu_{\min}(\mathcal{D})})$ and $\mu_l(\mathrm{HN}(\mathcal{E})) < \mu_l(\mathrm{HN}(\mathcal{D} \oplus \mathcal{F}))$. Let us choose a rational number μ with $\mu_l(\mathrm{HN}(\mathcal{E})) < \mu < \mu_l(\mathrm{HN}(\mathcal{D} \oplus \mathcal{F}))$. By concavity of HN polygons we find

$$\mathrm{rk}(\mathcal{E}^{<\mu}) = \mathrm{rk}(\mathcal{E}) - l \quad \text{ and } \quad \mathrm{rk}(\mathcal{F}^{<\mu}) > \mathrm{rk}(\mathcal{E}) - l.$$

However, we must have $\mathrm{rk}(\mathcal{E}^{<\mu}) \leq \mathrm{rk}(\mathcal{F}^{<\mu})$ as there exists an $\mathcal{E}$-permutation of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$. We thus have a desired contradiction, thereby completing the proof. □

Proposition 4.2 ([BFH+22, Theorem 1.1.2]). *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X_{E,F}$ such that $\mathcal{D}$ and $\mathcal{F}$ are semistable with $\mu(\mathcal{D}) \leq \mu(\mathcal{F})$. There exists a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

if and only if we have $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F}) \geq \mathrm{HN}(\mathcal{E})$.

Remark. While the cited result [BFH+22, Theorem 1.1.2] does not explicitly consider the case where $\mu(\mathcal{D})$ and $\mu(\mathcal{F})$ are equal, this case follows immediately from Proposition 2.6.

Theorem 4.3. *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X_{E,F}$ such that either $\mathcal{D}$ or $\mathcal{F}$ is semistable. There exists a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{4.1}$$

if and only if there exists an $\mathcal{E}$-permutation of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$.

Proof. Let $\mathcal{D}^\vee$, $\mathcal{E}^\vee$, and $\mathcal{F}^\vee$ respectively denote the duals of $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$. By Proposition 2.6 and Theorem 2.9, we observe that a vector bundle on $X_{E,F}$ is semistable if and only if its dual is semistable, and also find that $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ has an $\mathcal{E}$-permutation if and only if $\mathrm{HN}(\mathcal{D}^\vee \oplus \mathcal{F}^\vee)$ has an $\mathcal{E}^\vee$-permutation. Moreover, the existence of a short exact sequence (4.1) is equivalent to the existence of a short exact sequence

$$0 \longrightarrow \mathcal{F}^\vee \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{D}^\vee \longrightarrow 0.$$

Hence we may assume without loss of generality that $\mathcal{D}$ is semistable.

The necessity part is an immediate consequence of Theorem 3.11. For the sufficiency part, we henceforth assume that there exists an $\mathcal{E}$-permutation $\mathscr{P}$ of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$. We also fix a $\mathscr{P}$-partition pair $(S_\mathcal{D}, S_\mathcal{F})$ and assume in light of Lemma 3.8 that $\mathscr{P}$ is $(S_\mathcal{D}, S_\mathcal{F})$-sorted. Let us write r for the number of distinct slopes in $\mathrm{HN}(\mathcal{F})$ and proceed by induction on r.

We first consider the base case where $\mathrm{HN}(\mathcal{F})$ is a line segment. Note that $\mathcal{F}$ is semistable by Theorem 2.9. If we have $\mu(\mathcal{D}) \leq \mu(\mathcal{F})$, then Lemma 4.1 and Proposition 4.2 together yield a desired short exact sequence (4.1). If we have $\mu(\mathcal{D}) > \mu(\mathcal{F})$, then Lemma 4.1 implies that $\mathcal{E}$ is isomorphic to $\mathcal{D} \oplus \mathcal{F}$ and thus gives rise to a desired (splitting) exact sequence (4.1).

For the induction step, we assume from now on that $\mathrm{HN}(\mathcal{F})$ has at least two distinct slopes. We have a decomposition

$$\mathcal{F} \simeq \bigoplus_{i=1}^{r} \mathcal{F}_i$$

where $\mathcal{F}_1, \cdots, \mathcal{F}_r$ are semistable with $\mu(\mathcal{F}_1) > \cdots > \mu(\mathcal{F}_r)$, and thus obtain a decomposition

$$\mathcal{F} \simeq \overline{\mathcal{F}} \oplus \mathcal{F}_r$$

where $\overline{\mathcal{F}}$ is a vector bundle on $X_{E,F}$ with $r-1$ distinct slopes in $\mathrm{HN}(\overline{\mathcal{F}})$ and $\mu_{\min}(\overline{\mathcal{F}}) > \mu(\mathcal{F}_r)$.

We consider the case where $\mu(\mathcal{D})$ is greater than $\mu(\mathcal{F}_r)$. The readers may refer to Figure 5 for a visual illustration of our argument. Lemma 4.1 yields a decomposition $\mathcal{E} \simeq \overline{\mathcal{E}} \oplus \mathcal{F}_r$ for some vector bundle $\overline{\mathcal{E}}$ over $X_{E,F}$, and also implies that $\mathrm{HN}(\mathcal{E})$, $\mathscr{P}$, and $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ must coincide on the interval $[\mathrm{rk}(\overline{\mathcal{E}}), \mathrm{rk}(\mathcal{E})]$. Let us take $\overline{\mathscr{P}}$ to be the restriction of $\mathscr{P}$ on the interval

$[0, \mathrm{rk}(\overline{\mathcal{E}})]$. Then $\overline{\mathscr{P}}$ is an $\overline{\mathcal{E}}$-permutation of $\mathrm{HN}(\mathcal{D} \oplus \overline{\mathcal{F}})$, and thus gives rise to a short exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow \overline{\mathcal{E}} \longrightarrow \overline{\mathcal{F}} \longrightarrow 0$$

by the induction hypothesis. Now we get a desired exact sequence (4.1) using the decompositions $\mathcal{E} \simeq \overline{\mathcal{E}} \oplus \mathcal{F}_r$ and $\mathcal{F} \simeq \overline{\mathcal{F}} \oplus \mathcal{F}_r$.

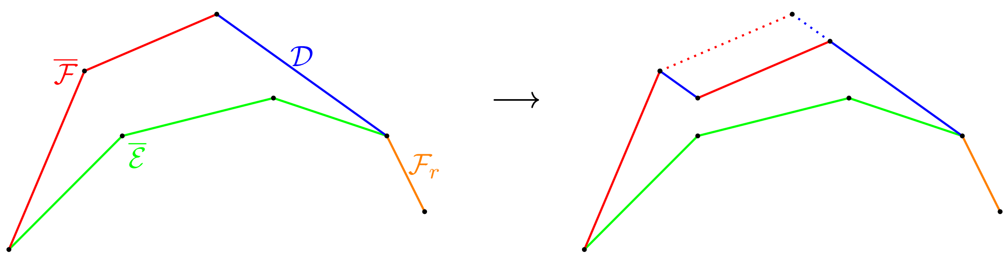

FIGURE 5. Construction of $\overline{\mathcal{E}}$ and $\overline{\mathscr{P}}$ in the case $\mu(\mathcal{D}) > \mu(\mathcal{F}_r)$

It remains to consider the case where we have $\mu(\mathcal{D}) \leq \mu(\mathcal{F}_r)$. The readers may refer to Figure 6 for a visual illustration of our argument. By Lemma 4.1 we find $\mu(\mathcal{D}) = \mu_{\min}(\mathcal{D} \oplus \mathcal{F}) \leq \mu_{\min}(\mathcal{E})$. In addition, we have $\mathrm{rk}(\mathcal{E}^{>\mu(\mathcal{F}_r)}) + \mathrm{rk}(\mathcal{F}_r) \leq \mathrm{rk}(\mathcal{E})$ as the surjective map $\mathcal{E} \twoheadrightarrow \mathcal{F} \twoheadrightarrow \mathcal{F}_r$ factors through $\mathcal{E}^{\leq \mu(\mathcal{F}_r)}$ by Proposition 2.6. Take $\overline{\mathcal{E}}$ to be the vector bundle on $X_{E,F}$ such that $\mathrm{HN}(\overline{\mathcal{E}} \oplus \mathcal{F}_r)$ is the upper convex hull of $\mathrm{HN}(\mathcal{E}^{>\mu(\mathcal{F}_r)} \oplus \mathcal{F}_r)$ and $\mathrm{HN}(\mathcal{E})$. Let $\mathcal{E}'$ denote the maximal common direct summand of $\overline{\mathcal{E}}$ and $\mathcal{E}$. Then we find

$$\mathcal{E} \simeq \mathcal{E}' \oplus \mathcal{E}'' \quad \text{and} \quad \overline{\mathcal{E}} \simeq \mathcal{E}' \oplus \mathcal{K} \tag{4.2}$$

where $\mathcal{E}''$ and $\mathcal{K}$ are vector bundles on $X_{E,F}$ with $\mathcal{K}$ being semistable. By construction, we have $\mathrm{HN}(\mathcal{F}_r \oplus \mathcal{K}) \geq \mathrm{HN}(\mathcal{E}'')$ and $\mu(\mathcal{F}_r) \geq \mu(\mathcal{K})$. Hence Proposition 4.2 and the decompositions (4.2) together yield a short exact sequence

$$0 \longrightarrow \overline{\mathcal{E}} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}_r \longrightarrow 0. \tag{4.3}$$

Let us now set $S_{\overline{\mathcal{F}}} := \{i \in S_{\mathcal{F}} : \mu_i(\mathscr{P}) \neq \mu(\mathcal{F}_r)\}$ and take $\overline{\mathscr{P}}$ to be the rationally tuplar polygon with $(\mu_i(\overline{\mathscr{P}})) = (\mu_i(\mathscr{P}))_{i \in S_{\overline{\mathcal{F}}} \cup S_{\mathcal{D}}}$; in other words, we obtain $\overline{\mathscr{P}}$ by removing all line segments of slope $\mu_i(\mathscr{P})$ from $\mathscr{P}$. Since $S_{\mathcal{F}} \backslash S_{\overline{\mathcal{F}}}$ does not contain any integer less than or equal to $\mathrm{rk}(\mathcal{E}^{>\mu(\mathcal{F}_r)})$, the polygons $\mathscr{P}$ and $\overline{\mathscr{P}}$ coincide on the interval $[0, \mathrm{rk}(\mathcal{E}^{>\mu(\mathcal{F}_r)})]$. In addition, as $\mathrm{HN}(\overline{\mathcal{E}} \oplus \mathcal{F}_r)$ is the upper convex hull of $\mathrm{HN}(\mathcal{E}^{>\mu(\mathcal{F}_r)} \oplus \mathcal{F}_r)$ and $\mathrm{HN}(\mathcal{E})$, we find

$$\mu(\mathcal{D}) \leq \mu_{\min}(\mathcal{E}) \leq \mu_i(\overline{\mathcal{E}}) \leq \mu(\mathcal{F}_r) = \mu_{\min}(\mathcal{F}) \qquad \text{for } i = \mathrm{rk}(\mathcal{E}^{>\mu(\mathcal{F}_r)}) + 1, \cdots, \mathrm{rk}(\overline{\mathcal{E}}).$$

Hence we obtain an $(\overline{\mathcal{E}} \oplus \mathcal{F}_r)$-permutation $\mathscr{P}'$ of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ by concatenating $\overline{\mathscr{P}_1}$, $\mathrm{HN}(\mathcal{F}_r)$, and $\overline{\mathscr{P}_2}$, where $\overline{\mathscr{P}_1}$ and $\overline{\mathscr{P}_2}$ are the rationally tuplar polygons with

$$(\mu_i(\overline{\mathscr{P}_1})) = (\mu_i(\overline{\mathscr{P}}))_{i \leq \mathrm{rk}(\mathcal{E}^{>\mu(\mathcal{F}_r)})} \quad \text{and} \quad (\mu_i(\overline{\mathscr{P}_2})) = (\mu_i(\overline{\mathscr{P}}))_{i > \mathrm{rk}(\mathcal{E}^{>\mu(\mathcal{F}_r)})}.$$

It follows that $\overline{\mathscr{P}}$ is an $\overline{\mathcal{E}}$-permutation of $\mathrm{HN}(\mathcal{D} \oplus \overline{\mathcal{F}})$; indeed, $\overline{\mathscr{P}}$ and $\overline{\mathcal{E}}$ are respectively obtained from $\mathcal{E}'$ and $\overline{\mathcal{E}} \oplus \mathcal{F}_r$ by removing $\mathrm{HN}(\mathcal{F}_r)$ over the same interval. Hence we obtain a short exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow \overline{\mathcal{E}} \longrightarrow \overline{\mathcal{F}} \longrightarrow 0 \tag{4.4}$$

by the induction hypothesis. Now the exact sequences (4.3) and (4.4) together yield a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{D} \arrow[r, equal] \arrow[d, hook] & \mathcal{D} \arrow[r] \arrow[d, hook, "\alpha"] & 0 \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \overline{\mathcal{E}} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{F}_r \arrow[r] & 0
\end{tikzcd}$$

which, by the snake lemma, induces a short exact sequence

$$0 \longrightarrow \overline{\mathcal{F}} \longrightarrow \operatorname{coker}(\alpha) \longrightarrow \mathcal{F}_r \longrightarrow 0.$$

Since this sequence is split by Proposition 2.6, we obtain a desired exact sequence (4.1). $\square$

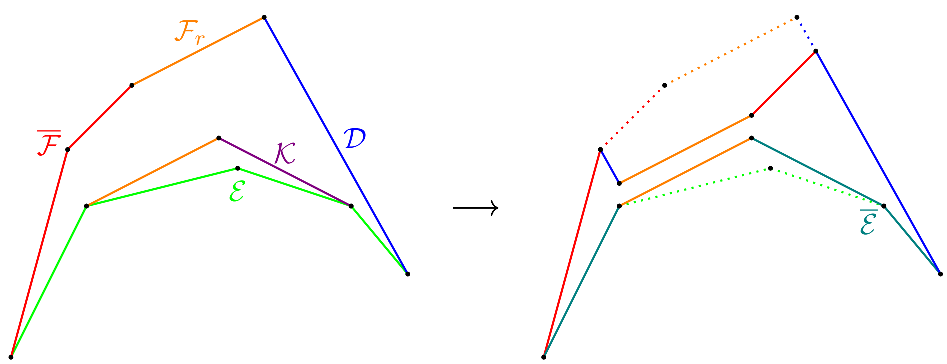

FIGURE 6. Construction of $\overline{\mathcal{E}}$ and $\overline{\mathscr{P}}$ in the case $\mu(\mathcal{D}) \leq \mu(\mathcal{F}_r)$

Theorem 4.4. *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X_{E,F}$. Let $\mathcal{F}_1, \cdots, \mathcal{F}_r$ denote the semistable vector bundles on $X_{E,F}$ such that $\mathrm{HN}(\mathcal{F}_i)$ represents the i-th line segment in $\mathrm{HN}(\mathcal{F})$. There exists a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{4.5}$$

if and only if the following equivalent conditions are satisfied:

(i) There exists a filtration

$$\mathcal{D} = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$$

with $\mathcal{E}_i/\mathcal{E}_{i-1} \simeq \mathcal{F}_i$ for each $i = 1, \cdots, r$.

(ii) There exists a sequence of vector bundles $\mathcal{D} = \mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_r = \mathcal{E}$ such that the polygon $\mathrm{HN}(\mathcal{E}_{i-1} \oplus \mathcal{F}_i)$ has an $\mathcal{E}_i$-permutation for each $i = 1, \cdots, r$.

Proof. The equivalence of the conditions (i) and (ii) is evident by Theorem 4.3. In addition, the sufficiency part is an immediate consequence of Proposition 2.6. For the necessity part, we henceforth assume that there exists an exact sequence (4.5).

Let us proceed by induction on r. The assertion is trivial for $r = 0$. For the induction step, we now assume that r is not zero. Let us set

$$\overline{\mathcal{F}} := \bigoplus_{i=1}^{r-1} \mathcal{F}_i.$$

Then we have $\mathcal{F} \simeq \overline{\mathcal{F}} \oplus \mathcal{F}_r$. Take $\mathcal{E}_{r-1}$ to be the kernel of the map $\mathcal{E} \twoheadrightarrow \mathcal{F} \twoheadrightarrow \mathcal{F}_r$. By construction, $\mathcal{E}_{r-1}$ contains $\mathcal{D}$ as a subsheaf. Hence we get a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{D} \arrow[r] \arrow[d, hook] & \mathcal{E} \arrow[r] \arrow[d, equal] & \mathcal{F} \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & \mathcal{E}_{r-1} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{F}_r \arrow[r] & 0 \end{tikzcd}$$

and consequently find by the snake lemma that the cokernel of the left vertical map is isomorphic to $\overline{\mathcal{F}}$. The desired assertion now follows by the induction hypothesis. □

Remark. We may check the condition (ii) in a finite amount of time. In fact, if we start with $i = r$ and inductively proceed with descending indices, we get finitely many candidates for each $\mathcal{E}_{i-1}$ just by the following simple observations:

(a) $\mathrm{HN}(\mathcal{E}_{i-1})$ and $\mathrm{HN}(\mathcal{E}_i)$ coincide on the interval $[0, \mathrm{rk}(\mathcal{E}_i^{>\mu(\mathcal{F}_i)})]$.
(b) We have $\mathrm{HN}(\mathcal{F}_i \oplus \mathcal{E}_{i-1}^{\leq\mu(\mathcal{F}_i)}) \geq \mathrm{HN}(\mathcal{E}_i^{\leq\mu(\mathcal{F}_i)})$.
(c) All breakpoints in $\mathrm{HN}(\mathcal{E}_{i-1})$ are integer points.

It is also worthwhile to note that the filtration in the condition (i) lifts the Harder-Narasimhan filtration of $\mathcal{F}$. Our proof indeed shows that, for any Harder-Narasimhan category with splitting Harder-Narasimhan filtrations, every extension between two arbitrary objects $\mathcal{D}$ and $\mathcal{F}$ gives rise to a filtration that lifts the Harder-Narasimhan filtration of $\mathcal{F}$.

Example 4.5. We present an example showing that Theorem 4.3 does not hold without the semistability assumption on either $\mathcal{D}$ or $\mathcal{F}$. For ease of notation, we will write $\mathcal{O}(\lambda) = \mathcal{O}_{E,F}(\lambda)$ for each $\lambda \in \mathbb{Q}$. Let us take

$$\mathcal{D} := \mathcal{O}(1/2) \oplus \mathcal{O}(-1), \quad \mathcal{E} := \mathcal{O}(3/2) \oplus \mathcal{O}(2/5), \quad \mathcal{F} := \mathcal{O}(3) \oplus \mathcal{O}(2/3).$$

Then it is not hard to verify the following facts:

(a) $\mathcal{E}_1 = \mathcal{O}(3/2) \oplus \mathcal{O}^{\oplus 2}$ is the only vector bundle on $X_{E,F}$ such that $\mathrm{HN}(\mathcal{E}_1 \oplus \mathcal{O}(2/3))$ has an $\mathcal{E}$-permutation.
(b) $\mathrm{HN}(\mathcal{D} \oplus \mathcal{O}(3))$ does not have an $\mathcal{E}_1$-permutation.

Hence Theorem 4.4 implies that there does not exist a short exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$

while $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ has an $\mathcal{E}$-permutation $\mathscr{P}$ with

$$(\mu_i(\mathscr{P})) = (3, 1/2, 2/3, 2/3, 2/3, 1/2, -1)$$

as illustrated in Figure 7.

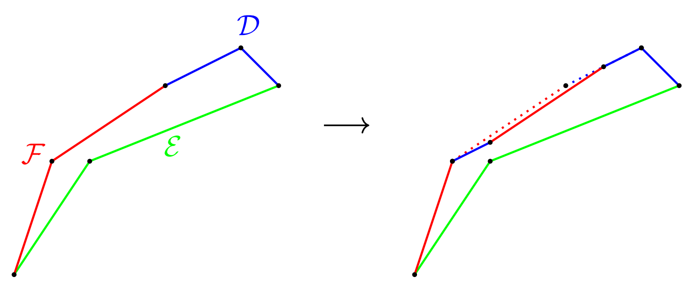

FIGURE 7. An $\mathcal{E}$-permutation of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ in Example 4.5

Remark. Example 4.5 suggests that the condition (ii) in Theorem 4.4 is unlikely to have an equivalent statement which is easy to check in the general case. On the other hand, such a statement exists under some additional assumptions.

(1) If all slopes in $\mathrm{HN}(\mathcal{D})$, $\mathrm{HN}(\mathcal{E})$, and $\mathrm{HN}(\mathcal{F})$ are integers, then the condition (ii) in Theorem 4.4 is satisfied if and only if $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ has a $\mathcal{E}$-permutation $\mathscr{P}$.

(2) If $\mathcal{E}$ is semistable, then the condition (ii) in Theorem 4.4 is satisfied if and only if we have $\mu_{\max}(\mathcal{D}) \leq \mu(\mathcal{E}) \leq \mu_{\min}(\mathcal{F})$.

The first statement can be proved by an induction argument similar to the proof of Theorem 4.3. The second statement immediately follows from the previous result of the author [Hon20, Theorem 1.1.1]. We also note that the first statement is comparable to the main result of Schlesinger [Sch00] which classifies all extension of two given vector bundles on $\mathbb{P}^1$.

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