Liquidity Effects of Trading Frequency (extended version).*

Roman Gayduk and Sergey Nadtochiy†‡

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Abstract

In this work, we present a discrete time modeling framework, in which the shape and dynamics of a Limit Order Book (LOB) arise endogenously from an equilibrium between multiple market participants (agents). The new framework captures very closely the true, micro-level, mechanics of an auction-style exchange. At the same time, it uses the standard abstractions of a continuum-player game to obtain a tractable macro-level description of the LOB. We use the proposed modeling framework to analyze the effects of trading frequency on market liquidity in a very general setting. In particular, we demonstrate the dual effect of high trading frequency. On the one hand, the higher frequency increases market efficiency, if the agents choose to provide liquidity in equilibrium. On the other hand, the higher trading frequency also makes markets more fragile, in the sense that the agents choose to provide liquidity in equilibrium only if they are market-neutral (i.e., their beliefs satisfy certain martingale property). Even a very small deviation from market-neutrality may cause the agents to stop providing liquidity, if the trading frequency is sufficiently high, which represents a self-inflicted liquidity crisis (aka flash crash) in the market. This framework allows us to provide more insight into how such a liquidity crisis unfolds, connecting it to the so-called adverse selection effect.

1 Introduction

The technological development presents new challenges to the mathematical modeling of social and economic phenomena. In particular, the rapid growth of electronic trading has changed significantly the existing approaches to modeling financial markets. The classical mathematical models used to focus on the macroscopic description of the financial processes, often, taking as input the price levels at which certain assets can be purchased or sold. However, in reality, the price arises as an outcome of the interaction between market participants, and understanding the mechanics of the price formation process has become an important problem on its own. A famous example of a problem that arises in this context is how to characterize the effects of trading frequency on an electronic exchange. On the one hand, the higher trading frequency provides more opportunities for the market participants to trade, hence, improving the liquidity of the market and increasing the market efficiency. On the other hand, the higher trading frequency also provides more opportunities for some participants to manipulate the price and disrupt the market liquidity. Such a manipulation creates a new type of risk, which reveals itself in the unusually high price deviations, that cannot be explained by the changes in the present, or projected, fundamental value of the asset. The most famous example of this phenomenon is

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*Partial support from the NSF grant DMS-1411824 is acknowledged by both authors.
†Address the correspondence to: Mathematics Department, University of Michigan, 530 Church St, Ann Arbor, MI 48104; sergeyn@umich.edu.
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the “flash crash” of 2010. This example motivates the need for a comprehensive study of the tradeoff between the liquidity providing role of the strategic players and the liquidity risk they generate, and its relation to trading frequency. In this paper, we analyze market microstructure in the context of an auction-style exchange (as most exchanges currently are), in which the participating agents can post limit or market orders. The collective liquidity effect of the agents (i.e., their liquidity providing vs. liquidity risk role) is captured by the shape and dynamics of the Limit Order Book (LOB), which contains all the limit buy and sell orders.

The goal of the present paper is two-fold. First, we develop a new framework for modeling market microstructure, in which the shape of the LOB, and its dynamics, arise endogenously from the interactions between the agents. This is in contrast to many of the existing results on market microstructure, which assume that the shape and dynamics of the LOB are given exogenously. Among the many advantages of our approach is the possibility of modeling the reaction of a market to the changes in the rules of the exchange: e.g., limited trading frequency, transaction tax, etc. The second, and most important, goal of the present work is to investigate the liquidity effects of trading frequency, using the proposed modeling framework. In particular, the main results of this paper (cf. the discussion in Section 3, as well as Theorems 1, 2 and Corollary 1, in Section 4) describe the dual effect of high trading frequency. On the one hand, if the agents choose to provide liquidity in equilibrium, the higher trading frequency decreases the bid-ask spread and makes the expected profits of all market participants converge to the same (fundamental) value, thus, improving the market efficiency. On the other hand, the higher trading frequency also makes the LOB more sensitive to the deviations of the participants from market-neutrality. It is, of course, clear that a strong bullish or bearish signal makes the market participants trade at a higher or lower price. However, the novelty of our observation is in the role that the trading frequency plays in amplifying this effect. Namely, we show that, if the trading frequency is high, and the agents have plenty of inventory, even a very small deviation from market-neutrality may cause the agents to stop providing liquidity, by either withdrawing from the market completely, or by posting the limit orders very far away from the fundamental price. Such actions cause disproportional deviations in the LOB, which cannot be explained by any fundamental reasons: they are much higher than the trading signal (i.e. the expected change in the fundamental price), and they occur without any shortage of supply or demand for the asset. We refer to such a deviation as an internal (or, self-inflicted) liquidity crisis, because it is due to the trading mechanism (i.e., the rules by which the market participants interact), rather than any fundamental reasons (note the similarity with the flash crash). Our framework allows us to provide more insight into how such liquidity crisis unfolds, connecting it to the so-called adverse selection effect. In particular, the example in Section 3 illustrates that an internal liquidity crisis may not be due to an abnormally large market order, wiping out the liquidity on one side of the LOB, but it is more likely to occur because it becomes optimal for the agents to stop providing liquidity on this side of the LOB. On the mathematical side, our analysis makes use of the properties of conditional tails of the increments of a general Itô process, with the corresponding result stated in Lemma 2. This lemma provides a uniform exponential bound on the conditional tails of the increments of a general Itô process. We believe that this result is useful in its own right, and, to the best of our knowledge, it is not available in existing literature.

In the recent years, we observed an explosion in the amount of literature devoted to the study of market microstructure. In addition to various empirical studies, a large part of the existing theoretical work focuses on the problem of optimal execution: see, among others, [37], [3], [41], [42], [24], [36], [20], [6], [5], [21], [39], [28], [18], [29], [45], and references therein. In these publications, the dynamics and shape of the LOB are modeled exogenously, or, equivalently, the arrival processes of the limit and market orders are specified exogenously. In particular, none of these works attempt to explain the shape and dynamics of the LOB, arising directly from the interaction between the market participants. A different approach to the analysis of market microstructure has its roots in the economic literature. For example, [38], [23], [27], [16], [33], [40] introduce the models of endogenous formation of LOB, and, to the best of our knowledge, are the closest available results to the present work. However, the models proposed in the aforementioned papers do not aim to represent the mechanics of an auction-style exchange with sufficient precision, and, in particular, they are not well suited for
analyzing the liquidity effects of trading frequency, which is the main focus of the present paper. A thorough analysis of an equilibrium-based model for LOB in a single period (i.e., in a static model) is provided in [7], [8], [9], [10], but without addressing the specific question of liquidity effects. The liquidity role of the agents is analyzed, e.g., in [11], [13], [30], [19], but not in the context of market microstructure. The results of the latter papers demonstrate that, depending on the model parameters, the agents may either serve as liquidity providers for each other, or attempt to manipulate the price, reducing the overall liquidity. There also exists a fairly large amount of literature on a related topic: namely, the endogenous formation of LOB in a market with a designated market maker: see e.g. [26], [32], [22], [17], [1]. In these papers, the LOB is not an outcome of a multi-agent equilibrium: instead, it is controlled by a single agent, the market maker, which is not the case in many modern exchanges and, in particular, is not assumed in the present paper (as we study the auction-style exchanges). Finally, several recent papers have applied an equilibrium-based approach to the problem of optimal execution (cf. [43], [31]). These papers describe an equilibrium between several agents solving an optimal execution problem, with the LOB (or, the market), against which these agents trade, being specified exogenously, rather than being modeled as an output of the equilibrium. In the present paper, we model the entire LOB as an output of the equilibrium between a large number of agents, each of whom is allowed to both consume and provide liquidity (in particular, we have no designated market maker). We formulate the problem as a continuum-player game – this abstraction allows us to obtain computationally tractable results (cf. [4], [44], [14] for more on the concept of a continuum-player game). The connection between our approach and the finite-player games and mean field games is discussed in Subsection 2.3 (cf. [35], [12], [15], [34], for more on mean field games).

The paper is organized as follows. Subsection 2.1 describes the probabilistic setting, along with the execution rules of the exchange and the resulting state processes of the agents. Subsection 2.2 defines the equilibrium and introduces the notion of degeneracy of the market (which represents an internal liquidity crisis). Subsection 2.3 discusses the connection of our approach to other modeling frameworks. In Section 3, we construct an equilibrium in a simple model, illustrating how an internal liquidity crisis unfolds, and how it is connected to the adverse selection effect. Theorems 1, 2, and Corollary 1, in Section 4, are the main results of the paper: they formalize and generalize the conclusions of Section 3. In Section 5, we prove the key technical results on the (conditional) tails of marginal distributions of Itô processes. Sections 6, 7 contain the proofs of the main results. In Section 8, we show how to construct equilibria in the models that are “not too far away” from the model considered in Section 3, and so that the resulting LOB possesses additional desirable properties. We conclude in Section 9.

2 Modeling framework for a finite-frequency auction-style exchange

2.1 Mechanics of the exchange

We consider an exchange in which the trading can only occur at discrete times \( n = 0, 1, \ldots, N \). We also assume that the market participants are split into two groups: the external investors, who are “impatient”, in the sense that they only submit market orders, which have to execute immediately, and the strategic players, who can submit both market and limit orders, and who are willing to optimize their actions over a given (short) time horizon, in order to get a better execution price.\(^1\) In our study, we focus on the strategic players, who are referred to as agents, and we model the behavior of the external investors exogenously, via the exogenous demand. The interpretation of the external investors is clear: these are the investors who either have a longer-term view on the market, or who simply need to buy or sell the asset for reasons other than the short-term profits. The strategic players (i.e., agents), on contrary, are the short-term traders, who attempt to maximize

\(^1\)We do not distinguish the “aggressive” limit orders, which are posted at the price level of an opposite limit order, and refer to them as market orders. This causes no loss of generality, as the market participants in our setting have a perfect observation of the LOB.
their objective at a shorter time horizon $N$. During every time period $[n, n+1)$, all the orders coming to the exchange are split into the limit and market orders. The limit orders are collected in the so-called Limit Order Book (LOB), and the market orders form the demand curve. At time $n+1$, the market orders in the demand curve are executed against the limit orders in the LOB. Then, the same is repeated in the next time interval. In particular, during a time period $[n, n+1)$ (for simplicity, we say “at time $n$”), an agent is allowed to submit a market order, post a limit buy or sell order, or wait (i.e. do nothing). If a limit order is not executed in a given time period, it costs nothing to cancel or re-position it for the next time period. Notice that such discrete-time framework does not allow to model the time-priority of the limit orders. However, this does not change the agents’ maximum objective value, as the “tick size” is assumed to be zero (i.e. the set of possible price levels is $\mathbb{R}$), and, hence, an agent can always achieve a priority by posting her order “infinitesimally” above or below the given competing order. Further details on modeling the formation of an LOB and the execution rules are presented below.

The demand curves are modeled exogenously by a random field $D_n(p) = \max(D_n(p), 0)$ on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^N, \mathbb{P}\right)$, such that $\mathcal{F}_0$ is a trivial sigma-algebra, completed w.r.t. $\mathbb{P}$. The random variable $D_n^+(p) = \max(D_n(p), 0)$ denotes the amount of asset that the external investors and the agents submitting market orders are willing to purchase at or below the price $p$, accumulated over the time period $[n-1, n]$, and $D_n^-(p) = -\min(D_n(p), 0)$ denotes the amount of asset that the external investors and the agents submitting market orders are willing to sell at or above the price $p$, in the same time period. We assume that $D_n(\cdot)$ is a.s. nonincreasing and measurable w.r.t. $\mathcal{F}_n \otimes B(\mathbb{R})$. We denote by $\mathcal{A}$ a Borel space of beliefs, and, for each $\alpha \in \mathcal{A}$, there exists a subjective probability measure $\mathbb{P}^\alpha$ on $(\Omega, \mathcal{F}_N)$, which is absolutely continuous with respect to $\mathbb{P}$. We assume that, for any $n = 0, \ldots, N$ and any $\alpha \in \mathcal{A}$, there exists a regular version of the conditional probability $\mathbb{P}^\alpha$ given $\mathcal{F}_n$, denoted $\mathbb{P}_n^\alpha$. We denote by $\mathbb{E}_n^\alpha$ the associated conditional expectations by $\mathbb{E}_n^\alpha$. We also need to assume that, for any $\alpha \in \mathcal{A}$, there exists a modification of the family $\{\mathbb{P}_n^\alpha\}_{n=0}^N$ which satisfies the tower property with respect to $\mathbb{P}$, in the following sense: for any $n \leq m$ and any r.v. $\xi$, such that $\mathbb{E}_n^\alpha \xi^+ < \infty$, we have

$$\mathbb{E}_n^\alpha \mathbb{E}_m^\alpha \xi = \mathbb{E}_n^\alpha \xi, \quad \mathbb{P}\text{-a.s.}$$

There exists such a modification, for example, if $\mathbb{P}^\alpha \sim \mathbb{P}$. In any market model, for every $\alpha$, we fix such a modification of conditional probabilities (up to a set of $\mathbb{P}$-measure zero) and assume that all conditional expectations $\{\mathbb{E}_n^\alpha\}$ are taken under this family of measures. The Limit Order Book (LOB) is given by a pair of adapted process $\nu = (\nu_n^+, \nu_n^-)_{n=0}^N$, such that every $\nu_n^+$ and $\nu_n^-$ is a finite sigma-additive random measure on $\mathbb{R}$ (w.r.t. $\mathcal{F}_n \otimes B(\mathbb{R})$). Herein, $\nu_n^+$ corresponds to the cumulative limit sell orders, and $\nu_n^-$ corresponds to the cumulative limit buy orders, posted at time $n$. The bid and ask prices at any time $n = 0, \ldots, N$ are given by the random variables

$$p_n^b = \sup \text{supp}(\nu_n^-), \quad p_n^a = \inf \text{supp}(\nu_n^+),$$

respectively. Notice that these extended random variables are always well defined but may take infinite values.

We define the state space of an agent as $S = \mathbb{R} \times \mathcal{A}$, where the first component denotes the inventory of an agent (i.e. how much asset she currently holds), and the second component denotes her beliefs. Every agent in state $(s, \alpha)$ models the future outcomes using the subjective probability measure $\mathbb{P}^\alpha$. There are infinitely many agents, and their distribution over the state space is given by the empirical distribution process $\mu = (\mu_n)_{n=0}^N$, such that every $\mu$ is a finite sigma-additive random measure on $S$ (w.r.t. $\mathcal{F}_n \otimes B(S)$). In particular, the total mass of agents in the set $S \subset S$ at time $n$ is given by $\mu_n(S)$. The inventory level $s$ represents the number of shares per agent, held by the agents at state $(s, \alpha)$. In particular, the total number of shares held by all agents in the set $S \subset S$ is given by $\int_S s \mu_n(ds, da)$. The interpretation of this definition in a finite-player game is discussed in Subsection 2.3. We refer the reader to [14] for more on the general concept a continuum-player

\footnote{This assumption holds, for example, if $\mathcal{F}_N$ is generated by a random element with values in a standard Borel space.}
game. As the parameter $\alpha$ does not change over time, the state process of an agent, denoted $(S_n)$, is an adapted $\mathbb{R}$-valued process, representing her inventory.\footnote{Note that, although $P^\alpha$ does not change over time, the conditional distribution of the future demand, as perceived by the agent, changes dynamically, according to the new information received.} The control of every agent is given by a triplet of adapted processes $(p, q, r) = (p_n, q_n, r_n)_{n=0}^{N-1}$ on $(\Omega, \mathcal{F})$, with values in $\mathbb{R}^2 \times \{0, 1\}$. The first coordinate, $p_n$, indicates the location of a limit order placed at time $n$, and $q_n$ indicates the size of the order (measured in shares per agent, and with negative values corresponding to buy orders). The last coordinate $r_n$ shows whether the agent submits a market order (if $r_n = 1$) or a limit order (if $r_n = 0$). Assume that an agent posts a limit sell order at a price level $p_n$. If the incremental demand to buy the asset at this price level, $D_{n+1}(p_n)$, exceeds the amount of all limit sell orders posted below $p_n$ at time $n$, then (and only then) the limit sell order of the agent is executed. Market orders are always executed at the bid or ask price available at the time when the order is submitted. We interpret an internal market order (i.e. the one submitted by an agent) as the decision of an agent to join the external investors, in the given time period. Summing up the above, we obtain the following dynamics for the state process of an agent, starting with initial inventory $s \in \mathbb{R}$ at time $m = 0, \ldots, N - 1$:

$$S_m^{(p,q,r)}(m, s, \nu) = s, \quad \Delta S_{n+1}^{(p,q,r)}(m, s, \nu) = S_{n+1}^{(p,q,r)}(m, s, \nu) - S_n^{(p,q,r)}(m, s, \nu) = -q_n 1_{(r_n = 1)}$$

The above dynamics represent an optimistic view on the execution by the agents. In particular, they imply that all limit orders at the same price level are executed in full, once the demand reaches them: i.e. each agent believes that her market order will be executed first among all orders at a given price level. In addition, all agents’ market orders are executed at the bid and ask prices: i.e. each agent believes that her market order will be executed first, when the demand curve is cleared against the LOB, at the end of the given time period. These assumptions can be partially justified by the fact that the agents’ orders are infinitesimal: $q_n$ is measured in shares per agent, and an individual agent has zero mass. However, if a non-zero mass of agents submit limit orders at the same price level, or execute market orders, at the same time, then, the above state dynamics may violate the market clearance condition: the total size of executed market orders (both in shares and in dollars) may not coincide with the total size of executed limit orders (at least, as viewed by the agents). Nevertheless, this issue is resolved if, at any time, the mass of the agents posting limit orders at the same price level is zero, as well as the mass of the agents posting market orders. In other words, $(\nu, p, q, r)$ satisfy, $P^\alpha$-a.s.: $\nu_n$ is continuous, as a measure on $\mathbb{R}$ (i.e. it has no atoms), and $r_n = 0$. Such an equilibrium is constructed in Section 8. The general definition of a continuum-player game and its connection to a finite-player game can be found, e.g., in [14] and in the references therein. We discuss such a connection for the proposed game in Subsection 2.3.

### 2.2 Equilibrium

The objective function of an agent, starting at the initial state $(s, \alpha) \in S$, at any time $m = 0, \ldots, N$, and using the control $(p, q, r)$, is given by the $\mathcal{F}_m$-measurable random variable:

$$J^{(p,q,r)}(m, s, \alpha, \nu) = \mathbb{E}^\alpha_m \left( S_N^{(p,q,r)}(m, s, \nu) \right)^{+} p_N^b - \left( S_N^{(p,q,r)}(m, s, \nu) \right)^{-} p_N^a$$

\footnote{Note each agent is only allowed to place her limit order at a single price level, at any given time. However, this results in no loss of optimality. Indeed, using the Dynamic Programming Principle derived in Lemma 9 and Corollaries 2, 3, one can show, by induction, that, in equilibrium, an agent does not benefit from posting multiple limit orders at the same time. As shown in [44], this is typical for a continuum-player game.}
where we assume that $0 \cdot \infty = 0$. In the above expression, we assume that, at the final time $n = N$, each agent is forced to liquidate her position at the bid or ask price available at that time. Alternatively, one can think of it as marking to market.

**Definition 1.** For a given LOB $\nu$, integer $m = 0, \ldots, N - 1$, and state $(s, \alpha) \in S$, the triplet of adapted processes $(p, q, r)$ is an admissible control if the positive part of the expression inside the expectation in (2) has a finite expectation under $P$.

For a given LOB $\nu$, an initial condition $(m, s, \alpha)$, and a triplet of $\mathbb{F} \times \mathcal{B}(S)$-adapted random fields $(p, q, r)$, we identify the latter (whenever it causes no confusion) with stochastic processes $(p, q, r)$ via:

$$
 p_n = p_n \left( S_{n+1}^{(p,q,r)}(m, s, \nu), \alpha \right), \quad q_n = q_n \left( S_{n+1}^{(p,q,r)}(m, s, \nu), \alpha \right), \quad r_n = r_n \left( S_{n+1}^{(p,q,r)}(m, s, \nu), \alpha \right),
$$

and the state dynamics (1), for $n = m, \ldots, N$. This system determines $(p, q, r)$ and $S_{n}^{(p,q,r)}$ recursively.

**Definition 2.** For a given LOB $\nu$, we call the triplet of progressively measurable random fields $(p, q, r)$ an optimal control if, for any $m = 0, \ldots, N$ and $(s, \alpha) \in S$, we have:

- $(p, q, r)$ is admissible,
- $J(p,q,r)(m, s, \alpha, \nu) \geq J(p', q', r')(m, s, \alpha, \nu)$, $P$-a.s., for any admissible control $(p', q', r')$.

In the above, we make the standard simplifying assumption of continuum-player games: each agent is too small to affect the empirical distribution of cumulative controls (reflected in $\nu$) when she changes her control (cf. [14]). Note also that our definition of the optimal control implies that it is time consistent: re-evaluation of the optimality at any future step, using the same terminal criteria, must lead to the same optimal strategy. Next, we discuss the notion of equilibrium in the proposed game. First, we notice that, if $p_N^b$ or $p_N^a$ becomes infinite, the agents with positive or negative inventory may face the objective value of “$-\infty$”, for any control they use. In such a case, their optimal controls may be chosen in an arbitrary way, resulting in unrealistic equilibria. To avoid this, we impose the additional regularity condition on $\nu$.

**Definition 3.** A given LOB $\nu$ is admissible if, for any $m = 0, \ldots, N - 1$ and any $\alpha \in A$, we have $P$-a.s.:

$$
 E_m^a|p_N^a|_\vee |p_N^b| < \infty.
$$

Let us consider the (stochastic) value function of an agent for a fixed $(m, s, \alpha, \nu)$:

$$
 V_m^\nu(s, \alpha) = \text{esssup}_{(p, q, r)} J^{(p,q,r)}(m, s, \alpha, \nu), \quad (3)
$$

where the essential supremum is taken under $P$, over all admissible controls $(p, q, r)$, and $J^{(p,q,r)}$ is given by (2). Appendix A shows that, for any admissible $\nu$, $V_m^\nu(\cdot, \alpha)$ has a continuous modification under $P$, which we refer to as the value function of an agent with beliefs $\alpha$. Using the Dynamic Programming Principle, Appendix A provides an explicit system of recursive equations that characterize optimal strategies and the value function. In particular, the results of Appendix A (cf. Corollary 2) yield the following proposition.

**Proposition 1.** Assume that, for an admissible LOB $\nu$, there exists an optimal control $(p, q, r)$. Then, for any $(s, \alpha) \in S$, the following holds $P$-a.s., for all $n = 0, \ldots, N - 1$:

$$
 V_n^\nu(s, \alpha) = s^+ \lambda^a_n(\alpha) - s^- \lambda^b_n(\alpha)
$$

with some adapted processes $\lambda^a(\alpha)$ and $\lambda^b(\alpha)$, such that $\lambda^a_N(\alpha) = p_N^b$ and $\lambda^b_N(\alpha) = p_N^a$. 

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The values of $\lambda^a(\alpha)$ and $\lambda^b(\alpha)$ can be interpreted as the expected execution prices of the agents with beliefs $\alpha$, who are long and short the asset, respectively.

**Definition 4.** Consider an empirical distribution process $\mu = (\mu_n)_{n=0}^N$ and a market model, as described in Subsection 2.1. We say that a given LOB process $\nu$ and a control $(p, q, r)$ form an equilibrium, if there exists a Borel set $\hat{\AA} \subset \AA$, called the support of the equilibrium, such that:

1. $\mu_n(\mathbb{R} \times (\AA \setminus \hat{\AA})) = 0$, $\mathbb{P}$-a.s., for all $n$,

2. $\nu$ is admissible, and $(p, q, r)$ is an optimal control for $\nu$, on the state space $\tilde{S} = \mathbb{R} \times \hat{\AA}$,

3. and, for any $n = 0, \ldots, N - 1$, we have, $\mathbb{P}$-a.s.:

$$
\mu _n^+((-\infty, x]) = \int_{\tilde{S}} 1_{\{p_n(s, \alpha) \leq x, r_n(s, \alpha) = 0\}} q_n^+(s, \alpha) \mu_n(ds, d\alpha), \quad \forall x \in \mathbb{R},
$$

$$
\mu _n^-((-\infty, x]) = \int_{\tilde{S}} 1_{\{p_n(s, \alpha) \leq x, r_n(s, \alpha) = 0\}} q_n^-(s, \alpha) \mu_n(ds, d\alpha), \quad \forall x \in \mathbb{R}.
$$

**Remark 1.** It follows from Proposition 1 that, in equilibrium, it is optimal for an agent with zero initial inventory to do nothing. Hence, in equilibrium, roundtrip strategies are impossible. To allow for roundtrip strategies in equilibrium, one can e.g. introduce an upper bound on $|\nu|$. However, we do not believe that such a modification would change the qualitative behavior of market liquidity as a function of trading frequency, which is the main focus of the present paper.

Notice that, because the optimal controls are required to be time consistent under $\mathbb{P}$, the above definition, in fact, defines a sub-game perfect equilibrium. It is worth mentioning that Definition 4 defines a partial equilibrium, as the empirical distribution process $\mu$ is given exogenously. A more traditional version of equilibrium would require $\mu$ to be determined by the initial distribution and the values of the state processes:

$$
\mu_n = \mu_0 \circ (s, \alpha) \mapsto \left( S_n^{(p, q, r)}(0, s, \nu), \alpha \right)^{-1},
$$

which must hold $\mathbb{P}$-a.s., for all $n = 0, \ldots, N$, with $S_n^{(p, q, r)}(0, s, \nu)$ defined via (1), in addition to the other conditions in Definition 4. Nevertheless, we choose not to enforce the condition (6) in the definition of equilibrium, in order to allow new agents to enter the game, which, in effect, amounts to modeling $\mu$ exogenously. If one assumes that no new agents arrive to the market, then, the fixed-point condition (6) has to be enforced. In fact, in Section 8, we construct an equilibrium which satisfies this condition. However, it is important to notice that the main results of this work (cf. Section 4) provide necessary conditions for all equilibria: those satisfying the condition (6) and the ones that do not.

**Remark 2.** Let us comment on the information structure of the game. In the present setting, all agents observe the same information, given by the filtration $\mathbb{F}$. We consider an open-loop equilibrium, in which the agent’s strategy is viewed as an adapted stochastic process (rather than a function of the states and controls of other players), and the definition of optimality is chosen accordingly. In addition, as $\mu$ is adapted to $\mathbb{F}$, each agent has a complete information about the present and past states of other agents, and their beliefs. However, as the agents use different (subjective) measures $\{\mathbb{P}^\alpha\}$, their views on the future values of $\mu$ may be different. Of course, it would be more realistic to assume that the agents do not have a complete information about each other’s current states, but this would make the problem significantly more complicated. In the present setting, the agents also have complete information about the current location of a fundamental price. In our follow-up paper, [25], we relax this assumption, which allows us to develop a more realistic model for the “local” behavior of an individual agent. However, such a relaxation does not seem necessary for the questions analyzed herein.
Next, we need to add another condition to the notion of equilibrium. Notice that equations (4)–(5) should serve as the fixed-point constraints that allow one to obtain the optimal controls \((p, q, r)\), along with the LOB \(\nu\). However, these equations only hold for \(n = 0, \ldots, N - 1\): indeed, the agents do not need to choose their controls at time \(n = N\), as the game is over and their residual inventory is marked to the bid and ask prices. However, the terminal bid and ask prices are determined by the LOB \(\nu_N\), which, in turn, can be chosen arbitrarily. To avoid such ambiguity, we impose an additional constraint on the equilibria studied herein. First, we introduce the notion of a fundamental price.

**Definition 5.** Assume that \(\mathbb{P}\text{-a.s.},\) for any \(n = 1, \ldots, N\), there exists a unique \(p_0^n\) satisfying \(D_n(p_0^n) = 0\). Then, the adapted process \((p_0^n)_{n=1}^N\) is called the fundamental price process.

Whenever the notion of a fundamental price is invoked, we assume that it is well defined. The intuition behind \(p_0\) is clear: it is a price level at which the external demand is balanced. However, it is important to stress that we do not assume that the asset can be traded at the fundamental price level. Rather, \(p_0^n\) denotes some fundamental forecast about external demand, whereas all actual trading happens on the exchange, against the current LOB. This makes our setting different from many other approaches existing in the literature.

**Definition 6.** Assume that the fundamental price is well defined and denote \(\xi_N = p_0^N - p_0^{N-1}\). Then, an equilibrium with LOB \(\nu\) is linear at terminal crossing (LTC) if

\[
\nu_N = \nu_{N-1} \circ (x \mapsto x + \xi_N)^{-1}, \quad \mathbb{P}\text{-a.s.}
\]

(7)

The above definition assumes that the terminal LOB \(\nu_N\) is obtained from \(\nu_{N-1}\) by a simple shift, with the size of the shift equal to the increment of the fundamental price. This definition connects the LOB at the terminal time with the demand process, ruling out many unnatural equilibria. In particular, the question of existence of an equilibrium becomes non-trivial. However, the mere existence of an equilibrium is not the main focus of the present work. As it turns out (and is demonstrated, for example, in Section 3), in many models, the agents may reach an equilibrium in which one side of the LOB becomes empty. We call such LOB, and the associated equilibrium, degenerate.

**Definition 7.** We say that an equilibrium with LOB \(\nu\) is non-degenerate if \(\nu_+^n(\mathbb{R}) > 0\) and \(\nu_-^n(\mathbb{R}) > 0\), for all \(n = 0, \ldots, N - 1\), \(\mathbb{P}\text{-a.s.}\).

Intuitively, the degeneracy of LOB refers to a situation where, with positive probability, one side of the LOB disappears from the market: i.e. \(\nu_+^n(\mathbb{R})\) or \(\nu_-^n(\mathbb{R})\) becomes zero. Clearly, this happens when the agents who are supposed to provide liquidity choose to post market orders (i.e. consume liquidity) or wait (neither provide nor consume liquidity). Such a degeneracy can be interpreted as the internal (or, self-inflicted) liquidity crisis – the one that arises purely from the interaction between the agents, and cannot be justified by any fundamental economic reasons (e.g. the external demand for the asset may still be high, on both sides). Taking an optimistic point of view, we assume that the agents choose a non-degenerate equilibrium, whenever one is available. Hence, if there is no non-degenerate equilibrium available, an internal liquidity crisis may occur with positive probability. The main contribution of this paper is the characterization of the effects of trading frequency on the existence or non-existence of a non-degenerate equilibrium (i.e. occurrence or non-occurrence of an internal liquidity crisis).

### 2.3 Connections to finite-player and mean field games

The continuum-player game defined in the previous subsections can be related to a finite-player game as follows.\(^5\) Consider a measure on \(\mathbb{R} \times A\), which is represented as a finite linear combination of Dirac measures:

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\(^5\) See e.g. [14], and the references therein, for a similar connection in a general case.
\[ \hat{\mu}_0 = \frac{1}{M} \sum_{i=1}^M \delta_{(s_i^0, \alpha_i^0)}. \]

Clearly, \( \hat{\mu}_0 \) can be interpreted as an empirical distribution of the agents’ states in the proposed continuum-player game. In this case, \( s_i^0 \) is the number of shares per agent held by all agents in the \( i \)th group. Let us explain how this notion is related to the actual inventory levels (i.e., actual numbers of shares held by the agents) in the associated finite-player game. To this end, consider a collection of \( M \) agents, whose states are given by their inventories and beliefs, \((s, \alpha)\), with the initial states \( \{(s_i^0 = s_i^0/M, \alpha_i^0)\} \). Define the “unit mass” of agents to be \( M \). In this finite-player collection, the mass of agents (measured relative to the unit mass, \( M \)) at any state \((M s, \alpha)\) is precisely \( \hat{\mu}_0(ds, d\alpha) \), and their total inventory is \( M s \hat{\mu}_0(ds, d\alpha) \). The number of shares per agent is, then, defined as the total inventory held by these agents divided by their mass, and it is equal to \( M s \). Choosing \( s = s_i^0 \), we conclude that, in the finite-player collection, the number of shares per agent held by the agents at state \((s_i^0, \alpha_i)\) is given by \( M s_i^0 = \hat{s}_i^0 \), which coincides with our interpretation of \( \hat{s}_i^0 \) in the continuum-player game.

Let us, now, discuss the connection between the equilibria in the finite-player and the continuum-player games. Denote the agents’ strategies in the finite-player game by \( \{p_i, q_i, r_i\} \) – they have the same interpretation as in the continuum-player game (but with \( q_i^0 \) measured in shares, rather than shares per agent). Given this interpretation, we define the time-\( \eta \) LOB \( \nu_\eta \) via

\[ \nu_\eta^+ = \sum_{i=1}^M |q_i^0| \mathbf{1}_{\{r_i^\eta = 0, q_i^\eta > 0\}} \delta_{p_i^\eta}, \quad \nu_\eta^- = \sum_{i=1}^M |q_i^0| \mathbf{1}_{\{r_i^\eta = 0, q_i^\eta < 0\}} \delta_{p_i^\eta}. \]

(8)

The beliefs of the agents do not change over time, but their inventory levels, \( \{(S_i^\eta)\} \), change according to (1), with \( (p', q', r') \) in lieu of \( (p, q, r) \) and with \( \nu \) given by (8). Notice that choosing the dynamics of the state process according to (1) implies that no partial execution reaches a limit order, this order is executed in full. Similarly, every market order is executed in full at the bid or ask price. Clearly, these assumptions become more realistic if the order sizes are small.\(^6\)

Having fixed \( \{p', q', r'\} \), we define, for any index \( j \) and any admissible control \( (p', q', r') \), the random measure \( \nu_{\alpha_j}^{(p', q', r')} \) by the equation (8), in which we replace \( (p_i^0, q_i^0, r_i^0) \) in the \( j \)th term of the sum, by \( (p'_i, q'_i, r'_i) \). Clearly, \( \nu_{\alpha_j}^{(p', q', r')} \) is the LOB in the market in which all agents follow the strategies \( \{p', q', r'\} \), except the \( j \)th one, who follows \( (p'_j, q'_j, r'_j) \). In addition, we define \( \nu_{\alpha_j}^{(0)} \) via (8), with the \( j \)th term excluded from the sum – this is the LOB constructed form the limit orders of all agents except the \( j \)th one. The main difference of the finite-player game is in the notion of optimality: in a finite-player game, every player knows that changing her actions would have a non-negligible effect on the LOB, and she takes this fact into account when searching for an optimal strategy. Therefore, for a fixed set of strategies \( \{p', q', r'\} \), we introduce the “perturbed objective function” of the \( i \)th player:

\[ \tilde{j}^{(p', q', r')} = E^{\alpha_i} \left[ \left( E_N^{(p', q', r')} \right)^+ p_N^b - \left( E_N^{(p', q', r')} \right)^- p_N^a \right. 
\]

\[ \left. - \sum_{k=0}^{N-1} \left( p_k^b \mathbf{1}_{\{r_k=0\}} + p_k^a \mathbf{1}_{\{r_k=1, q_k < 0\}} + p_k^a \mathbf{1}_{\{r_k=1, q_k > 0\}} \right) \Delta S_{k+1}^{(p', q', r')} \right] \]

where the new (perturbed) state process \( S^{(p', q', r')} = S^{(p', q', r')} \left( 0, s_i^0, \nu_i^{(p', q', r')} \right) \) follows (1), and \( \tilde{p}^b \) and \( \tilde{p}^a \) are the bid and ask prices of the LOB without the \( i \)th agent present:

\[ \tilde{p}_k^b = \sup \mathrm{supp}(\hat{s}_k^{(0)}), \quad \tilde{p}_k^a = \inf \mathrm{supp}(\hat{s}_k^{(0)}) \]

\(^6\)We discuss the relaxation of these assumptions at the end of this subsection.
Note that we exclude the $i$th agent from the LOB, in the computation of the bid and ask prices, because an agent cannot mark her position to her own limit order. Comparing to (2), we notice that
\[
\hat{j}_i(p',q',r') \leq j(p',q',r') \left(0, s^i_0, \omega^i, \nu^i(p',q',r')\right)
\]
Naturally, the strategies $\{p^n, q^n, r^n\}$ form an equilibrium in the finite-player game, if, for any $i$, the control $(p^n, q^n, r^n)$ is optimal for the $i$th player, i.e.
\[
\hat{j}_i(p',q',r') \geq \hat{j}_i(p',q',r'),
\]
for all admissible $(p', q', r')$.

Assume that there exists an equilibrium $(\nu, p, q, r)$ of the associated continuum-player game, in the sense of Definition 4, with $\mu$ satisfying the additional constraint (6), and with $\mu_0(ds, d\alpha) = M\mu_0(d(Ms), \alpha) = \sum_{i=1}^M \delta(d_i(s^n_0, \alpha^n))(ds, d\alpha)$. Let us, first, show how the equilibrium strategies of a continuum-player game generate the strategies for a finite-player game. Namely, consider a finite-player game, in which the strategy $\{s^n_0, \alpha^n\}$, which represent the agents’ states in a finite-player game. The latter observation, along with (4)–(5) and (10), imply that $\nu$ satisfies (8) and, hence, the dynamics of the state processes $\{S^n\}$ are consistent with the definition of a finite-player game. Finally, notice that, in the associated continuum-player game, if an agent starts from the initial state $(s^n_0, \alpha^n)$ and follows the feedback strategy $(p, q, r)$, given the LOB $\nu$, her actual strategy (viewed as a process, not as a random field) coincides with $(p^n, q^n, r^n)$, and her state process coincides with $S^n$.

Next, we show that the strategies $\{p^n, q^n, r^n\}$, constructed from the continuum-player equilibrium $(p, q, r)$, produce an approximate equilibrium in the finite-player game. In order to do this, we need to make few additional assumptions. First, as it is shown in the subsequent sections (cf. Corollary 3), a continuum-player equilibrium can always be constructed so that $q^n(s, \alpha)$ takes values in $[0, s]$. Hence, we assume that this is the case for the continuum-player equilibrium $(\nu, p, q, r)$, which implies that the associated $q^n_i$ has the same sign as $s^n_0$ and cannot exceed it in absolute value. In addition, as we are interested in small agents, we assume that the admissible strategies of the agents in the finite-player game are constrained, so that their inventory is absolutely bounded by $\bar{s} > 0$ (which, in turn, implies that the sizes of their orders are absolutely bounded by $2\bar{s}$). We also assume, for simplicity, that all admissible price levels of limit orders, in the finite-player game, are absolutely bounded by a constant $P > 0$ (naturally, we assume that the same condition holds for the continuum-player control $p$). Finally, we assume that the model for future external demand is such that the cumulative distribution function of $D_{n+1}(p)$, under $\mathbb{P}^{\alpha^i}(\omega)$, satisfies the global Lipschitz property, uniformly over all $p \in \mathbb{R}$, all $i$ and $n$, and $\mathbb{P}^{\alpha^i}$-a.e. $\omega$ (i.e. the conditional distribution of the future demand size is uniformly continuous).\footnote{It is clear that the last two assumptions can be relaxed, and the desired conclusions will still hold. However, this part of the paper serves merely as an illustration of the connection between the two games, hence, we do not aim for the highest generality of the results.}

Let us show that, if the agents in the finite-player game follow the strategy $(p^n, q^n, r^n)$, the $i$th player can only improve her objective function (choosing a different admissible strategy) by at most $C\bar{s}n_0$, where $C > 0$ is a constant, independent of $\{(s^n_0, \alpha^n)\}$ and of the number of agents $M$. The main idea of this argument is to
notice that, for any admissible \((p', q', r')\), the values of \(\nu_n^+((-\infty, p'_n))\) and \(\nu_n^+((-\infty, p'_n),+)((-\infty, p'_n))\) cannot be too different, due to the small size of \(q^n_i\) (analogous statement holds for the buy side of the LOB). As a result, for any admissible \((p', q', r')\), the execution probabilities, with the original LOB and with the perturbed one, are very close. More formally, we have

\[
\mathbb{P}^{\nu}(D^+_n(p'_n) \in \left(\nu^+_n((p', q', r'),+)((-\infty, p'_n)), \nu_n^+((-\infty, p'_n))\right)]
\]

\[
\leq \mathbb{E}^{\nu} \mathbb{P}^{\alpha_i}(D^+_n(p'_n) \in [0, |q^n_i|]) \leq C_1 s^n_0,
\]

with some constant \(C_1 > 0\). A similar estimate applies to the buy side of the LOB. Thus, we obtain

\[
\tilde{j}^{i,(p', q', r')} \leq J^{i,(p', q', r')}(0, s^n_0, \alpha^i, \nu^{i,(p', q', r')}) \leq J^{i,(p', q', r')}(0, s^n_0, \alpha^i, \nu) + C_2 \bar{s} s^n_0
\]

\[
\leq J^{(p, q, r)}(0, s^n_0, \alpha^i, \nu) + C_2 \bar{s} s^n_0 = \tilde{j}^{i,(p', q', r')} + C_2 \bar{s} s^n_0,
\]

where the third inequality is due to the optimality of \((p, q, r)\) in the continuum-player game, and the equality follows from the fact that the strategies \(\{(p^n_i, q^n_i, r^n_i)\}\) are such that the long agents never post limit orders on the buy side of the LOB, and the short agents never post on the sell side. As a result, whenever the \(i\)-th agent needs to mark her position to the bid or ask price of the adjusted LOB \(\nu^{i,(0)}\), i.e. \(p^b\) or \(p^a\) (cf. (9)), the corresponding price coincides with the bid or ask price of the LOB \(\nu\), i.e. \(p^b\) or \(p^a\), respectively. The above estimate shows that the strategy \((p', q', r')\) generates the relative objective value, \(\tilde{j}^{i,(p', q', r')} / \bar{s} s^n_0\), which is \(\varepsilon\)-close to its maximum value, with \(\varepsilon = C_2 \bar{s}\). Note that it is important to consider the relative objective value (as opposed to the actual objective value), as it follows from the subsequent sections (cf. Corollary 2) that \(\tilde{j}^{i,(p', q', r')} = J^{(p, q, r)}(0, s^n_0, \alpha^i, \nu)\) is proportional to \(s^n_0\), which goes to zero as \(\bar{s} \to 0\). It is also important that the constant \(C_2\) does not depend on \(\{(s^n_0, \alpha^i)\}\) and on the number of agents \(M\), which implies that the above strategy is approximately optimal even if the total initial inventory of the agents is fixed, as long as every individual inventory is small enough.

The above \(\varepsilon\)-optimality can be easily established for a finite-player model with partial execution: i.e. with the state dynamics (1) modified so that only \(D^+_n(p_n) - \nu^+_n((-\infty, p_n))\) \& \(q^n_i\) shares (as opposed to \(q^n_i\) shares) are sold when \(D^+_n(p_n) > \nu^+_n((-\infty, p_n))\) (and similarly for the limit buy orders). Using the same arguments as above, one can easily show that the continuum-player equilibrium (with the original state dynamics (1)) also provides \(\varepsilon\)-optimal strategies for the finite-player model with partial execution. However, to ensure that the modified dynamics truly reproduce the effect of partial execution, we need to ensure, for example, that no two agents post limit orders at the same price level. This, in turn, follows from the assumption that all \(\{(s^n_0, \alpha^i)\}\) are distinct, and that the associated continuum-player equilibrium strategies, \((p, q, r)\), are such that \(p_n(s, \alpha) \neq p_n(s', \alpha')\) for \((s, \alpha) \neq (s', \alpha')\). Otherwise, the excessive demand would need to be shared between the two agents in the finite-player game, who post their orders at the same price level. The latter is not reflected in the modified state dynamics. Similarly, to avoid the unrealistic execution of market orders (i.e. always at the bid or ask prices), we can, for example, search for a continuum-player equilibrium in which the agents never post market orders. An equilibrium satisfying both of these conditions is constructed in Section 8.

Finally, we would like to comment on the connection between the proposed continuum-player game and the mean field games. The mean field games provide another framework for modeling continuum-player games. However, in the mean field game approach, the common factor in the individual state dynamics (i.e. the LOB \(\nu\), in the present case) is a function of the probability distribution of the states and/or controls of a representative player, rather than the empirical distribution of a continuum of players. To formally construct a corresponding mean field game model, one would have to extend the probability space to \(\Omega \times \Omega'\), with \(\Omega' = \mathbb{R} \times \Lambda\), and consider a new probability measure \(\tilde{\mathbb{P}}(d\omega, d\omega') = \mathbb{P}^{\alpha}(d\omega)(d\omega') \mu_0(d\omega')\), where \(\alpha(\omega')\) is
the canonical projection and $\mu_0$ is a probability measure on $\mathbb{R} \times \mathcal{A}$ (and we assume that $\alpha \mapsto \mathbb{P}^\alpha$ is $\mathcal{B}(\mathcal{A})$-measurable). Next, consider a single agent, whose state process evolves according to (1), with the given processes $(\nu_\omega, p_\omega, q_\omega, r_\omega)$, with the demand $D(\omega)$, defined on the original probability space $\Omega$, and with the initial condition $(s(\omega'), \alpha(\omega'))$. For a fixed $(\omega, n)$, denote the distribution of the random vector $\omega' \mapsto (p_n(\omega, \omega'), q_n(\omega, \omega'), r_n(\omega, \omega'))$ by $\hat{\mu}_n(\omega; dp, dq, dr)$. The collection $(\nu, p, q, r)$ is a mean field equilibrium if $(p, q, r)$ is optimal for $\nu$ (i.e. it maximizes the objective $J^{(p, q, r)}(0, s(\omega'), \alpha(\omega'), \nu)$, given in (2), for $\mu_0$-a.e. $\omega'$), and the following fixed-point condition holds:

$$\nu_n^+(\omega; (-\infty, x]) = \int_{\mathbb{R}_+} q_1(p \leq x, r = 0) \hat{\mu}_n(\omega; dp, dq, dr), \quad \forall x \in \mathbb{R}, \forall n, \mathbb{P}^\alpha\text{-a.e. } \omega,$$

and similarly for $\nu^-$. Consider a continuum-player equilibrium $(\nu, p, q, r)$, in the sense of Definition 4, constructed on the original probability space $\Omega$, with $\mu$ satisfying the additional constraint (6), and with the given initial measure $\mu_0$. We conjecture that any such equilibrium (perhaps, under some mild technical assumptions) yields a mean field equilibrium, where the dependence on $\omega'$ in the optimal control process is introduced through the initial condition $(s(\omega'), \alpha(\omega'))$. However, such a connection does not seem to help the analysis conducted in the present paper. Indeed, the mean field game formulation (as an approximation of a game with a large number of players) simplifies the analysis of equilibrium only if the dependence of a state process on randomness is relatively simple. In such a case, the mean field approach allows one to "average out" the dependence on the idiosyncratic source of randomness. In the present case, the idiosyncratic source is given by the initial condition $(s(\omega'), \alpha(\omega'))$, whose effect on the associated state process and on the optimal control is well understood, provided we know the optimal control in a feedback form. On the other hand, the dependence on the common source, $D$, is rather complicated, and it represents the main difficulty in describing the equilibrium. Therefore, herein, we do not prove the above conjecture, and we only provide this discussion to better illustrate the connection of our setting to other existing modeling approaches.

### 3 Example: a Gaussian random walk model

In this section, we consider a specific market model for the external demand $D$ to construct a non-degenerate LTC equilibrium. More importantly, using this model, we illustrate the liquidity effects of trading frequency, which, as mentioned in the introduction, is the main goal of the present work. The present example, albeit very simplistic, allows us to identify certain important phenomena that occur to the optimal strategies of the agents (and, hence, to the LOB) as the trading frequency increases. In the rest of the paper, we show that these phenomena are not due to the particular choice of a model made in the present section, but, in fact, occur in any model in which the (limiting) price process is an Itô process.

On a complete stochastic basis $(\Omega, \mathbb{F} = (\tilde{\mathcal{F}}_t)_{t \in [0, T]}, \mathbb{P})$, we consider a continuous time process $\tilde{p}_t$:

$$\tilde{p}_t^0 = p_0^0 + \alpha t + \sigma W_t, \quad p_0^0 \in \mathbb{R}, \quad t \in [0, T],$$

(11)

where $\alpha \in \mathbb{R}$ and $\sigma > 0$ are constants, and $W$ is a Brownian motion. We also consider an arbitrary progressively measurable random field $(\tilde{D}_t(p))$, s.t., $\mathbb{P}$-a.s., the function $\tilde{D}_t(\cdot) - \tilde{D}_s(\cdot)$ is strictly decreasing and vanishing at zero, for any $0 \leq s < t \leq T$. Finally, we introduce the empirical distribution process $(\tilde{\mu}_t)$, with values in the space of finite sigma-additive measures on $\mathbb{S}$. We partition the time interval $[0, T]$ into $N$ subintervals of size $\Delta t = T/N$. A discrete time model is obtained by discretizing the continuous time one:

$$\mathcal{F}_n = \tilde{\mathcal{F}}_{n\Delta t}, \quad p_n^0 = \tilde{p}_{n\Delta t}^0, \quad D_n(p) = (\tilde{D}_{n\Delta t} - \tilde{D}_{(n-1)\Delta t})(p - p_n^0), \quad \mu_n = \tilde{\mu}_{n\Delta t}.$$

\[\text{In order to ensure the existence of regular conditional probabilities for the discrete time model, we can, for example, assume that } \mathcal{F}_T \text{ is generated by a random element with values in a standard Borel space.}\]
In this section, for simplicity, we assume that the set of agents’ beliefs is a singleton: \( \mathcal{A} = \{ \alpha \} \) and \( \mathbb{P}^\alpha = \mathbb{P} \). We also assume that (at least, from the agents’ point of view) there are always some long and short agents present in the market: \( \mu_n((0, \infty) \times \mathcal{A}), \mu_n((-\infty, 0) \times \mathcal{A}) > 0 \), \( \mathbb{P} \)-a.s., for all \( n \). Clearly, \( N \) represents the trading frequency, and the continuous time model represents the “limiting model”, which the agents use as a benchmark, in order to make consistent predictions in the markets with different trading frequencies. We assume that the benchmark model is fixed, and \( N \) is allowed to vary. In the remainder of this section, we propose a method for constructing a non-degenerate LTC equilibrium in the above discrete time model. We show that the method succeeds for any \((N, \sigma)\) if \( \alpha = 0 \). However, for \( \alpha \neq 0 \), we demonstrate numerically that the method fails as \( N \) becomes large enough. We show why, precisely, the proposed construction fails, providing an economic interpretation of this phenomenon. Moreover, we analyze the market close to the moment when a non-degenerate equilibrium fails to exist and demonstrate that the agents’ behavior at this time follows the pattern typical for an internal (or, self-inflicted) liquidity crisis.

In view of Proposition 1, in order to construct a non-degenerate LTC equilibrium, we need to find a control \((\hat{p}, \hat{q}, \hat{\nu})\), and the expected execution prices \((\hat{\lambda}^a, \hat{\lambda}^b)\), s.t. the value function of an agent with inventory \( s \) is given by \( V_n(s) = s^+ \hat{\lambda}^a_n + s^- \hat{\lambda}^b_n \), and it is attained by the strategy \((\hat{p}, \hat{q}, \hat{\nu})\). In addition, we need to find a non-degenerate LOB \( \nu \), s.t. (4), (5) and (7) hold. Our ansatz is as follows:

\[
\nu_n = h_n^a \delta_{p_n^a} + h_n^b \delta_{p_n^b}, \quad p_n^a = \hat{p}_n^a + p_n^0, \quad p_n^b = \hat{p}_n^b + p_n^0, \quad -\infty < \hat{p}_n^a, \hat{p}_n^b < \infty, \\
\hat{p}_n(s) = p_n^a 1_{\{s > 0\}} + p_n^b 1_{\{s < 0\}}, \quad \hat{q}_n(s) = s, \quad \hat{r}_n(s) = 0, \quad \lambda_n^a = \hat{\lambda}_n^a + p_n^0, \quad \lambda_n^b = \hat{\lambda}_n^b + p_n^0, 
\]

where \( \delta \) is the Dirac measure, \((\hat{p}^a, \hat{p}^b, \hat{\lambda}^a, \hat{\lambda}^b)\) are deterministic processes, and \( h_n^a = \int_0^\infty s \mu_n(ds) > 0, h_n^b = \int_{-\infty}^0 \mu_n(ds) > 0 \). With such an ansatz, the conditions (4), (5) are satisfied automatically. Thus, we only need to choose finite deterministic processes \((\hat{p}^a, \hat{p}^b, \hat{\lambda}^a, \hat{\lambda}^b)\) s.t.: \( \hat{p}_n^a = \hat{p}_n^a(N-1), \hat{p}_n^b = \hat{p}_n^b(N-1) \) (so that the equilibrium is LTC) and the associated \((\hat{p}, \hat{q}, 0)\) form an optimal control, producing the value function \( V_n(s) = s^+ \hat{\lambda}^a_n + s^- \hat{\lambda}^b_n \). Appendix A contains necessary and sufficient conditions for characterizing such families \((p^a, p^b, \lambda^a, \lambda^b)\). In particular, we deduce from Corollaries 2 and 3 that \((\hat{p}_N^{a,N-1}, \hat{p}_N^{b,N-1}, \hat{\lambda}_N^a, \hat{\lambda}_N^b)\) form a suitable family in a single-period case, \([N-1, N]\), if they solve the following system:

\[
\begin{align*}
\hat{p}_N^{a,N-1} &\in \arg\max_{p \in \mathbb{R}} \mathbb{E} \left[ (p - \hat{p}_N^{a,N-1} - \xi) 1_{\{\xi > p\}} \right], \quad \hat{p}_N^{a,N-1} < 0, \\
\hat{p}_N^{b,N-1} &\in \arg\max_{p \in \mathbb{R}} \mathbb{E} \left[ (\hat{p}_N^{b,N-1} - p + \xi) 1_{\{\xi < p\}} \right], \quad \hat{p}_N^{b,N-1} > 0, \\
\hat{\lambda}_N^a &\geq \hat{p}_N^{a,N-1} + \alpha \Delta t + \mathbb{E} \left[ (\hat{p}_N^{a,N-1} - \hat{p}_N^{b,N-1} - \xi) 1_{\{\xi > \hat{p}_N^{b,N-1}\}} \right], \\
\hat{\lambda}_N^b &\leq \hat{p}_N^{b,N-1} + \alpha \Delta t - \mathbb{E} \left[ (\hat{p}_N^{a,N-1} - \hat{p}_N^{b,N-1} + \xi) 1_{\{\xi < \hat{p}_N^{b,N-1}\}} \right], \\
\hat{p}_N^{b,N-1} &\geq \hat{p}_N^{a,N-1} + \alpha \Delta t, \quad \hat{\lambda}_N^a \leq \hat{\lambda}_N^b, \quad \hat{\lambda}_N^a = \hat{\lambda}_N^b \equiv \hat{\lambda}_N \\
\hat{p}_N^{a,N-1} &\leq \hat{p}_N^{b,N-1}, \quad \hat{\lambda}_N^a \leq \hat{\lambda}_N^b, \quad \hat{p}_N^{b,N-1} \geq \hat{p}_N^{a,N-1} + |\alpha| \Delta t,
\end{align*}
\]

where \( \xi = \Delta p^a_N \sim \mathcal{N}(\alpha \Delta t, \sigma^2 \Delta t) \). Let us comment on the economic meaning of the equations in (12). The expectations in the first two lines represent the relative expected profit from executing a limit order at time \( N \), at the chosen price level \( p + p_N^{0,N-1} \) versus marking the inventory to market at time \( N \), at the best price available on the other side of the book: i.e. \( p_N^b = \hat{p}_N^{b,N-1} + \xi + p_N^{0,N-1} \) or \( p_N^a = \hat{p}_N^{a,N-1} + \xi + p_N^{0,N-1} \). Notice that a limit order is executed if and only if the fundamental price at time \( N \) is above or below the chosen level of agent’s limit order: i.e. if \( p_N^a + \xi > p + p_N^{0,N-1} \) or \( p_N^b + \xi < p + p_N^{0,N-1} \). Clearly, it is only optimal for an agent to post a limit order if the relative expected profit is nonnegative, which is the case if and only if \( p_N^{a,N-1} < 0 < p_N^{b,N-1} \). The third and fourth lines in (12) represent the expected execution prices of the agents at

---

\(^9\)The execution of limit orders simplifies in the chosen ansatz, because the agents on each side of the book (i.e. long or short) post orders at the same prices level.
time \(N - 1\), assuming they use the controls given by \((\hat{p}_{N-1}^a, \hat{p}_{N-1}^b)\). Each of the right hand sides is a sum of two components: the relative expected profit from posting a limit order and the expected value of marking to market at time \(N\), measured relative to \(p_{N-1}^b\). Let us analyze the inequalities in the last line of (12). If the bid price at time \(N - 1\) exceeds the expected execution price of a long agent, i.e. \(\hat{p}_{N-1}^a + p_{N-1}^b > \hat{p}_{N-1}^a + p_{N-1}^b\), then every agent with positive inventory prefers to submit a market order, rather than a limit order, at time \(N - 1\), which causes the ask side of the LOB to degenerate. Similarly, we establish \(\hat{p}_{N-1}^b \leq \hat{p}_{N-1}^b\). Finally, if \(\alpha > 0\) and \(\hat{p}_{N-1}^a < \hat{p}_{N-1}^b + \alpha \Delta t\), an agent may buy the asset using a market order at time \(N - 1\), at the price \(\hat{p}_{N-1}^a + p_{N-1}^b\), and sell it at time \(N\), at the expected price \(\hat{p}_{N-1}^a + p_{N-1}^b + \alpha \Delta t > \hat{p}_{N-1}^a + p_{N-1}^b\) (a reverse strategy works for \(\alpha < 0\)). This strategy can be scaled to generate infinite expected profit and, hence, is excluded by the last inequality in the last line of (12).

We construct a solution to (12) by solving a fixed-point problem given by the first two lines of (12) and verifying that the desired inequalities hold. We implement this computation in MatLab, and the results can be seen as the right-most points on the graphs in Figure 2. From the numerical solution, we see that, whenever \(\Delta t\) is small enough, the conditions \(\hat{p}_{N-1}^a \leq \hat{\lambda}_{N-1}^a\) and \(\hat{p}_{N-1}^b \leq \hat{\lambda}_{N-1}^b\) are satisfied (cf. the right part of Figure 2). In addition, for \(\alpha \geq 0\), we have

\[
0 < \mathbb{E} \left[\hat{p}_{N-1}^a - \hat{p}_{N-1}^b - \xi | \xi > \hat{p}_{N-1}^b\right] = \hat{p}_{N-1}^a - \hat{p}_{N-1}^b - \mathbb{E} \left[\xi | \xi > \hat{p}_{N-1}^b\right] \leq \hat{p}_{N-1}^a - \hat{p}_{N-1}^b - \alpha \Delta t,
\]

which yields the last inequality in (12). The case of \(\alpha < 0\) is treated similarly. Notice that \(\hat{\lambda}_N^a = \hat{p}_N^a = \hat{p}_{N-1}^a\) and \(\hat{\lambda}_{N-1}^b = \hat{p}_{N-1}^b = \hat{\lambda}_N^b\). Thus, the single-period equilibrium we have constructed satisfies:

\[
\hat{p}_n^a \leq \hat{\lambda}_n^a, \quad \hat{\lambda}_n^b \leq \hat{\lambda}_n^a, \quad \hat{\lambda}_{n+1}^a < 0, \quad \hat{\lambda}_{n+1}^b > 0,
\]

(13)

for \(n = N - 1\). If one of the first two inequalities in (13) fails, the agents choose to submit market orders, as opposed to limit orders, which leads to degeneracy of the LOB – one side of it disappears. If one of the last two inequalities fails, the execution of a limit order, at any price level, yields a negative relative expected profit for the agents on one side of the book (given by the expectation in the first or second line of (12)). As a result, it becomes optimal for all such agents to not post any limit orders, and the LOB degenerates. The latter is interpreted as the adverse selection effect. For example, if the third inequality in (13) fails, then, every long agent believes that, no matter at which price level her limit order is posted, if it is executed in the next time period, her expected execution price at the next time step will be higher than the price at which the limit order is executed. Hence, it does not make sense to post a limit order at all.

In a single period \([N - 1, N]\), by choosing small enough \(\Delta t\), we can ensure that the inequalities in (13) are satisfied. However, it turns out that, as we progress recursively backwards, constructing an equilibrium, we may encounter a time step at which one of the inequalities in (13) fails, implying that a non-degenerate LTC equilibrium cannot be constructed for the given time period (at least, using the proposed method). To see this, consider the recursive equations for \((\hat{p}_n^a, \hat{\lambda}_n^a)\) (which are chosen to satisfy the conditions of Corollary 2, given our ansatz):

\[
\begin{align*}
\hat{p}_n^a \in \arg \max_{p \in \mathbb{R}} \mathbb{E} \left[\left(p - \hat{\lambda}_{n+1}^a - \xi\right) \mathbf{1}_{\{\xi > p\}}\right], \\
\hat{\lambda}_n^a = \hat{\lambda}_{n+1}^a + \alpha \Delta t + \mathbb{E} \left[\left(\hat{p}_n^a - \hat{\lambda}_{n+1}^a - \xi\right) \mathbf{1}_{\{\xi > \hat{p}_n^a\}}\right] < 0,
\end{align*}
\]

(14)

and similarly for \((\hat{p}_n^b, \hat{\lambda}_n^b)\). Using the properties of Gaussian distribution, it is easy to see that, if \(\hat{\lambda}_{n+1}^a < 0\), we have \(\hat{p}_n^a > 0\). Similar conclusion holds for \((\hat{\lambda}_n^b, \hat{p}_n^b)\). Thus, if \(\hat{\lambda}_n^a < 0 < \hat{\lambda}_n^b\), for \(k = n + 1, \ldots, N\), our method

\footnote{In fact, it is not difficult to prove rigorously that, for any \((\alpha, \sigma)\), there exists a unique solution to such system, provided \(\Delta t\) is small enough. We omit this result for the sake of brevity.}

\footnote{This is easy to explain intuitively, as the optimal objective values in the first two lines of (12) are of the form \(C \sqrt{\Delta t} + \alpha O(\Delta t)\).}
allows us to construct a non-degenerate LTC equilibrium on the time interval \([n, N]\), with \(\hat{p}^b < 0 < \hat{p}^a\). Such a construction always succeeds if the agents are market-neutral: i.e. \(\alpha = 0\). Indeed, in this case, assuming \(\lambda^a_{n+1} < 0 < \lambda^b_{n+1}\), we have: \(\hat{p}^b_n < 0 < \hat{p}^a_n\), and

\[
\lambda^a_{n+1} + \left( E \left[ \left( \hat{p}^a_n - \lambda^a_{n+1} - \xi \right) \mathbf{1}_{\{\xi > \hat{p}^a_n\}} \right] \right) + E \left[ \left( \hat{p}^a_{n+1} - \xi \right) \mathbf{1}_{\{\xi > \hat{p}^a_n\}} \right] \leq 0.
\]

Hence, \(\hat{\lambda}^a_n < 0\), and, similarly, we deduce that \(\hat{\lambda}^b_n > 0\). By induction, we obtain a non-degenerate LTC equilibrium on \([0, N]\), for any \((N, \sigma)\), as long as \(\alpha = 0\). Corollary 1 shows that, as \(N \to \infty\), the processes \((\hat{\lambda}^a, \hat{\lambda}^b)\) converge to zero, which means that the expected execution prices converge to the fundamental price. The latter is interpreted as market efficiency in the high-frequency trading regime: any market participant expects to buy or sell the asset at the fundamental price. The left hand side of Figure 3 shows that the bid and ask prices also converge to the fundamental price if \(\alpha = 0\). This can be interpreted as a positive liquidity effect of increasing the trading frequency.

However, the situation is quite different if \(\alpha \neq 0\). Assume, for example, that \(\alpha > 0\). Then, the second line of (14) implies that \(\hat{\lambda}^a\) increases by, at least, \(\alpha \Delta t\) at each step of the (backward) recursion. Recall that the number of steps is \(N = T / \Delta t\), hence, \(\hat{\lambda}^a_0 \geq \hat{\lambda}^b_0 + \alpha T\). If \(|\lambda^b_0|\) is small (which is typically the case if \(N\) is large), then, we may obtain \(\lambda^a_{n+1} \geq 0\), at some time \(n\), which violates the third inequality in (13), or, equivalently, implies that the objective in the first line of (14) is strictly negative for all \(p\). The latter implies that it is suboptimal for the agents with positive inventory to post limit orders, and the proposed method fails to produce a non-degenerate LTC equilibrium in the interval \([n, N]\). Figure 2 shows that this does, indeed, occur. Figures 2 and 3 also show that, for a given (finite) frequency \(N\), if \(|\alpha|\) is small enough, a non-degenerate equilibrium may still be constructed. Nevertheless, for any \(|\alpha| \neq 0\), however small it is, there exists a large enough \(N\), s.t. the non-degenerate LTC equilibrium fails to exist (at least, within the class defined by the proposed method). This is illustrated in Figure 3.

It is important to provide an economic interpretation of why such degeneracy occurs. A careful examination of Figure 2 reveals that, around the time when \(\hat{\lambda}^a\) becomes nonnegative, the ask price \(\hat{p}^a\) explodes. This means that the agents who want to sell the asset are only willing to sell it at a very high price. Notice also that this price is several magnitudes larger than the expected change in the fundamental price (represented by the black dashed line in the left hand side of Figure 2). Hence, such a behavior cannot be justified by the fundamental reasons. Indeed, this is precisely what is called an internal (or, self-inflicted) liquidity crisis. So, what causes such a liquidity crisis? Recall that there are two potential reasons for the market to degenerate: the agents may choose to submit market orders (if \(\hat{p}^b_n > \lambda^b_n\), or \(\hat{p}^a_n < \lambda^a_n\)), or they may choose to wait and do nothing (if \(\lambda^a_{n+1} \geq 0\) or \(\lambda^b_{n+1} \leq 0\)). The right hand side of Figure 2 shows that the degeneracy is caused by the second scenario. This means that the naive explanation of the internal liquidity crisis, based on the claim that, in a bullish market, those who need to buy the asset will submit market orders wiping out liquidity on the sell side of the book, is wrong. Instead, if the agents on the sell side of the book have the same beliefs, they will increase the ask price so that it is no longer profitable for the agents who want to buy the asset to submit market buy orders. In fact, the ask price may increase disproportionally to the expected change in the fundamental price (i.e. the signal), and this is what causes an internal liquidity crisis. The size of the resulting change in the bid or ask price depends not only on the signal, but also on the trading frequency, which demonstrates the negative liquidity effect of increasing the trading frequency: it makes the market more fragile with respect to deviations of the agents from market-neutrality. The latter, in turn, is explained by the fact that higher trading frequency makes the adverse selection effect more pronounced. To see this, consider e.g. an agent who is trying to sell one share of the asset. Increasing the trading frequency increases the expected execution value of this agent, bringing it closer to the fundamental price: this corresponds to \(\hat{\lambda}^a\) approaching zero (from below). Assume that the agent posts a limit sell order at a price level \(p\). If this order is executed in the next period, then, the agent receives \(p\), but, for this to happen, the fundamental price value at the next time step, \(\hat{p}^b_{n+1}\), has to be above \(p\).
On the other hand, the expected execution price of the agent at the next time step is \( p_{n+1}^0 + \hat{\lambda}_{n+1}^a \). Thus, the expected relative profit of the agent, given the execution of her limit order, is \( \mathbb{E}(p - p_{n+1}^0 - \hat{\lambda}_{n+1}^a | p_{n+1}^0 > p) \). The latter expression cannot be positive, unless \( \hat{\lambda}_{n+1}^a < 0 \) and \( |\hat{\lambda}_{n+1}^a| \) is sufficiently large. Therefore, if \( |\hat{\lambda}_{n+1}^a| \) is small relative to \( \mathbb{E}(p_{n+1}^0 - p | p_{n+1}^0 > p) \), the agent is reluctant to post a limit order at the price level \( p \). Hence, \( p \) needs to be sufficiently large, to ensure that \( \mathbb{E}(p_{n+1}^0 - p | p_{n+1}^0 > p) \) is smaller than \( |\hat{\lambda}_{n+1}^a| \) (in the Gaussian model of this section, the latter expectation vanishes as \( p \rightarrow \infty \)) – and smallest such level of \( p \) determines the effect of adverse selection. It turns out that, if the agents are market-neutral (i.e. \( \alpha = 0 \)), as the frequency \( N \) increases, the quantity \( \mathbb{E}(p_{n+1}^0 - p | p_{n+1}^0 > p) \), for any fixed \( p \), converges to zero at the same rate as \( |\hat{\lambda}_{n+1}^a| \), hence, the above adverse selection effect does not get amplified. On contrary, if the agents are not market-neutral, \( \hat{\lambda}_{n+1}^a \) reaches zero at some high enough (but finite) frequency, while \( \mathbb{E}(p_{n+1}^0 - p | p_{n+1}^0 > p) \) remains strictly positive, for any \( p \), which amplifies the adverse selection effect infinitely and causes the market to degenerate. Of course, so far, these conclusions are based on a very specific example and on a particular method of constructing an equilibrium. The next section shows that all these conclusions remain valid in any model (with, possibly, heterogeneous beliefs) in which the fundamental price is given by an Itô process.

4 Main results

In this section, we generalize the conclusions made in the previous section, so that they hold in a general model and for any choice of an equilibrium. As before, we begin with the “limiting” continuous time model. Consider a terminal time horizon \( T > 0 \) and a complete stochastic basis \((\Omega, \mathbb{F}, \mathbb{P}) = (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P}\), with a Brownian motion \( W \) on it.\(^{12}\) We define the adapted process \( \tilde{p}^0 \) via

\[
\tilde{p}_t^0 = p_0^0 + \int_0^t \sigma_s dW_s, \quad p_0^0 \in \mathbb{R},
\]

where \( \sigma \) is a progressively measurable locally square integrable process.

**Assumption 1.** There exists a constant \( C > 1 \), such that, \( 1/C \leq \sigma_t \leq C \), for all \( t \in [0, T] \), \( \mathbb{P}\)-a.s.

Consider a Borel set of beliefs \( \mathcal{A} \) and the associated family of measures \( \{\mathbb{P}^\alpha\}_{\alpha \in \mathcal{A}} \) on \((\Omega, \mathcal{F}_T)\), absolutely continuous with respect to \( \mathbb{P} \). Then, for any \( \alpha \in \mathcal{A} \), we have

\[
\tilde{p}_t^0 = p_0^\alpha + A_t^\alpha + \int_0^t \sigma_s dW_s^\alpha, \quad p_0^\alpha \in \mathbb{R}, \quad \mathbb{P}^\alpha\text{-a.s., } \forall t \in [0, T],
\]

where \( W^\alpha \) is a Brownian motion under \( \mathbb{P}^\alpha \), and \( A^\alpha \) is a process of finite variation. We assume that \( A^\alpha \) is absolutely continuous: i.e. for any \( \alpha \in \mathcal{A} \), there exists a locally integrable process \( \mu^\alpha \), such that:

\[
A_t^\alpha = \int_0^t \mu_s^\alpha ds, \quad \mathbb{P}^\alpha\text{-a.s., } \forall t \in [0, T].
\]

**Assumption 2.** For any \( \alpha \in \mathcal{A} \), the process \( \mu^\alpha \) is \( \mathbb{P}\)-a.s. right-continuous, and there exists a constant \( C > 0 \), such that \( |\mu_t^\alpha| \leq C \), for all \( t \in [0, T] \), \( \mathbb{P}\)-a.s.

Thus, we can rewrite the dynamics of \( \tilde{p}^0 \), under each \( \mathbb{P}^\alpha \), as follows: \( \mathbb{P}^\alpha\)-a.s., the following holds for all \( t \in [0, T] \):

\[
\tilde{p}_t^0 = p_0^\alpha + \int_0^t \mu_s^\alpha ds + \int_0^t \sigma_s dW_s^\alpha, \quad p_0^\alpha \in \mathbb{R}.
\]

\(^{12}\)In order to ensure the existence of regular conditional probabilities for the discrete time model, we can, for example, assume that \( \tilde{F}_T \) is generated by a random element with values in a standard Borel space.
There exists a modification of regular conditional probabilities

\[ \tilde{P}^\alpha_t = P^\alpha (\cdot | \tilde{F}_t) \]

such that it satisfies the tower property with respect to \( P \) (as described in Section 2.1), and, \( P^- \)-a.s., for all \( \alpha \in \mathcal{A} \) and all \( t \in [0, T] \), the future price process \( (\tilde{p}^\alpha_s)_{s \in [t, T]} \) satisfies (16), \( P^\alpha^-\)-a.s.

Throughout the rest of the paper, \( \tilde{P}^\alpha_t \) refers to a member of the family appearing in Assumption 3. All conditional expectations \( \tilde{E}_t^\alpha \) are taken under such \( \tilde{P}^\alpha_t \). Assumption 3 is satisfied, for example, if \( \tilde{p}^\alpha_t \sim P \), for all \( \alpha \in \mathcal{A} \), and, \( P^-\)-a.s., (16) holds for all \( t \in [0, T] \) and all \( \alpha \in \mathcal{A} \). The main results of this section require the additional continuity assumptions on \( \sigma \) and \( \mu^\alpha \). The following assumption can be viewed as a stronger version of \( L^2 \)-continuity of \( \sigma \).

**Assumption 4.** There exists a function \( \varepsilon(\cdot) \geq 0 \), such that \( \varepsilon(\Delta t) \to 0 \), as \( \Delta t \to 0 \), and, \( P^-\)-a.s.,

\[ \tilde{P}^\alpha_t \left( \mathbb{E}^\alpha \left( (\sigma_{s \wedge \tau} - \sigma_\tau)^2 | \mathcal{F}_\tau \right) \leq \varepsilon(\Delta t) \right) = 1 \]

holds for all \( t \in [0, T - \Delta t] \), all \( s \in [t, t + \Delta t] \), all stopping times \( t \leq \tau \leq s \), and all \( \alpha \in \mathcal{A} \).

The above assumption is satisfied, for example, if \( \sigma \) is an Itô process with bounded drift and diffusion coefficients. Next, we state a continuity assumption on the drift, which can be interpreted as a uniform right-continuity in probability of the martingale \( \tilde{E}_t^\alpha \mu^\alpha_s \).

**Assumption 5.** For any \( \alpha \in \mathcal{A} \), and any \( t \in [0, T) \), there exists a deterministic function \( \varepsilon(\cdot) \geq 0 \), such that \( \varepsilon(\Delta t) \to 0 \), as \( \Delta t \to 0 \), and, \( P^\alpha^-\)-a.s.,

\[ \tilde{P}^\alpha_t \left( \left| \int_t^T \left( \tilde{E}_s^\alpha \mu^\alpha_s - \tilde{E}_t^\alpha \mu^\alpha_s \right) ds \right| \geq \varepsilon(\Delta t) \right) \leq \varepsilon(\Delta t) \]

holds for all \( t \leq t' \leq t'' \leq t + \Delta t \leq T \).

The above assumption is satisfied, for example, if \( \tilde{E}_t^\alpha \mu^\alpha_s \) has an integral representation with respect to some \( P^\alpha^-\)-Brownian motion \( B^\alpha \),

\[ \tilde{E}_t^\alpha \mu^\alpha_s = \mu^\alpha_s + \int_t^s \beta_u^\alpha \cdot dB_u^\alpha, \]

and \( \beta_u^\alpha \) is absolutely bounded by a constant, for all \( (u, s) \). The latter condition is satisfied, for example, in any diffusion-based model – i.e. where \( \mu^\alpha_s = \mu^\alpha(t, Y_s) \), with a diffusion process \( Y \) – in which the generator of \( Y \) is strictly elliptic, its coefficients are Hölder continuous, and the derivative of \( \mu^\alpha(t, \cdot) \) is absolutely bounded, uniformly over \( t \). This observation follows from the Feynman-Kac formula and the standard Schauder estimates.

As in Section 3, we also consider a progressively measurable random field \( D \), s.t. \( P^-\)-a.s. the function \( \tilde{D}_t(\cdot) - \tilde{D}_s(\cdot) \) is strictly decreasing and vanishing at zero, for any \( 0 \leq s < t \leq T \). We assume that the curve of the incremental demand size, \( \tilde{D}_t(\cdot) - \tilde{D}_s(\cdot) \), cannot be “too flat”.

**Assumption 6.** There exists \( \varepsilon > 0 \), s.t. for any \( 0 \leq t - \varepsilon \leq s < t \leq T \), there exists a \( \tilde{F}_s \otimes B(\mathbb{R}) \)-measurable random function \( \kappa_s(\cdot) \), s.t., \( P^-\)-a.s., \( \kappa_s(\cdot) \) is strictly decreasing and \( |\tilde{D}_t(p) - \tilde{D}_s(p)| \geq |\kappa_s(p)| \), for all \( p \in \mathbb{R} \).
Finally, we introduce the empirical distribution process \((\tilde{\mu}_t)\), with values in the space of finite sigma-additive measures on \(\mathbb{S}\). The next assumption states that every \(\tilde{\mu}_t\) is dominated by a deterministic measure.

**Assumption 7.** For any \(t \in [0, T]\), there exists a finite sigma-additive measure \(\mu^0_t\) on \((\mathbb{S}, \mathbb{B}(\mathbb{S}))\), s.t., \(\mathbb{P}\)-a.s., \(\tilde{\mu}_t\) is absolutely continuous w.r.t. \(\mu^0_t\).

We partition the time interval \([0, T]\) into \(N\) subintervals of size \(\Delta t = T/N\). A discrete time model is obtained by discretizing the continuous time one:

\[
F_n = \tilde{F}_{n\Delta t}, \quad p^0_n = \tilde{p}^0_{n\Delta t}, \quad D_n(p) = (\tilde{D}_{n\Delta t} - \tilde{D}_{(n-1)\Delta t})(p - p^0_n), \quad \mu_n = \tilde{\mu}_{n\Delta t}.
\]

Before we present the main results, let us comment on the above assumptions. These assumptions are important from a technical point of view, however, some of them have economic interpretation that may provide (partial) intuitive explanations of the results that follow. In particular, Assumption 1 ensures that the fundamental price remains “noisy”, which implies that an agent can execute a limit order very quickly by posting it close to the present value of \(p^0\), if there are no other orders posted there. In combination with Assumption 6, the latter implies that, when the frequency, \(N\), is high, an agent has a lot of opportunities to execute her limit order at a price close to the fundamental price (at least, if no other orders are posted too close to the fundamental price).

Intuitively, this means that the agent’s execution value should improve as the frequency increases. Assumption 5 ensures that, if an agent has a signal about the direction of the fundamental price, this signal is persistent – i.e. it is continuous in the appropriate sense. When the trading frequency \(N\) is large, such persistency means that an agent has a large number of opportunities to exploit the signal, implying that she is in no rush to have her order executed immediately. The main results of this work, presented below, along with their proofs, confirm that these heuristic conclusions are, indeed, correct.

As mentioned in the preceding sections, our main goal is to analyze the liquidity effects of increasing the trading frequency. Therefore, we fix a limiting continuous time model, and consider a sequence of discrete time models, obtained from the limiting one as described above, for \(N \to \infty\) (this can be interpreted as observing the same population of agents, with continuous time beliefs about future demand, in various exchanges that allow different trading frequency). We begin with the following theorem, which shows that, if every market model in a given sequence admits a non-degenerate equilibrium, then, the terminal bid and ask prices converge to the fundamental price, as the trading frequency goes to infinity.

**Theorem 1.** Let Assumptions 1, 2, 3, 4, 6, 7 hold. Consider a family of uniform partitions of a given time interval \([0, T]\), with diameters \(\{\Delta t = T/N > 0\}\) and with the associated family of discrete time models, and denote the associated fundamental price process by \(p^{0,\Delta t}\). Assume that every such model admits a non-degenerate LTC equilibrium, and denote the associated bid and ask prices by \(p^{a,\Delta t}\) and \(p^{b,\Delta t}\) respectively. Then, there exists a deterministic function \(\varepsilon(\cdot)\), s.t. \(\varepsilon(\Delta t) \to 0\), as \(\Delta t \to 0\), and, for all small enough \(\Delta t > 0\), the following holds \(\mathbb{P}\)-a.s.:

\[
|p^{a,n\Delta t}_N - p^{0,n\Delta t}_N| + |p^{b,n\Delta t}_N - p^{0,n\Delta t}_N| \leq \varepsilon(\Delta t)
\]

The above theorem has a useful corollary, which can be interpreted as follows: if the market does not degenerate as the frequency increases, then, such an increase improves market efficiency. Here, we understand the “improving efficiency” in the sense that the expected execution price (i.e. the price per share that an agent expects to receive or pay by the end of the game) of every agent converges to the fundamental price.

**Corollary 1.** Under the assumptions of Theorem 1, denote the support of every equilibrium by \(\lambda^{a,\Delta t}\) and the associated expected execution prices by \(\lambda^{a,n\Delta t}_n\) and \(\lambda^{b,n\Delta t}_n\). Then, there exists a deterministic function \(\varepsilon(\cdot)\), such that \(\varepsilon(\Delta t) \to 0\), as \(\Delta t \to 0\), and, \(\mathbb{P}\)-a.s.,

\[
\sup_{n=0,\ldots,N, \alpha \in \lambda^{a,\Delta t}} (|\lambda^{a,n\Delta t}_n(\alpha) - p^{0,n\Delta t}_n| + |\lambda^{b,n\Delta t}_n(\alpha) - p^{0,n\Delta t}_n|) \leq \varepsilon(\Delta t),
\]
for all small enough \( \Delta t > 0 \).

**Proof:** Denote \( \mathbb{E}_n^\alpha = \mathbb{E}_n^{b_n}\Delta t \). It follows from Corollary 2 and the definition of LTC equilibrium that \( \lambda_{n,\Delta t}^{a_n}(\alpha) = P_{N}^{a_n,\Delta t} \) and \( \lambda_{n,\Delta t}^{b_n}(\alpha) = P_{N}^{b_n,\Delta t} \). It also follows from Corollary 2 (or, more generally, from the definition of a value function) that \( \lambda_{n,\Delta t}^a(\alpha) \) is a supermartingale, and \( \lambda_{n,\Delta t}^b(\alpha) \) is a submartingale, under \( \mathbb{P}^\alpha \). Thus, we have: \( \lambda_{n,\Delta t}^a(\alpha) \geq \mathbb{E}_n^\alpha P_{N}^{a_n,\Delta t} \) and \( \lambda_{n,\Delta t}^b(\alpha) \leq \mathbb{E}_n^\alpha P_{N}^{a_n,\Delta t} \). On the other hand, notice that we must have: \( \lambda_{n,\Delta t}^a(\alpha) \leq \mathbb{E}_n^\alpha P_{N}^{b_n,\Delta t} \) and \( \lambda_{n,\Delta t}^b(\alpha) \geq \mathbb{E}_n^\alpha P_{N}^{b_n,\Delta t} \). Assume, for example, that \( \lambda_{n,\Delta t}^a(\alpha) > \mathbb{E}_n^\alpha P_{N}^{a_n,\Delta t} \) on the event \( \Omega \) of positive \( \mathbb{P}^\alpha \)-probability. Consider an agent at state \((0, \alpha)\), who follows the optimal strategy of an agent at state \( (1, \alpha) \), starting from time \( n \) and onward, on the event \( \Omega' \) (otherwise, she does not do anything). It is easy to see that the objective value of this strategy is

\[ \mathbb{E}^\alpha 1_{\Omega'} \left( \lambda_{n,\Delta t}^a(\alpha) - \mathbb{E}_n^\alpha P_{N}^{a_n,\Delta t} \right) > 0, \]

which contradicts Corollary 2. The second inequality is shown similarly. Thus, we conclude that, for any \( n = 0, \ldots, N - 1 \), both \( \lambda_{n,\Delta t}^a(\alpha) \) and \( \lambda_{n,\Delta t}^b(\alpha) \) belong to the interval

\[ \left[ \mathbb{E}_n^\alpha P_{N}^{a_n,\Delta t}, \mathbb{E}_n^\alpha P_{N}^{b_n,\Delta t} \right], \]

which, in turn, converges to zero, as \( \Delta t \to 0 \), due to the deterministic bounds obtained in the proof of Proposition 1.

The above results demonstrate the positive role of high trading frequency. However, they are based on the assumption that the market does not degenerate as frequency increases. In the context of Section 3, we saw that the markets do not degenerate only if the agents are market-neutral (i.e. \( \alpha = 0 \)). If this condition is violated and the frequency \( N \) is sufficiently high, the market admits no non-degenerate equilibrium (i.e. there exists no safe regime, in which the liquidity crisis would never occur). It turns out that this conclusion still holds in the general setting considered herein.

**Theorem 2.** Let Assumptions 1, 2, 3, 4, 5, 6, 7 hold. Consider a family of uniform partitions of a given time interval \([0, T]\), with diameters \( \{\Delta t = T/N > 0\} \), containing arbitrarily small \( \Delta t \), and with the associated family of discrete time models. Assume that every such model admits a non-degenerate LTC equilibrium, with the same support \( \tilde{A} \). Then, for all \( \alpha \in \tilde{A} \), we have: \( \tilde{p}^0 \) is a martingale under \( \mathbb{P}^\alpha \).

The above theorem shows that the market degenerates even if the signal \( \mu^\alpha \) is very small (but non-zero), provided the trading frequency \( N \) is large enough. Therefore, as discussed at the end of Section 3, such degeneracy cannot be attributed to any fundamental reasons, and we refer to it as the internal (or, self-inflicted) liquidity crisis. The proof of Theorem 2 also reveals what causes the degeneracy. It turns out that, similar to Section 3, the higher trading frequency amplifies the adverse selection effect, forcing the agents to withdraw liquidity from the market. Note that, in the present setting, the agents may have different beliefs, the LOB may have a complicated shape and dynamics, and the expected execution prices are no longer deterministic. All this makes it difficult to provide a simple description of how the high frequency amplifies the adverse selection. Nevertheless, the proof is still based on the idea discussed at the end of Section 3: it has to do with how fast \( \mathbb{E}_n^\alpha (p_{n+1}^0 - p | p_{n+1}^0 > p) \) vanishes (as the frequency increases), relative to the rate at which the expected execution prices approach the fundamental price.

5 **Conditional tails of the marginal distributions of Itô processes**

As follows form the discussion in the preceding sections, in order to prove the main results of the paper, we need to investigate the properties of marginal distributions of the fundamental price \( \tilde{p}^0 \) (more precisely, the
distributions of its increments). In order to prove Theorem 1, we need to show that the difference between the fundamental price and the bid or ask price converges to zero, as the frequency $N$ increases to infinity. It turns out that, for this purpose, it suffices to show that the distribution of a normalized increment of $\tilde{p}^\alpha_0$ converges to the standard normal distribution. The following lemma summarizes these results. It is rather simple, but technical, hence, its proof is postponed to Appendix B. In order to formulate the result (and to facilitate the derivations in subsequent sections), we introduce additional notation. For convenience, we drop the superscript $\Delta t$ in many variables which do, in fact, depend on $\Delta t$, hoping it causes no confusion (we emphasize this dependence whenever it is important). For any market model on the time interval $[0, T]$, associated with a uniform partition with diameter $\Delta t = T/N > 0$, and having a fundamental price process $p^\alpha_0$, we define

$$\xi_n = p^0_n - p^0_{n-1} = \tilde{p}^0_n - \tilde{p}^0_{n-1}, \quad \mathbb{E}_n^\alpha = \tilde{\mathbb{E}}_n^\alpha, \quad \mathbb{P}_n^\alpha = \tilde{\mathbb{P}}_n^\alpha, \quad t_n = n\Delta t, \quad n = 1, \ldots, NT/\Delta t. \quad (17)$$

We denote by $\eta_0$ a standard normal random variable (on a, possibly, extended probability space), which is independent of $\mathcal{F}_N$ under every $\mathbb{P}^\alpha$.

**Lemma 1.** Let Assumptions 1, 2, 3, 4 hold. Then, there exists a function $\varepsilon(\cdot) \geq 0$, s.t. $\varepsilon(\Delta t) \to 0$, as $\Delta t \to 0$, and the following holds $\mathbb{P}$-a.s., for all $p \in \mathbb{R}$, all $\alpha \in \mathcal{A}$, and all $n = 1, \ldots, N$:

1. $(|p| \vee 1) \left| \mathbb{P}^\alpha_{n-1} \left( \frac{\xi_n}{\sqrt{\Delta t}} > p \right) \right| - \mathbb{P}^\alpha_{n-1} (\sigma_{t_n-1} \eta_0 > p) \leq \varepsilon(\Delta t)$
2. $\left| \mathbb{P}^\alpha_{n-1} \left[ \frac{\xi_n}{\sqrt{\Delta t}} \mathbf{1}_{\left\{ \frac{\xi_n}{\sqrt{\Delta t}} > p \right\}} \right] - \mathbb{E}^\alpha_{n-1} \left[ \sigma_{t_n-1} \eta_0 \mathbf{1}_{\left\{ \sigma_{t_n-1} \eta_0 > p \right\}} \right] \right| \leq \varepsilon(\Delta t)$

In addition, the above estimates hold if we replace $(\tilde{\xi}_n, \eta_0, p)$ by $(-\tilde{\xi}_n, -\eta_0, -p)$.

In order to prove Theorem 2 we need to compare the rates at which the conditional expectations $\mathbb{E}^\alpha_n (p^0_{n+1} - p^0_n | p^0_{n+1} > p)$ vanish (as the frequency $N$ goes to infinity) to the rate at which the expected execution prices converge to the fundamental price. This requires a more delicate analysis – in particular, the mere proximity of the distribution of a (normalized) fundamental price increment to the Gaussian distribution is no longer sufficient. In fact, what we need is a precise uniform estimate of the conditional tail of the distribution of a fundamental price increment. The desired property is formulated in the following lemma, which, we believe, is valuable in its own right. This result allows us to estimate the tails of the conditional marginal distribution of an Itô process $X$ uniformly by an exponential. To the best of our knowledge, this result is new. The main difficulties in establishing the desired estimates are: (a) the fact that we estimate the conditional, as opposed to the regular, tail, and (b) the fact that the estimates need to be uniform over the values of the argument. Note that, even in the case of a diffusion process $X$, the classical Gaussian-type bounds for the tails of the marginal distributions of $X$ are not sufficient to establish the desired estimates. The reason is that, in general, the Gaussian estimates of the regular tails from above and from below have different orders of decay, for the large values of the argument, which makes them useless for estimating the conditional tail (which is a ratio of two regular tails).

**Lemma 2.** Consider the following continuous semimartingale on a stochastic basis $(\hat{\Omega}, (\hat{\mathcal{F}}_t)_{t \in [0, 1]}, \hat{\mathbb{P}})$:

$$X_t = \int_0^t \hat{\mu}_u du + \int_0^t \hat{\sigma}_u dB_u, \quad t \in [0, 1],$$

where $B$ is a Brownian motion (with respect to the given stochastic basis), $\hat{\mu}$ and $\hat{\sigma}$ are progressively measurable processes, such that the above integrals are well defined. Assume that, for any stopping time $\tau$ with values in $[0, 1]$, $c \leq |\hat{\sigma}_\tau| \leq C$ holds a.s. with some constants $c, C > 0$. Then, there exists $\varepsilon > 0$, depending only on $(c, C)$, s.t., if

$$\hat{\mu}_\tau^2 \leq \varepsilon, \quad \hat{\mathbb{E}} \left( (\hat{\sigma}_{\sqrt{\tau}} - \hat{\sigma}_\tau)^2 \left| \hat{\mathcal{F}}_\tau \right) \right) \leq \varepsilon \ a.s.,$$

20
for all \( s \in [0,1] \) and all stopping time \( \tau \), with values in \([0,1]\), then, for any \( c_1 > 0 \), there exists \( C_1 > 0 \), depending only on \((c,C,\varepsilon,c_1)\), s.t. the following holds:

\[
\hat{P}(X_1 > x + z \mid X_1 > x) \leq C_1 e^{-c_1 z}, \quad \forall x, z \geq 0.
\]

**Proof:** In the course of this proof, we will use the shorthand notation, \( \hat{E} \) and \( \hat{P} \), to denote the conditional expectation and the conditional probability w.r.t \( \hat{F}_\tau \). We also denote

\[
A_t = \int_0^t \hat{\mu}_u du, \quad G_t = \int_0^t \hat{\sigma}_u dB_u.
\]

For any \( x \geq 0 \), let us introduce \( \tau_x = 1 \wedge \inf \{ t \in [0,1] : X_t = x \} \). Then

\[
\hat{P}(X_1 > x + z) \leq \hat{P}\left( \sup_{t \in [0,1]} X_t > x + z \right) = \hat{E}\left( 1_{\{\tau_1 < 1\}} \hat{P}_{\tau_x} \left( \sup_{s \in [\tau_1,1]} (X_s - x) > z \right) \right)
\]

Notice that, on \( \{ \tau_x \leq s \} \), we have: \( X_s - x = A_s \vee \tau_x - A_x + G_s \vee \tau_x - G_{\tau_x} \). In addition, the process \( (Y)_s \in [0,1] \), with \( Y_s = A_s \vee \tau_x - A_{\tau_x} \), is adapted to the filtration \( (\hat{F}_{\tau_x}) \), while the process \( (Y)_s \in [0,1] \), with \( Z_s = G_s \vee \tau_x - G_{\tau_x} \), is a martingale with respect to it. Next, we observe:

\[
\hat{P}_{\tau_x} \left( \sup_{s \in [\tau_x,1]} (X_s - x) > z \right) = \hat{P}_{\tau_x} \left( \sup_{s \in [0,1]} (Y_s + Z_s) > z \right)
\]

\[
\leq \hat{P}_{\tau_x} \left( \sup_{s \in [0,1]} \exp \left( c_1 Z_s - \frac{1}{2} c_1^2 \langle Z \rangle_s \right) > \exp \left( c_1 z - c_1 \sqrt{\varepsilon} - \frac{1}{2} c_1^2 C^2 \right) \right),
\]

where we used the fact that \( \langle Z \rangle_s \leq \langle X \rangle_s \leq C^2 \), for all \( s \in [0,1] \). Using the Novikov’s condition, it is easy to check that

\[
M_s = \exp \left( c_1 Z_s - \frac{1}{2} c_1^2 \langle Z \rangle_s \right), \quad s \in [0,1],
\]

is a true martingale, and, hence, we can apply the Doob’s martingale inequality:

\[
\hat{P}_{\tau_x} \left( \sup_{s \in [0,1]} \exp \left( c_1 Z_s - \frac{1}{2} c_1^2 \langle Z \rangle_s \right) > \exp \left( c_1 z - c_1 \sqrt{\varepsilon} - \frac{1}{2} c_1^2 C^2 \right) \right) \leq \exp \left( -c_1 z + c_1 \sqrt{\varepsilon} + \frac{1}{2} c_1^2 C^2 \right).
\]

Collecting the above inequalities, we obtain

\[
\hat{P}(X_1 > x + z) \leq \hat{P}\left( \sup_{t \in [0,1]} X_t > x + z \right) \leq C_2(\varepsilon) e^{-c_1 z} \hat{P}(\tau_x < 1) = C_2(\varepsilon) e^{-c_1 z} \hat{P}\left( \sup_{t \in [0,1]} X_t > x \right).
\]

(18)

The next step is to estimate the distribution tails of a running maximum via the tails of the distribution of \( X_1 \). To do this, we proceed as before:

\[
\hat{P}(X_1 > x) = \hat{E}\left( 1_{\{\tau_1 < 1\}} \hat{P}_{\tau_x} (Y_1 + Z_1 > 0) \right),
\]

(19)

with \( Y \) and \( Z \) defined above. Notice that

\[
\hat{P}_{\tau_x} (Y_1 + Z_1 > 0) = \hat{P}_{\tau_x}\left( \hat{\sigma}_{\tau_x} \frac{B_1 - B_{\tau_x}}{\sqrt{1 - \tau_x}} + \frac{1}{\sqrt{1 - \tau_x}} \int_{\tau_x}^1 \hat{\mu}_u du + \frac{1}{\sqrt{1 - \tau_x}} \int_{0}^1 (\hat{\sigma}_u \vee \tau_x - \hat{\sigma}_{\tau_x}) dB_u > 0 \right)
\]
where \( B_s^r = B_{s \lor \tau_\nu} \) is a continuous square-integrable martingale with respect to \((\hat{\mathcal{F}}_{s \lor \tau_\nu})\). Denote
\[
R_s = \int_0^s (\hat{\sigma}_{u \lor \tau_\nu} - \hat{\sigma}_{\tau_\nu}) dB_u^r, \quad s \in [0, 1],
\]
and notice that it is a square-integrable martingale with respect to \((\hat{\mathcal{F}}_{s \lor \tau_\nu})\). Then, on \( \{ \tau_\nu < 1 \} \) (possibly, without a set of measure zero), we have:
\[
\hat{\mathbb{E}}_{\tau_\nu} \left( \frac{1}{\sqrt{1 - \tau_\nu}} R_1 \right)^2 = \frac{1}{1 - \tau_\nu} \hat{\mathbb{E}}_{\tau_\nu} R_1^2 \leq \frac{1}{1 - \tau_\nu} \int_{\tau_\nu}^1 \hat{\mathbb{E}}_{\tau_\nu} (\hat{\sigma}_{u \lor \tau_\nu} - \hat{\sigma}_{\tau_\nu})^2 du \leq \varepsilon.
\]
In addition,
\[
\hat{\mathbb{E}}_{\tau_\nu} \left( \frac{1}{\sqrt{1 - \tau_\nu}} \int_{\tau_\nu}^1 \hat{\mu}_u du \right)^2 \leq \varepsilon.
\]
Collecting the above and using Chebyshev’s inequality, we obtain
\[
\left| \hat{\mathbb{P}}_{\tau_\nu} (Y_1 + Z_1 > 0) - \hat{\mathbb{P}}_{\tau_\nu} \left( \hat{\sigma}_{\tau_\nu} \frac{B_1 - B_{\tau_\nu}}{\sqrt{1 - \tau_\nu}} \leq -\varepsilon^{1/3} \right) \right| \leq 2\varepsilon^{1/6}.
\]
On the other hand, due to the strong Markov property of Brownian motion, on \( \{ \tau_\nu < 1 \} \), we have, a.s.:
\[
\hat{\mathbb{P}}_{\tau_\nu} \left( \hat{\sigma}_{\tau_\nu} \frac{B_1 - B_{\tau_\nu}}{\sqrt{1 - \tau_\nu}} \leq -\varepsilon^{1/3} \right) = \hat{\mathbb{P}} \left( \xi \leq -\varepsilon^{1/3} \right) \bigg|_{\sigma = \hat{\sigma}_{\tau_\nu}},
\]
where \( \xi \) is a standard normal. As \( \hat{\sigma}_{\tau_\nu} \in [c, C] \), we conclude that the right hand side of the above converges to 1/2, as \( \varepsilon \to 0 \), uniformly over almost all random outcomes in \( \{ \tau_\nu < 1 \} \). In particular, for all small enough \( \varepsilon > 0 \), we have:
\[
\mathbf{1}_{\{\tau_\nu < 1\}} \left| \hat{\mathbb{P}}_{\tau_\nu} (Y_1 + Z_1 \leq 0) - \hat{\mathbb{P}}_{\tau_\nu} (Y_1 + Z_1 > 0) \right| \leq \mathbf{1}_{\{\tau_\nu < 1\}} \delta(\varepsilon) < 1,
\]
and, in view of (19),
\[
\hat{\mathbb{P}}(X_1 > x) \geq \hat{\mathbb{E}} \left( \mathbf{1}_{\{\tau_\nu < 1\}} \hat{\mathbb{P}}_{\tau_\nu} (Y_1 + Z_1 \leq 0) \right) - \delta(\varepsilon)\hat{\mathbb{P}}(\tau_\nu < 1)
\]
Summing up the above inequality and (19), we obtain
\[
2\hat{\mathbb{P}}(X_1 > x) \geq (1 - \delta(\varepsilon))\hat{\mathbb{P}}(\tau_\nu < 1) = (1 - \delta(\varepsilon))\hat{\mathbb{P}}\left( \sup_{t \in [0, 1]} X_t > x \right),
\]
which, along with (18), yields the statement of the lemma.

6 Proof of Theorem 1

Within the scope of this proof, we adopt the notation introduced in (17) and use the following convention.

**Notational Convention 1.** The LOB, the bid and ask prices, the expected execution prices, and the demand, are all measured relative to \( p_0 \). Namely, we use \( \nu_n \) to denote \( \nu_n \circ (x \mapsto x + p_0^0)^{-1} \), \( p_n^b \) to denote \( p_n - p_0^b \), \( p_n^b \) to denote \( p_n^b - p_0^0 \), \( \lambda_n^b \) to denote \( \lambda_n^b - p_0^0 \), \( \lambda_n^0 \) to denote \( \lambda_n^0 - p_0^0 \), and \( D_n(p) \) to denote \( D_n(p_0^0 + p) \).
Herein, we are only concerned with what happens in the last trading period – at time \((N - 1)\), where \(N = T/\Delta t\). Hence, we omit the subscript \(N - 1\) whenever it is clear from the context. In particular, we write \(p^a\) and \(p^b\) for \(p^a_{N-1}\) and \(p^b_{N-1}\), \(\nu\) for \(\nu_{N-1}\), and \(\xi\) for \(\xi_N\). Note also that, in an LTC equilibrium, we have: \(p^a = p^a_N = p^b_{N-1}\), with similar equalities for \(p^b\) and \(\nu\). For convenience, we also drop the superscript \(\Delta t\) in the LOB and the associated bid and ask prices. Finally, we denote by \(\hat{\mu}\) the support of a given equilibrium, and by \(\mu\) the empirical measure at time \(N - 1\). As the roles of \(p^a\) and \(p^b\) in our model are symmetric, we will only prove the statement of the proposition for \(p^b\). We are going to show that, under the assumptions of the theorem, there exists a constant \(C_0 > 0\), depending only on the constant \(C\) in Assumptions 1 and 2, such that, for all small enough \(\Delta t\), we have, \(\mathbb{P}\)-a.s.:

\[-C_0 \leq p^b/\sqrt{\Delta t} < 0\]  \hspace{1cm} (20)

First, we introduce \(\hat{A}^\alpha(p; x)\), which we refer to as the simplified objective:

\[\hat{A}^\alpha(p; x) = \mathbb{E}_{N-1}^\alpha \left[ (p - x - \xi) 1_{\{\xi > p\}} \right].\]  \hspace{1cm} (21)

Recall that the expected relative profit from posting a limit sell order at price level \(p\) in the last time period,\(^{13}\) is given by \(A^\alpha(p; p^b_N)\), where

\[A^\alpha(p; x) = \mathbb{E}_{N-1}^\alpha \left[ (p - x - \xi) 1_{\{D^+(p - \xi_N) > \nu^+((-\infty, p))\}} \right].\]  \hspace{1cm} (22)

The simplified objective is similar to \(A^\alpha\), but it assumes that there are no orders posted at better prices than the one posted by the agent. In particular, \(\hat{A}^\alpha(p; x) = A^\alpha(p; x)\) for \(p \leq p^a\). Recall Corollary 2, which states that, in equilibrium, \(\mathbb{P}\)-a.s., if the agents in the state \((s, \alpha)\) post limit sell orders, then they post them at a price level \(p\) that maximizes the true objective \(A^\alpha(p; p^b)\). The following lemma shows that the value of the modified objective becomes close to the value of the true objective, for the agents posting limit sell orders close to the ask price.

**Lemma 3.** \(\mathbb{P}\)-a.s., either \(\nu^+(\{p^a\}) > 0\) or we have:

\[\left| A^\alpha(p; p^b) - \hat{A}^\alpha(p^a; p^b) \right| \to 0,\]

as \(p \downarrow p^a\), uniformly over all \(\alpha \in \hat{\alpha}\).

**Proof:** If \(\nu^+(\{p^a\}) = 0\), then \(\nu^+((-\infty, p]) \to 0\), as \(p \downarrow p^a\). Then, we have

\[\left| A^\alpha(p; p^b) - \hat{A}^\alpha(p^a; p^b) \right| = \left| \mathbb{E}_{N-1}^\alpha (p - p^b - \xi_N) 1_{\{D^+(p - \xi_N) > \nu^+((-\infty, p))\}} - \mathbb{E}_{N-1}^\alpha (p^a - p^b - \xi_N) 1_{\{\xi_N > p^a\}} \right| \leq |p - p^a| + \|p^a - p^b - \xi_N\|_{L^2(p^a_{N-1})} \mathbb{E}_{N-1}^\alpha [\xi_N > p^a, D^+(p - \xi_N) \leq \nu^+((-\infty, p))]\]

Thus, it suffices to show that: (i) \(\|p^a - p^b - \xi_N\|_{L^2(p^a_{N-1})}\) is bounded by a finite random variable independent of \(\alpha\), and (ii) \(\mathbb{P}_{N-1}^\alpha [\xi_N > p^a, D^+(p - \xi_N) \leq \nu^+((-\infty, p))] \to 0\), \(\mathbb{P}\)-a.s., as \(p \downarrow p^a\), uniformly over \(\alpha\). For (i), we have:

\[\|p^a - p^b - \xi\|_{L^2(p^a_{N-1})} \leq |p^a - p^b| + \|\xi\|_{L^2(p^a_{N-1})} \leq |p^a - p^b| + 2C\sqrt{\Delta t},\]

\(^{13}\)Recall that everything is measured relative to the fundamental price, according to the Notational Convention 1
where the constant $C$ appears in Assumptions 1 and 2. For (ii), we note that
\[
\{\xi_N > p^a, D^+(p - \xi_N) \leq \nu^+((-\infty,p))\} = \{\xi_N > p^a, \xi_N \leq p - D^{-1}(\nu^+((-\infty,p)))\},
\]
as $D(\cdot)$ is strictly decreasing, with $D(0) = 0$. Assumption 6 implies that
\[
\kappa^{-1}(\nu^+((-\infty,p))) \leq D^{-1}(\nu^+((-\infty,p))) < 0,
\]
where $\kappa$ is known at time $N - 1$. Therefore,
\[
\mathbb{P}_{N-1}^{\alpha} \left[ \xi_N > p^a, D^+(p - \xi_N) \leq \nu^+((-\infty,p)) \right] \leq \mathbb{P}_{N-1}^{\alpha} \left( \xi_N \in (p^a, p - \kappa^{-1}(\nu^+((-\infty,p)))) \right)
\]
It remains to show that, $\mathbb{P}$-a.s., the right hand side of the above converges to zero, uniformly over all $\alpha$. Assume that it does not hold. Then, with positive probability $\mathbb{P}$, there exists $\varepsilon > 0$ and a sequence of $(p_k, \alpha_k)$, such that $p_k \downarrow p^a$ and
\[
\mathbb{P}_{N-1}^{\alpha_k} \left( \xi_N \in (p^a, p_k - \kappa^{-1}(\nu^+((-\infty,p))) \right) \geq \varepsilon
\]
Notice that, $\mathbb{P}$-a.s., the family of measures $\{\hat{\mu}_k = \mathbb{P}_{N-1}^{\alpha_k} \circ \xi_N^{-1}\}$ is tight. The latter follows, for example, from the fact that, $\mathbb{P}$-a.s., the conditional second moments of $\xi_N$ are bounded uniformly over all $\alpha$ (which, in turn, is a standard exercise in stochastic calculus). Prokhorov’s theorem, then, implies that there is a subsequence of these measures that converges weakly to some measure $\hat{\mu}$ on $\mathbb{R}$. Next, notice that, for any fixed $k$ in the chosen subsequence, there exists a large enough $k'$, such that
\[
|\mu \left( (p^a, p_k - \kappa^{-1}(\nu^+((-\infty,p))) \right) - \mu_{k'} \left( (p^a, p_k - \kappa^{-1}(\nu^+((-\infty,p))) \right)| \leq \varepsilon/2
\]
Thus, for any $k$ in the subsequence, we have
\[
\mu \left( (p^a, p_k - \kappa^{-1}(\nu^+((-\infty,p))) \right) \geq \varepsilon/2
\]
The above is a contradiction, as the intersection of the corresponding intervals, $(p^a, p_k - \kappa^{-1}(\nu^+((-\infty,p))))$, over all $k$ is empty. ■

Now we are ready to prove the upper bound in (20).

**Lemma 4.** In any non-degenerate LTC equilibrium, $p^b < 0 < p^a$, $\mathbb{P}$-a.s.

**Proof:** We only show that $p^b < 0$ hold, the other inequality being very similar. Assume that $p^b \geq 0$ on some positive $\mathbb{P}$-probability set $\Omega' \in \mathcal{F}_{N-1}$. We are going to show that this results in a contradiction. First, Corollary 2 implies that, $\mathbb{P}$-a.s., if the agents in state $(s, \alpha)$ post a limit sell order, then we must have: $\sup_{p \in \mathbb{R}} A^\alpha(p; p^b) \geq 0$.

In addition, on $\Omega'$, we have: $A^\alpha(p^a; p^b) < 0$ for all $\alpha \in \tilde{A}$, as $\xi_N$ has full support in $\mathbb{R}$ under every $\mathbb{P}_{N-1}^{\alpha} \mathbb{P}$-a.s., that $p$ is bounded uniformly away from zero. Then, Lemma 3 implies that there exists a $\mathcal{F}_{N-1}$-measurable $\bar{p} \geq p^a$, such that, on $\Omega'$, the following holds a.s.: if $\nu^+\left(\{p^a\}\right) = 0$ then $\bar{p} > p^a$, and, in all cases,
\[
A^\alpha(p; p^b) < 0, \quad \forall p \in [p^a, \bar{p}], \quad \forall \alpha \in \tilde{A}.
\]
Clearly, it is suboptimal for an agent to post a limit sell order below $\bar{p}$. However, an agent’s strategy only needs to be optimal up to a set of $\mathbb{P}$-measure zero, and these sets can be different for different $(s, \alpha)$. Therefore, a little more work is required to obtain the desired contradiction. Consider the set $B \subset \Omega' \times \mathbb{R} \times \tilde{A}$:
\[
B = \left\{ (\omega, s, \alpha) \mid \bar{q}(s, \alpha) > 0, \quad \bar{p}(s, \alpha) \leq \bar{p} \right\}.
\]
This set is measurable with respect to $\mathcal{F}_{N-1} \otimes \mathcal{B} \left( \mathbb{R} \times \hat{\mathcal{A}} \right)$, due to the measurability properties of $\hat{q}$ and $\hat{p}$.

Notice that, due to the above discussion and the optimality of agents’ actions (cf. Corollary 2), for any $\rho$ where

$$\rho \in \text{argmin} \{ \mathbb{E}\omega \mathbb{P}(\{\omega \mid (\omega, s, \alpha) \in B\}) = 0,$$

and hence

$$\mathbb{E}_{N-1} \int_{\mathbb{R} \times \hat{\mathcal{A}}} 1_B(\omega, s, \alpha)\mu_{N-1}(ds, d\alpha) = \int_{\mathbb{R} \times \hat{\mathcal{A}}} \mathbb{E}_{N-1} (1_B(\omega, s, \alpha)\rho_{N-1}(\omega, s, \alpha))\mu_{N-1}^0(ds, d\alpha) = 0,$$

where $\rho_{N-1}$ is the Radon-Nikodym density of $\mu_{N-1}$ w.r.t. to the deterministic measure $\mu_{N-1}^0$ (cf. Assumption 7). The above implies that, $\mathbb{P}_{N-1}$-a.s., $1_B(\omega, s, \alpha)\rho_{N-1}(\omega, s, \alpha) = 0$, for $\mu_{N-1}^0$-a.e. $(s, \alpha)$. Notice also that, for all $(\omega, s, \alpha) \in \Omega' \times \mathbb{R} \times \hat{\mathcal{A}}$,

$$1_{\{\hat{p}(s, \alpha) \leq \hat{p}\}} \hat{q}^+(s, \alpha)1_B = 0.$$

From the above observations and the condition (4) in the definition of equilibrium (cf. Definition 4), we conclude that, on $\Omega'$, the following holds a.s.:

$$\nu_{N-1}^+(\{p^a, \hat{p}\}) = 0,$$

where $\hat{p} \geq p^a$, and, if $\nu^+(\{p^a\}) = 0$, then $\hat{p} > p^a$. This contradicts the definition of $p^a$ (recall that $p^a$ is $\mathbb{P}$-a.s. finite, due to non-degeneracy of the LOB).

It only remains to prove the lower bound on $p^b$ in (20). Assume that it does not hold. That is, assume that there exists a family of equilibria, with arbitrary small $\Delta t$, and positive $\mathbb{P}$-probability $\mathcal{F}_{N-1}$-measurable sets $\Omega^{\Delta t}$, such that $p^b < -C_0\sqrt{\Delta t}$ on $\Omega^{\Delta t}$. We are going to show that this leads to a contradiction with $p^a > 0$. To this end, assume that the agents maximize the simplified objective function, $\hat{A}^a$, instead of the true one, $A^a$. Then, it turns out that, if $p^b$ is negative enough, the optimal price levels become negative for all $\alpha$. The precise formulation of this is given by the following lemma.

**Lemma 5.** There exists a constant $C_0 > 0$, s.t., for any small enough $\Delta t$, there exist constants $\epsilon, \delta > 0$, s.t., $\mathbb{P}$-a.s., we have:

$$\hat{A}^a(-\delta; x) \geq \epsilon + \sup_{y \geq 0} \hat{A}^a(y; x),$$

for all $\alpha \in \hat{\mathcal{A}}$ and all $x \leq -C_0\sqrt{\Delta t}$.

**Proof:** Denote $\xi = \xi_N / \sqrt{\Delta t}$ and consider the random function

$$\hat{A}^a(p; x) = \mathbb{E}_{N-1}^a \left[ (p - x - \xi)1_{\{\xi > p\}} \right].$$

Notice that

$$\hat{A}^a(p; x) = \sqrt{\Delta t}\hat{A}^a \left( \frac{p}{\sqrt{\Delta t}}; x / \sqrt{\Delta t} \right),$$

and, hence, we can reformulate the statement of the lemma as follows: there exists a constant $C_0 > 0$, s.t., for any small enough $\Delta t$, there exist constants $\epsilon, \delta > 0$, s.t., $\mathbb{P}$-a.s., we have:

$$\hat{A}^a(-\delta; x) \geq \epsilon + \sup_{y \geq 0} \hat{A}^a(y; x),$$

for all $\alpha \in \hat{\mathcal{A}}$ and all $x \leq -C_0$. Notice that
\[ \tilde{A}^\alpha(-\delta; x) - \tilde{A}^\alpha(y; x) = -x E_N^{\alpha} \left[ 1_{\{-\delta < \xi \leq y\}} \right] - E_N^{\alpha} \left[ 1_{\{-\delta < \xi \leq y\}} \right] - E_N^{\alpha} \left[ 1_{\{\xi > -\delta\}} \right] - y E_N^{\alpha} \left[ 1_{\{\xi > y\}} \right] \]

is non-increasing in \( x \), and, hence, such is \( \tilde{A}^\alpha(-\delta; x) - \sup_{y \geq 0} \tilde{A}^\alpha(y; x) \). Hence, it suffices to prove the above statement for \( x = -C_0 \). Next, consider the deterministic function \( A_\sigma(p; x) \), defined via

\[ A_\sigma(p; x) = \mathbb{E} \left[ (p - x - \sigma \eta_0) 1_{\{\sigma \eta_0 > p\}} \right], \tag{24} \]

where \( \eta_0 \) is a standard normal random variable on some auxiliary probability space \( (\hat{\Omega}, \hat{\mathbb{P}}) \). It follows from Lemma 1 that there exists a function \( \varepsilon_2(\cdot) \geq 0 \), s.t. \( \varepsilon_2(\Delta t) \to 0 \), as \( \Delta t \to 0 \), and, \( \mathbb{P} \text{-a.s., we have:} \)

\[ \left| \tilde{A}^\alpha(p; -C_0) - A_{\sigma_{N-1}}(p; -C_0) \right| \leq \varepsilon_2(\Delta t), \]

for all \( \alpha \in \hat{\mathcal{A}} \) and all \( p \in \mathbb{R} \). Then, as we can always choose \( \Delta t \) small enough, so that \( \varepsilon_2(\Delta t) < \varepsilon \), the statements of the lemma would follow if we can show that there exist constants \( \varepsilon, \delta, C_0 > 0 \), s.t., \( \mathbb{P} \text{-a.s.,} \)

\[ A_{\sigma_{N-1}}(-\delta; -C_0) \geq 3\varepsilon + \sup_{y \geq 0} A_{\sigma_{N-1}}(y; -C_0) \]

As \( \sigma_{N-1}(\omega) \in [1/C, C] \), \( \mathbb{P} \text{-a.s., it suffices to find} \varepsilon, \delta, C_0 > 0 \), s.t.

\[ A_\sigma(-\delta; -C_0) \geq 3\varepsilon + \sup_{y \geq 0} A_\sigma(y; -C_0), \quad \forall \sigma \in [1/C, C]. \]

Note that the above inequality does not involve \( \omega \) or \( \xi \), and it is simply a property of a deterministic function. Notice also that \( A_\sigma(p; x) = \sigma A_1(p/\sigma; x/\sigma) \), with \( A_1 \) given in (24). Then, if we denote by \( F(x) \) and \( f(x) \), respectively, the cdf and pdf of a standard normal, we obtain:

\[ A_1(p; x) = (p - x)(1 - F(p)) - \int_p^\infty tf(t)dt. \]

A straightforward calculation gives us the following useful properties of \( A_1 \) and \( A_\sigma \):

(i) For any \( \sigma > 0 \) and any \( x < 0 \), the function \( p \mapsto A_\sigma(p; x) \) has a unique maximizer \( p_\sigma(x) \), in particular, it is increasing in \( p \leq p_\sigma(x) \) and decreasing in \( p \geq p_\sigma(x) \).

(ii) The function

\[ x \mapsto p_\sigma(x) = \sigma p_1(x/\sigma) = \sigma ((1 - F)/f)^{-1} (-x/\sigma) \]

is increasing in \( x < 0 \) and converges to \( -\infty \), as \( x \to -\infty \).

Then, choosing \( C_0 \) large enough, so that \( p_1(-C_0/C) < 0 \), ensures \( p_\sigma(-C_0) < 0 \), for all \( \sigma \in [1/C, C] \). Setting \( \delta = -p_1(-C_0/C)/C \) guarantees that \( p_\sigma(-C_0) \leq -\delta \), for all \( \sigma \in [1/C, C] \). Then, by property (i) above, we have, for all \( \sigma \in [1/C, C] \):

\[ A_\sigma(-\delta; -C_0) > A_\sigma(0; -C_0) = \sup_{y \geq 0} A_\sigma(y; -C_0). \]

Finally, as \( A_\sigma(-\delta; -C_0) - A_\sigma(0; -C_0) \) is a continuous function of \( \sigma \in [1/C, C] \), we can find \( \varepsilon \), such that

\[ A_\sigma(-\delta; -C_0) \geq 3\varepsilon + \sup_{y \geq 0} A_\sigma(y; -C_0), \quad \forall \sigma \in [1/C, C]. \]
Recall that our assumption is that \( p^b < -C_0 \sqrt{\Delta t} \) holds on a set \( \Omega^\Delta_t \) of positive \( \mathbb{P} \)-measure. Recall also that \( p^a > 0 \), \( \mathbb{P} \)-a.s., due to Lemma 4. Then, Lemmas 3 and 5 imply that there exists \( F_{n-1} \)-measurable \( \bar{p} \geq p^a \), s.t., on \( \Omega^\Delta_t \), we have a.s.: if \( \nu^+(\{p^a\}) = 0 \) then \( \bar{p} > p^a \), and, in all cases,

\[
A^\alpha(p; p^a) < \sup_{p' \in \mathbb{R}} A^\alpha(p'; p^a), \quad \forall p \in [p^a, \bar{p}], \forall \alpha \in \hat{\alpha}.
\]

It is intuitively clear that posting limit sell orders at the above price levels \( p \) must be suboptimal for the agents. However, the above inequality, on its own, does not yield a contradiction, as the agents’s strategies are only optimal up to a set of \( \mathbb{P} \)-probability zero, and these sets may be different for different states \((s, \alpha)\). To obtain a contradiction with the definition of \( p^a \), we simply repeat the last part of the proof of Lemma 4 (following equation (23)). This ensures that (20) holds and completes the proof of the theorem.

7 Proof of Theorem 2

Within the scope of this proof, we adopt the notation introduced in (17) and use Notational Convention 1 (i.e. we measure the LOB, the expected execution prices, and the demand, relative to \( p^a \), but keep the same variables’ names). Assume that the statement of the theorem does not hold: i.e. there exists \( \alpha_0 \in \hat{\alpha} \), such that \( \tilde{p}^0 \) is not a martingale under \( \mathbb{P}^{\alpha_0} \). Then, there exists \( s \in [0, T) \), s.t., with positive probability \( \mathbb{P}^{\alpha_0} \), we have:

\[
\tilde{\mathbb{E}}^{\alpha_0}_{s}[\tilde{p}^0_T] \neq \tilde{p}^0_s.
\]

Without loss of generality, we assume that there exists a constant \( \delta > 0 \) and a set \( \Omega' \subset \mathcal{F}_s \), having positive probability \( \mathbb{P}^{\alpha_0} \) (and hence \( \mathbb{P} \)), s.t., for all random outcomes in \( \Omega' \), we have:

\[
\tilde{\mathbb{E}}^{\alpha_0}_{s}[\tilde{p}^0_T - \tilde{p}^0_s] \geq \delta
\]

(25)

(the case of negative values is analogous). Next, we fix an arbitrary \( \Delta t \) from a given family and consider the associated non-degenerate LTC equilibrium.

Lemma 6. There exists a deterministic function \( \varepsilon(\cdot) \geq 0 \), s.t. \( \varepsilon(\Delta t) \to 0 \), as \( \Delta t \to 0 \), and, for any small enough \( \Delta t > 0 \), there exists \( n = 0, \ldots, N - 3 \) and \( \Omega'' \subset \mathcal{F}_n \), s.t. \( \mathbb{P}^{\alpha_0}_{n}(\Omega'') > 0 \) and the following holds on \( \Omega'' \):

\[
\mathbb{P}^{\alpha_0}_{n+2}(\mathbb{E}^{\alpha_0}_{n+3}(\tilde{p}^0_n - \tilde{p}^0_{n+3}) \leq \delta/2) \leq \varepsilon(\Delta t).
\]

Proof: The proof follows from Assumption 5. Consider \( t = t' = s \) and \( t'' = t_{n+2} \). Then, Assumption 5 implies

\[
\tilde{\mathbb{E}}^{\alpha_0}_{s}\left(\mathbb{E}^{\alpha_0}_{t_{n+2}} \int_s^T \mu^\alpha_u du - \tilde{\mathbb{E}}^{\alpha_0}_{s} \int_s^T \mu^\alpha_u du \right) \geq \varepsilon(\Delta t) \leq \varepsilon(\Delta t)
\]

on \( \Omega' \), a.s.. Notice also that

\[
\tilde{\mathbb{E}}^{\alpha_0}_{s} [\tilde{p}^0_T - \tilde{p}^0_s] = \tilde{\mathbb{E}}^{\alpha_0}_{s} \int_s^T \mu^\alpha_u du.
\]

Then, assuming that \( \varepsilon(\Delta t) \) is small enough and recalling (25), we obtain

\[
\tilde{\mathbb{E}}^{\alpha_0}_{s}\left(\mathbb{E}^{\alpha_0}_{t_{n+2}} \int_s^T \mu^\alpha_u du \leq 3\delta/4 \right) \leq \varepsilon(\Delta t),
\]

27
on $\Omega'$. Therefore, there exists a set $\Omega'' \in \mathcal{F}_s \subset \mathcal{F}_{t_n}$, s.t. $\hat{P}^\alpha_{t_n}(\Omega'') > 0$ and

$$\hat{P}^\alpha_{t_n+2} \int_s^T \mu_u^\alpha \, du \geq 3\delta/4,$$

on $\Omega''$. Next, we choose $t = s$, $t' = t_{n+2}$, $t'' = t_{n+3}$, and use Assumption 5 to obtain:

$$\hat{P}^\alpha_{t_{n+2}} \left( \left| \hat{P}^\alpha_{t_{n+3}} \int_s^T \mu_u^\alpha \, du - \hat{P}^\alpha_{t_{n+2}} \int_s^T \mu_u^\alpha \, du \right| \geq \varepsilon(\Delta t) \right) \leq \varepsilon(\Delta t),$$

on $\Omega''$, a.s.. Assuming that $\varepsilon(\Delta t)$ is small enough and using the last two inequalities, we obtain

$$\hat{P}^\alpha_{t_{n+2}} \left( \hat{P}^\alpha_{t_{n+3}} \int_s^T \mu_u^\alpha \, du \leq \delta/2 \right) \leq \varepsilon(\Delta t).$$

Finally, due to Assumption 2, and as $\Delta t$ is small, we can replace $\int_s^T \mu_u^\alpha \, du$ by $\int_s^T \mu_u^\alpha \, du$, and $\delta/2$ by $\delta/4$, in the above equation. This completes the proof of the lemma.  

Using the strategy at which the agent in state $(1, \alpha_0)$ waits until the last moment $n = N$, we conclude that the process $\left( \lambda^a_n(\alpha_0) + p^b_n \right)$ must be a supermartingale under $\hat{P}^\alpha$. More precisely, due to the definition of an optimal strategy, we have, $\hat{P}$-a.s.:

$$\lambda^a_{n+2}(\alpha_0) \geq \mathbb{E}^{\alpha}_{n+2} \lambda^a_N(\alpha_0) + \mathbb{E}^{\alpha}_{n+2} \left( \mathbb{E}^{\alpha}_{n+3} \left( p^0_N - p^0_{n+3} \right) + \xi_{n+3} \right).$$

Recall that $\lambda^a_N(\alpha_0) = p^b_N$ and, due to Theorem 1 (more precisely, it follows from the proof of the theorem), there exists a constant $C_0 > 0$, s.t., for all small enough $\Delta t > 0$, the following holds $\hat{P}$-a.s.:

$$-C_0 \sqrt{\Delta t} \leq p^b_N < 0 < p^a_n \leq C_0 \sqrt{\Delta t}.$$

Thus, we have, $\hat{P}$-a.s.:

$$\lambda^a_{n+2}(\alpha_0) \geq -C_0 \sqrt{\Delta t} + \mathbb{E}^{\alpha}_{n+2} \left( \mathbb{E}^{\alpha}_{n+3} \left( p^0_N - p^0_{n+3} \right) \right) + \mathbb{E}^{\alpha}_{n+2} \xi_{n+3}. \tag{26}$$

Due to Assumption 2, we have, $\hat{P}$-a.s.:

$$\mathbb{E}^{\alpha}_{n+2} \xi_{n+3} \leq C \Delta t, \quad \left| \mathbb{E}^{\alpha}_{n+3} \left( p^0_N - p^0_{n+3} \right) \right| \leq CT,$$

and, hence,

$$\lambda^a_{n+2}(\alpha_0) \geq -C_0 \sqrt{\Delta t} + CT + C \Delta t.$$

In addition, making use of Lemma 6, we conclude that, for any small enough $\Delta t$, there exist $n = 0, \ldots, N - 2$ and $\Omega'' \in \mathcal{F}_n$, s.t. $\hat{P}^\alpha_n(\Omega'') > 0$ and

$$\hat{P}^\alpha_{n+2} \left( \mathbb{E}^{\alpha}_{n+3} \left( p^0_N - p^0_{n+3} \right) \leq \delta/2 \right) \leq \varepsilon(\Delta t), \quad \text{on } \Omega''.$$

Using (26) and assuming that $\Delta t$ is small enough, we obtain:

$$\lambda^a_{n+2}(\alpha_0) \geq \delta/4, \quad \text{on } \Omega''.$$

Next, Corollary 2 implies that, $\hat{P}$-a.s.,

$$p^b_{n+1} \geq \mathbb{E}^{\alpha}_{n+1} \left[ \lambda^a_{n+2}(\alpha_0) + \xi_{n+2} \xi_{n+2} < p^b_{n+1} \right].$$
Thus, on $\Omega''$, we obtain:
\[
p_{n+1}^b - E_{n+1}^{\alpha_0} [\xi_{n+2} < p_{n+1}^b] \geq \delta/4.
\] (27)

The following lemma shows that, for any number $p$, the conditional expectation of the fundamental price increment, $E_{n+1}^{\alpha_0} [\xi_{n+2} < p]$, approaches $p$ as the size of the time interval vanishes. This result follows from Lemma 2.

**Lemma 7.** There exists a constant $C_3 > 0$, s.t., for all small enough $\Delta t > 0$, and for any $t \in [0, T - \Delta t]$, the following holds $P$-a.s.:

\[
\sup_{p \leq 0} \left| p - \tilde{E}_t^{\alpha_0} [p_{t+\Delta t}^0 - p_t^0] \right| \leq C_3 \sqrt{\Delta t}.
\]

**Proof:** Fix $t$ and $\Delta t > 0$ and consider the evolution of $\tilde{p}_s^0$, for $s \in [t, t + \Delta t]$, under $P_t^{\alpha_0}$:

\[
\tilde{p}_s^0 - \tilde{p}_t^0 = \int_t^s \mu_u du + \int_t^s \sigma_u dW_u,
\]

where $W^{\alpha_0}$ is a Brownian motion under $P^{\alpha_0}$. Rescaling by $\sqrt{\Delta t}$, we obtain

\[
\frac{\tilde{p}_s^0 - \tilde{p}_t^0}{\sqrt{\Delta t}} = X_{(s-t)/\Delta t}, \quad X_s = \int_0^s \mu_u du + \int_0^s \sigma_u dW_u, \quad s \in [0, 1],
\]

with

\[
\hat{\mu}_s = \sqrt{\Delta t} \mu_{t+s+\Delta t}, \quad \hat{\sigma}_s = \sigma_{t+s+\Delta t}, \quad \hat{W}_s = \frac{1}{\sqrt{\Delta t}} (W_{t+s+\Delta t} - W_t^{\alpha_0}), \quad s \in [0, 1].
\]

Notice that the above processes are adapted to the new filtration $\hat{\mathcal{F}}$, with $\hat{\mathcal{F}}_s = F_{t+s+\Delta t}$, and, $P$-a.s., under $P_t^{\alpha_0}$, $\hat{W}$ is a Brownian motion with respect to $\hat{\mathcal{F}}$. Next, due to Assumptions 1 and 4, for any small enough $\Delta t > 0$, $P$-a.s., the dynamics of $(-X_s)$, under $P_t^{\alpha_0}$, satisfy all the assumptions of Lemma 2. As a result, we obtain:

\[
\tilde{E}_t^{\alpha_0} (X_1 < -x - z) \leq C_1 e^{-\frac{z}{\sqrt{\Delta t}}} P_t^{\alpha_0} (X_1 < -x), \quad \forall x, z \geq 0.
\]

Finally, we notice that

\[
\sup_{p \leq 0} \left| p - \tilde{E}_t^{\alpha_0} [p_{t+\Delta t}^0 - p_t^0] \right| = \sqrt{\Delta t} \sup_{p \leq 0} \left| p - \tilde{E}_t^{\alpha_0} [X_1 | X_1 < p] \right| = \sqrt{\Delta t} \sup_{p \leq 0} \left| \frac{\int_{-p}^{\infty} x d\tilde{E}_t^{\alpha_0} (X_1 < x)}{\tilde{E}_t^{\alpha_0} (X_1 < p)} \right| \leq C_1 \sqrt{\Delta t},
\]

which completes the proof of the lemma.

Using (27) and Lemma 7, we conclude that, for all small enough $\Delta t$, we have: $p_{n+1}^b > 0$ on $\Omega''$, $P$-a.s.. In addition, Corollary 2 implies that, for any $\alpha \in \hat{A}$, the following holds $P$-a.s.:

\[
\lambda_{n+1}^{\alpha} \geq p_{n+1}^b.
\]

Next, with a slight abuse of notation (similar notation was introduced in the proof of Proposition 1), we consider the simplified objective of an agent who posts a limit sell order at the ask price $p_n^a$:

\[
\hat{A}^\alpha (p_n^a, \lambda_{n+1}^\alpha) = E_{\alpha_n}^\alpha (p_n^a - \lambda_{n+1}^\alpha - \xi_{n+1} | \xi_{n+1} > p_n^a)
\]
This convergence, along with (28), implies that there exists a $F_{n+1}$-measurable $\lambda^a_{n+1}$ such that, on $\Omega''$, we have, $\mathbb{P}$-a.s.:

$$\hat{A}^a(p_n^a; \lambda^a_{n+1}) \leq E_{n}^a(p_n^a - \xi_{n+1} | \xi_{n+1} > p_n^a) - E_{n}^a(p_{n+1}^b 1_{\Omega''} | \xi_{n+1} > p_n^a) < 0, \quad \forall \alpha \in \hat{A}. \quad (28)$$

To obtain the last inequality in the above, we recall that $\Omega'' \in \mathcal{F}_n$ and, $\mathbb{P}$-a.s., $1_{\Omega''} \mathbb{P}_n(\Omega \setminus \Omega'') = 0$, $p_{n+1}^b > 0$ on $\Omega''$, and $\mathbb{P}_n^a(\xi_{n+1} > p_n^a) > 0$, for all $\alpha \in \hat{A}$. Next, repeating the proof of Lemma 3 (and using the fact that $\lambda^a_{n+1}$ is absolutely bounded, as shown in Corollary 1), we conclude that, $\mathbb{P}$-a.s., either $\nu^*_a(\{p_n^a\}) > 0$, or we have:

$$\left| A^{\alpha}(p; \lambda^a_{n+1}) - \hat{A}^a(p_n^a; \lambda^a_{n+1}) \right| \to 0,$$

as $p \downarrow p^a$, uniformly over all $\alpha \in \hat{A}$, where we introduce the true objective,

$$A^{\alpha}(p; \lambda^a_{n+1}) = E_{n}^{\alpha} \left( (p - \lambda^a_{n+1} - \xi_{n+1}) 1_{(p_{n+1}^a + p_{n+1}^b - \xi_{n+1}) > \nu^*_a((-\infty,0))} \right).$$

This convergence, along with (28), implies that there exists a $\mathcal{F}_n$-measurable $\bar{p} \geq p_n^a$, such that, on $\Omega''$, the following holds $\mathbb{P}$-a.s.: if $\nu^*_a(\{p_n^a\}) = 0$ then $\bar{p} > p_n^a$, and, in all cases,

$$A^{\alpha}(p; \lambda^a_{n+1}) < 0, \quad \forall p \in [p_n^a, \bar{p}], \quad \forall \alpha \in \hat{A}.$$

Finally, we repeat the last part of the proof of Lemma 4 (following equation (23)), to obtain a contradiction with the definition of $p_n^a$, and complete the proof of the theorem. The last argument also shows that, when $\Delta t$ is small enough, it becomes suboptimal for the agents to post limit sell orders, as the expected relative profit from this action becomes negative, causing the market to degenerate.

8 Existence of a non-degenerate equilibrium for homogeneous beliefs

As discussed earlier, the main contribution of this work is a characterization of the liquidity effects of high trading frequency, which is stated as a necessary condition for all non-degenerate LTC equilibria. However, one may wonder if there exist any such equilibria, for an arbitrarily high trading frequency. In Section 3, we construct a specific equilibrium in a Gaussian random walk model – i.e. when $\sigma$ is constant. In this section, we show how to construct a non-degenerate equilibrium, for an arbitrarily high trading frequency, in any model in which $\sigma$ is only required to be deterministic around the terminal time horizon, and the agents have homogeneous beliefs (and, of course, the fundamental price has to be a martingale). It is worth mentioning that we do not claim that the assumptions made in this section are the most realistic ones. The proper construction of an equilibrium that produces realistic LOB is the subject of our follow-up paper [25]. Herein, we present the existence result, merely, as a sanity check, to show that the equilibria defined in Subsection 2.2 do exist, and are not restricted to the specific Gaussian random walk model, considered in Section 3. From a theoretical point of view, the equilibrium constructed herein (as opposed to the one considered in Section 3) is also interesting because it satisfies the condition (6). The latter condition is important if one assumes that no new agents arrive in the market, and, in particular, it allows one to connect the proposed continuum-player game to other existing modeling frameworks (cf. the discussion in Subsection 2.3). Thus, we consider the setting of Section 4 and adopt the same notation, along with the notation introduced in (17). Our goal is construct a non-degenerate LTC equilibrium for high enough frequency (i.e. when $N = T/\Delta t$ is large enough). In view of the results of Section 4, we need to assume that $\bar{p}^0$ is a martingale (i.e. all $\mu^a$ are zero). In order to construct the equilibrium, we make several additional assumptions.

Assumption 8. $\hat{A}$ is a singleton.

Assumption 9. One of the following two properties holds:
\begin{itemize}
  \item $\sigma_t$ is deterministic and non-increasing in $t \in [0, T]$, $\mathbb{P}$-a.s.;
  \item there exist constants $\bar{c}, \bar{\delta} > 0$, such that, $\mathbb{P}$-a.s., we have: $\sigma_t \geq \bar{c} > 0$, for all $t \in [0, T - \bar{\delta}]$, and $\sigma_t$ is deterministic and non-increasing in $t \in [T - \bar{\delta}, T]$, with $\sigma_{T^-} < \bar{c}$.
\end{itemize}

The following technical lemma is an integral part of the equilibrium construction. It explains why we need Assumption 9 to hold.

**Lemma 8.** Let Assumptions 1, 4, 8, 9 hold, and assume that $\bar{p}^0$ is a martingale. Then, there exist constants $\bar{p}^b < 0 < \bar{p}^a$, s.t., for any small enough $\Delta t > 0$, the following conditions hold $\mathbb{P}$-a.s.:

1. For $n = N - 1$,
   \[
   \bar{p}^a = \inf \left\{ p \in \mathbb{R} \mid \mathbb{E}_n \left[ (p - \bar{p}^b - \xi_N) 1_{\{\xi_N \geq p\}} \right] = 0 \right\}, \\
   \bar{p}^b = \sup \left\{ p \in \mathbb{R} \mid \mathbb{E}_n \left[ (\bar{p}^a - p + \xi_N) 1_{\{\xi_N < p\}} \right] = 0 \right\}.
   \]

2. For all $n = 0, \ldots, N - 1$,
   \[
   \bar{p}^a \leq \inf \left\{ p \in \mathbb{R} \mid \mathbb{E}_n \left[ (p - \bar{p}^b - \xi_{n+1}) 1_{\{\xi_{n+1} > p\}} \right] = 0 \right\}, \\
   \bar{p}^b \geq \sup \left\{ p \in \mathbb{R} \mid \mathbb{E}_n \left[ (\bar{p}^a - p + \xi_{n+1}) 1_{\{\xi_{n+1} < p\}} \right] = 0 \right\}.
   \]

**Proof:** First, we notice that the conditions 1–2 of the lemma hold for $(\bar{p}^a, \bar{p}^b, \{\xi\})$ if and only if they holds for $(\bar{p}^a/\sqrt{\Delta t}, \bar{p}^b/\sqrt{\Delta t}, \{\xi/\sqrt{\Delta t}\})$. Thus, we consider the conditions 1–2 rescaled by $\sqrt{\Delta t}$. Denote by $\eta$ a normal random variable, with zero mean and variance $\bar{\sigma}$, independent of $\mathcal{F}_N$. Notice that, due to Assumption 9, $\sigma_{(N-1)\Delta t}$ is deterministic and strictly below $\bar{c}$, when $\Delta t$ is small enough. Then, due to symmetry, the condition 1 of the lemma can be satisfied by choosing $\bar{p}^b = -\bar{p}^a = -\sqrt{\Delta t}\bar{p}^a$, solving

\[
\mathbb{E} \left[ (2\bar{p}^a - \eta) 1_{\{\eta > \bar{p}^a\}} \right] = 0,
\]

with $\bar{\sigma} = \sigma_{(N-1)\Delta t} < \bar{c}$. It is easy to see that there exists a unique root of the above equation, and it is strictly increasing in the variance of $\eta$. Let us verify condition 2 (rescaled by $\sqrt{\Delta t}$). It is easy to see that the unique roots $p$ of

\[
\mathbb{E} \left[ (p + \bar{p}^a - \eta) 1_{\{\eta > \bar{p}^a\}} \right] = 0, \quad \mathbb{E} \left[ (\bar{p}^a - p + \eta) 1_{\{\eta < \bar{p}^a\}} \right] = 0,
\]

are, respectively, increasing and decreasing in the variance of $\eta$, $\bar{\sigma}^2$. Due to Assumption 9, for $n = 0, \ldots, N - 2$, we have $\sigma_{n\Delta t} \geq \sigma_{(N-1)\Delta t}$. If $\sigma$ is deterministic and non-increasing, then, the above monotonicity implies condition 2 of the lemma. If $\sigma$ is not deterministic in the interval $[0, T - \bar{\delta}]$, then, there exists $\varepsilon > 0$, s.t., whenever $n \Delta t \in [0, T - \bar{\delta}]$, we have $\sigma_{n\Delta t} \geq \sigma_{(N-1)\Delta t} + \varepsilon$, $\mathbb{P}$-a.s.. Lemma 1, then, implies that the expectations in condition 2 of the lemma (rescaled by $\sqrt{\Delta t}$) can be approximated by the values of the expectations in (29) (viewed as functions of $\bar{\sigma}$), with $\bar{\sigma} = \sigma_{n\Delta t}$. In particular, it implies that, for all small enough $\Delta t$, all roots of the first equation in condition 2 (rescaled by $\sqrt{\Delta t}$) are above $\bar{p}^a$, and all roots of the second equation in condition 2 (rescaled by $\sqrt{\Delta t}$) are below $\bar{p}^b$. $\blacksquare$

We also make the following assumption on the conditional tails of the fundamental price increments.

**Assumption 10.** The following holds $\mathbb{P}$-a.s., for all $n = 0, \ldots, N - 1$ and all $p, \Delta p > 0$:

\[
\mathbb{E}_n (\xi_{n+1} \mid \xi_{n+1} > p + \Delta p) \leq \mathbb{E}_n (\xi_{n+1} \mid \xi_{n+1} > p) + \Delta p
\]

Assumption 10 is satisfied in Gaussian models (i.e. when $\sigma$ is deterministic). The next assumption controls the form of the demand process, as well as its size relative to the agents’ inventory. In order to formulate the assumption, we introduce $\mu^c_t$ as the continuous part of the empirical distribution $\hat{\mu}_t$. We define $\tilde{\mu}^c_t^{a,c}$ and $\tilde{\mu}^c_t^{b,c}$ as the restrictions of $\mu^c_t$ to $(-\infty, 0)$ and $(0, \infty)$, respectively.
**Assumption 11.** The demand size process \( \tilde{D} \) is deterministic. In addition, \( \mathbb{P} \)-a.s., we have:

\[
\tilde{\mu}_0^{a,c}(\mathbb{R}) > \sum_{n=1}^{N} \sup_{p \in \mathbb{R}} \left( \tilde{D}_{n\Delta t}(p) - \tilde{D}_{(n-1)\Delta t}(p) \right), \quad \tilde{\mu}_0^{b,c}(\mathbb{R}) > -\sum_{n=1}^{N} \inf_{p \in \mathbb{R}} \left( \tilde{D}_{n\Delta t}(p) - \tilde{D}_{(n-1)\Delta t}(p) \right).
\]

The above assumptions ensures that the total external demand cannot exceed the total internal supply of the agents (otherwise, the LOB may degenerate, even if the agents are market-neutral). Finally, we present the main result of this subsection. It shows that, under the above assumptions, there exists a non-degenerate LTC equilibrium with continuous LOB, satisfying the additional condition (6).

**Proposition 2.** Let Assumptions 1, 4, 8, 9, 10, 11 hold, and assume that \( \bar{p}_{\tilde{\nu}} \) is a martingale. Then, for any small enough \( \Delta t > 0 \), there exists an empirical distribution process \( \mu = (\mu_n) \), having the prescribed initial value \( \mu_0 \), such that the associated discrete time model and \( \mu \) admit a non-degenerate LTC equilibrium satisfying (6). Moreover, this equilibrium can be constructed so that the agents do not post market orders, and the LOB \( \nu_n \) is continuous (i.e. has no mass points in \( \mathbb{R} \)).

**Proof:**

Within the scope of this proof, we use the Notational Convention 1, introduced in the proof of Proposition 1 (i.e. we shift the LOB, the expected execution prices, and the demand curve, by \( \bar{p}_{\tilde{\nu}} \), without changing the variables’ names). Throughout most of the proof, we assume that we are given an empirical distribution \( \mu \) satisfying \( \mu_n^a((0, \infty)) > \sup_p D_n^a(p) \) and \( \mu_n^a((-\infty, 0)) > \sup_p D_n^b(p) \). In the last step we verify that there exists \( \mu \), s.t. \( \mu_0 = \tilde{\mu}_0 \) and (6) holds in the equilibrium constructed for this \( \mu \).

**Step 1.** We begin with a single-period sub-game, construct an equilibrium in the time period \( \{N - 1, N\} \). Denote the optimal strategies by \( (\tilde{p}(s), \tilde{q}(s), \tilde{r}(s)) \). We construct the equilibrium so that \( \tilde{r} = 0 \), but \( \tilde{q}(s) \) and \( \tilde{p}(s) \) may depend on \( s \). First, we construct it so that \( \tilde{q}(s) = s \) and \( \tilde{p}(s) = \tilde{p}(\text{sign}(s)) \); i.e. all agents post limit orders at the bid and ask prices. To achieve this, we find \( \mathcal{F}_{N-1} \)-measurable random variables \( \tilde{p}^b < 0 < \tilde{p}^a \), s.t.:

\[
\tilde{p}(-1) = p^b = \lambda^a < 0 < \lambda^b = \tilde{p}(1)
\]

Assume that all agents who are long the asset to post limit sell orders at the price level \( p^a \). Then, the total size of the LOB at \( p^a \) is \( \mu_{N-1}((0, \infty)) \). Assumption 11 and condition (6) imply that the incremental demand never exceeds \( \mu_{N-1}^a(\mathbb{R}) \leq \mu_{N-1}((0, \infty)) \), and, hence, the agents cannot benefit from posting their orders above \( p^a \). Similar argument applies to the agents who are short the asset. Then, due to Corollary 3, it suffices to find \( \mathcal{F}_{N-1} \)-measurable \( \tilde{p}^b \) and \( \tilde{p}^a \), s.t., \( \mathbb{P} \)-a.s., we have:

\[
\begin{aligned}
\tilde{p}^b &< 0 < \tilde{p}^a, \\
\tilde{p}^a &\equiv \inf \{ p \in \mathbb{R} | \mathbb{E}_{N-1} \left[ (p - p^b - \xi_N^a) 1_{\{\xi_N^a > p\}} \right] = 0 \}, \\
\tilde{p}^b &\equiv \sup \{ p \in \mathbb{R} | \mathbb{E}_{N-1} \left[ (p^a - p - \xi_N^a) 1_{\{\xi_N^a < p\}} \right] = 0 \},
\end{aligned}
\tag{30}
\]

Lemma 8 shows that \( (\tilde{p}^a = \bar{p}_{\tilde{\nu}}, \tilde{p}^b = \bar{p}_{\tilde{\nu}}) \) satisfies the above system, thus, producing an equilibrium with LOB \( \nu_{N-1} \) being a combination of two Dirac measures. Corollary 2 implies that, in such an equilibrium, we have:

\[
\lambda^a = \tilde{p}^b = \bar{p}^b < 0 < \tilde{p}^a = \xi^a = \bar{p}^a.
\]

Let us, now, modify the above construction, to obtain an equilibrium with continuous LOB, in the single-period sub-game. Denote \( \kappa_n(p) = \tilde{D}_{n\Delta t}(p) - \tilde{D}_{(n-1)\Delta t}(p) \). For every \( p \geq p^a \), let \( x = x(p) \geq 0 \) be the smallest nonnegative solution to

\[
\mathbb{E}_{N-1} \left[ \xi_N^a | \kappa_n(p - \xi_N^a) > x \right] - p = -\tilde{p}^b
\]

Notice that, for \( p = p^a \), we have \( x(p) = 0 \), due to the choice of \( \bar{p}_{\tilde{\nu}} \). Notice also that the left hand side of (31) is jointly continuous in \( (p, x) \), due to Assumptions 1 and 11. In addition, the value of the left hand side of (31) converges to infinity, as \( x \to \infty \), and, due to Assumption 10, for all \( p \geq p^a \), we have:

\[
\mathbb{E}_{N-1} \left[ \xi_N^a | \kappa_n(p - \xi_N^a) > 0 \right] - p = \mathbb{E}_{N-1} \left[ \xi_N^a | \xi_N^a > p \right] - p.
\]

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\[
\begin{align*}
\leq & \mathbb{E}_{N-1}[\xi_N | \xi_N > p^a] + p - p^a - p = -p^b.
\end{align*}
\]
Thus, a nonnegative solution to (31), denoted \( x = x(p) \), exists for any \( p \geq p^a \). Due to the maximum theorem, \( x(\cdot) \) is continuous. Let us show that this function is also non-decreasing. For any \( \Delta x, \Delta p > 0 \), consider
\[
\begin{align*}
\mathbb{E}_{N-1}[\xi_N | \xi_N > p + \Delta p - \kappa_N^{-1}(x - \Delta x)] - p - \Delta p \\
\leq & \mathbb{E}_{N-1}[\xi_N | \xi_N > p - \kappa_N^{-1}(x - \Delta x)] - p.
\end{align*}
\]
If the right hand side of the above is equal to \(-p^b\), for some \( \Delta x > 0 \), then, we have a contradiction with the minimality of \( x \) as a solution to (31). Therefore, there exists no such \( \Delta x \) and, in turn, \( x(p + \Delta p) \geq x(p) \).

Finally, we define the LOB \( \nu^+_{N-1} \) via
\[
\nu^+_{N-1}((\infty, p]) = x(p) \land \mu^{a,c}_{N-1}(\mathbb{R}), \quad \forall p \in \mathbb{R}.
\]
Analogous construction is used to obtain \( \nu^-_{N-1} \). It is easy to check that, with such a LOB, the optimal action for every agent who is long the asset is to post a limit sell order at or above \( p^a \), or wait, and the agents are indifferent between these choices. Similar conclusion applies to the agents who are short the asset. It only remains to show that such a LOB, indeed, results from the aggregated actions of the agents. First, consider the agents who are long the asset. Let us choose \( \bar{s} \leq \infty \), such that
\[
\int_{(0, \bar{s})} s \mu^{a,c}_{N-1}(ds) = \nu^+_{N-1}(\mathbb{R}).
\]
Notice that such \( \bar{s} \) exists, due to the choice of \( \nu^+ \). We assume that the agents whose states \( s \) correspond to the atoms of \( \mu_{N-1} \), and those with \( s > \bar{s} \), choose to wait. Every remaining agent who is long the asset and is at state \( s \), posts a limit sell order of size \( s \) at the price level \( \hat{p}(s) \), for \( s \in [0, \bar{s}] \). We choose \( \hat{p} \) to be nondecreasing, starting from \( \hat{p}(0) = p^a \), and defined for all other values of \( s \) via
\[
\hat{p}(s) = \sup \left\{ p \in \mathbb{R} | \int_0^s s' \mu^{a,c}_{N-1}(ds') = x(p) \right\}.
\]
It is easy to see that, with such a choice, (4) holds. The actions of agents who are short the asset are defined analogously. Notice that, in the new equilibrium, the bid and ask prices, as well as the expected execution prices, remain the same as in the original equilibrium (constructed in this proof), in which the LOB is a combination of two Dirac measures.

**Step 2.** Here, we use the induction argument. Assume that we have constructed an equilibrium with a LOB \( \nu \), for the sub-games with initial times \( m = n+1, \ldots, N-1 \), such that the value function and the LOB satisfy, \( \mathbb{P} \)-a.s.: \( \lambda^a_m = \tilde{p}^a_m \) and \( \lambda^b_m = \tilde{p}^b_m \). Step 1 shows that the above assumptions are satisfied for \( n = N - 2 \). We need to show that there is an equilibrium in the sub-game with initial time \( n \). As before, we, first, construct an equilibrium in which the LOB is a combination of two Dirac measures. Notice that, if all agents who are long the asset post limit sell orders at the price level
\[
p^a_n = \inf \left\{ p \in \mathbb{R} | \mathbb{E}_n \left[ (p - \lambda^a_{n+1} - \xi_{n+1})1_{\{\xi_{n+1} > p\}} \right] = 0 \right\},
\]
then an agent never benefits from posting a limit sell order at \( p \neq p^a_n \). This, again, follows from the fact that the incremental demand never exceeds \( \mu^{a,c}(\mathbb{R}) \leq \mu_{n+1}(\mathbb{R}) \). The above \( p^a_n \) is finite, due to Lemma 1 and the fact that \( \lambda^a_{n+1} = \tilde{p}^b < 0 \). Due to Lemma 8, we have \( p^a_n \geq \tilde{p}^b \). Similarly, we define the bid price \( p^b_n \leq \tilde{p}^b \). Then, it is easy to see that the expected execution prices are given by \( \lambda^a_n = \lambda^a_{n+1} = \tilde{p}^b \geq p^a_n \) and \( \lambda^b_n = \lambda^b_{n+1} = \tilde{p}^b \leq p^a_n \). According to Corollary 3, we have constructed an equilibrium, in which \( q_n(s) = s \) and \( p_n(s) \) is equal to \( p^a_n \).
or $\mu^b_n$, depending on the sign of $s$. The induction assumptions are verified above. Finally, we modify this equilibrium in exactly the same way as in Step 1, to obtain an equilibrium with continuous LOB at time $n$. The bid and ask prices, as well as the expected execution prices, remain the same in the new equilibrium, hence, the induction assumptions are verified for the new equilibrium as well.

**Step 3.** Next, we need to show that there exists $\mu$, s.t. (6) holds in the equilibrium constructed in Steps 1–2. The proof follows from the fact that the LOB constructed in Steps 1–2 depends on $\mu$ in a very simple way. Namely, the $\mu_n$ is “cut off” at a level that depends on $\mu_n^c((0, \infty))$ or $\mu_n^c((\infty, 0))$ – and this is the only source of dependence of the LOB on $\mu$. Thus, we can start with an arbitrary $\mu$, satisfying $\mu^c_n((0, \infty)) > \sup_p D^+_n(p)$ and $\mu^c_n((\infty, 0)) > \sup_p D^-_n(p)$. We construct the equilibrium via Steps 1–2. Then, we proceed forward, for $n = 0, \ldots, N - 1$, to construct $\bar{\mu}$ via (6),

$$\bar{\mu}_n = \bar{\mu}_0 \circ \left( s \mapsto \left( S_n^0(\bar{\mu}, \bar{\mu})_n((0, s)) \right) \right)^{-1},$$

along with the new candidate optimal strategies $(\bar{p}, \bar{q}, \bar{r})$, defined via (32)–(33), using the original function $x(p)$, and $\bar{\mu}$ in place of $\mu$. We also construct the new LOB, using the function $x(p)$ and the cut-off points $\bar{\mu}^c_n((0, \infty))$ and $\bar{\mu}^c_n((\infty, 0))$, as shown in Steps 1–2. The fixed-point constraints (4)–(5) are satisfied with such a construction. Notice also that, given such a LOB, the agents with positive inventory are indifferent between posting limit orders at any level above the ask price. Moreover, it is not strictly optimal for them to post limit orders below the ask price, or submit a market order, or wait (this follows from the construction in Steps 1–2). Similar conclusion holds for the agents with negative inventory. Thus, the new candidate optimal strategies are, indeed, optimal, and we obtain an equilibrium with continuous LOB, and with $\bar{\mu}$ as the empirical distribution process.

9 **Summary and future work**

In this paper, we have presented a new framework for modeling the market microstructure, which does not assume the existence of a designate market maker, and in which the LOB arises endogenously, as a result of equilibrium between multiple strategic players (aka agents). This framework is based on a continuum-player game. It reproduces the mechanics of an auction-style exchange very closely, so that, in particular, it can be used to analyze the liquidity effects of changes in a relevant market factor or in the rules of the exchange. We used the proposed modeling framework to study the liquidity effects of high trading frequency. In particular, we have demonstrated the dual nature of high trading frequency. On the one hand, in the absence of a bullish or bearish signal about the asset, the higher trading frequency makes the market more efficient. On the other hand, at a sufficiently high trading frequency, even a very small trading signal may amplify the adverse selection effect, creating a disproportionally large change in the LOB, which is interpreted as an internal (or, self-inflicted) liquidity crisis.

The present work raises many questions for further research. In particular, it is desirable to model more carefully the internal market orders of the agents. In the current setting, the agents ignore the potential market orders of other agents, when predicting the future demand, and expect execution from the external orders only. Such setting is consistent if, in equilibrium, no agents choose to submit market orders (which is the case in the equilibria we construct herein). However, in general, it would be more realistic to incorporate the anticipation of internal market orders in the agent’s models for future demand. Notice that the main results of the present work are of a qualitative nature: they demonstrate the general behavior of the LOB, as a function of trading frequency, but do not immediately allow for any computations. It would also be interesting to develop quantitative results. In particular, we would like to construct an equilibrium in a more realistic, and more concrete, model than the ones used in Sections 3 and 8. Such a model would allow for heterogeneous beliefs, and it would prescribe the specific sources of information (i.e. relevant market indicators) used by the agents.
Figure 1: On the left: bid price $\hat{p}_b$ (negative) and ask price $\hat{p}_a$ (positive). On the right: expected execution prices, $\hat{\lambda}_a$ (negative) and $\hat{\lambda}_b$ (positive). Different curves correspond to different trading frequencies ($N = 20, \ldots, 500$). All prices are measured relative to the fundamental price and are plotted as functions of time. Zero drift case: $\alpha = 0, \sigma = 1, T = 1$.

to form their beliefs. A model of this type could be calibrated to market data and used to study the effects of changes in relevant market parameters on the LOB. Finally, it is interesting to develop the continuous time version of the proposed framework, in order to better capture the present state of the markets, where the trading frequency is not restricted. All these questions are the subject of our follow-up paper [25].

10 Appendix A

This section contains several useful technical results on the representation of the value function of an agent in the proposed game. Notice that (1) and (2) imply that, if $\nu$ is admissible, then, for any $(\alpha, m, p, q, r)$, we have, $P$-a.s.:

$$|J^{(p,q,r)}(m,s,\alpha,\nu) - J^{(p,q,r)}(m,s',\alpha,\nu)| \leq |s-s'| E_m^{\alpha}|p_n^b| \lor |p_n^a|, \quad \forall s, s' \in \mathbb{R}$$

This implies that every $J^{(p,q,r)}(m,\cdot,\alpha,\nu)$ and $V_\nu^{\alpha}(\cdot,\alpha)$ has a continuous modification under $P$. Thus, whenever $\nu$ is admissible, we define the value function of an agent as the aforementioned continuous modification of the left hand side of (3).

Lemma 9. Assume that an optimal control exists for an admissible LOB $\nu$. Assume also that, for any $\alpha \in \Lambda$, the associated value function $V_\nu^{\alpha}(\cdot,\alpha)$, defined in (3), is measurable with respect to $\mathcal{F}_n \otimes \mathcal{B}(\mathbb{R})$. Then, it satisfies the following Dynamic Programming Principle.

- For $n = N$ and all $(s, \alpha) \in \mathbb{S}$, we have, $P$-a.s.:
  $$V_N^{\nu}(s, \alpha) = s^+ p_N^b - s^- p_N^a$$

- For all $n = N-1, \ldots, 0$ and all $(s, \alpha) \in \mathbb{S}$, we have:
  $$V_n^{\nu}(s, \alpha) = \text{esssup}_{p,q,r} \left\{ 1_{\{r_n=0\}} E_n^{\alpha} \left[ V_{n+1}^{\nu}(s, \alpha) + (q_n p_n + V_{n+1}^{\nu}(s-q_n, \alpha) - V_{n+1}^{\nu}(s, \alpha)) \right] \right\}.$$
Figure 2: On the left: ask price $\hat{p}^a$ (in red) and the associated expected execution prices $\hat{\lambda}^a$ (in blue); different curves correspond to different trading frequencies ($N = 20, \ldots, 500$); black dashed line is the expected change in the fundamental price $\alpha(T - t)$. On the right: ask price $\hat{p}^a$ (in red) and the associated expected execution price $\hat{\lambda}^a$ (in blue), bid price $\hat{p}^b$ (in orange) and the associated expected execution price $\hat{\lambda}^b$ (in green), for $N = 100$. Non-degenerate equilibrium exists only on a time interval where $\hat{\lambda}^a < 0$. All prices are measured relative to the fundamental price and are plotted as functions of time. Positive drift: $\alpha = 0.1$, $\sigma = 1$, $T = 1$.

Figure 3: The horizontal axis represents trading frequency, measured in the number of steps $N$. Left: time-zero bid-ask spread in the zero-drift case ($\alpha = 0$). Right: the maximum value of drift $\alpha$ for which a non-degenerate equilibrium exists on the entire time interval. Parameters: $\sigma = 1$, $T = 1$. 
where the essential supremum is taken under \( \mathbb{P} \), over all admissible controls \((p, q, r)\).

\[ J^{(p, q, r)}(n, s, \alpha, \nu) = \text{esssup}_{p, q, r} \mathbb{E}_n^\alpha \left( V_n^{\nu} \left( S_{n+1}^{m, s, (p, q, r)}, \alpha \right) - g_n^{\nu}(p_n, q_n, r_n, D_{n+1}) \right), \]

Notice also that, for any \( n \leq k \leq n \), we have, \( \mathbb{P} \)-a.s.:

\[ \mathbb{E}_m^\alpha J^{(p, q, r)}(n, S_{m+1}^{m, s, (p, q, r)}, \alpha, \nu) = J^{(p, q, r)}(k, S_k^{m, s, (p, q, r)}, \alpha, \nu) + \mathbb{E}_k^\alpha \sum_{j=k}^{n-1} g_j^{\nu}(p_j, q_j, r_j, D_{j+1}). \]

Notice also that, for any \((p, q, r)\) we have, \( \mathbb{P} \)-a.s.:

\[ J^{(p, q, r)}(m, s, \alpha, \nu) \leq V_m^{\nu}(s, \alpha), \quad \forall s \in \mathbb{S} \]

Let us show that the left hand side of (36) is less than its right hand side:

\[ V_m^{\nu}(s, \alpha) = \text{esssup}_{p, q, r} J^{(p, q, r)}(m, S_{m+1}^{m, s, (p, q, r)}, \alpha, \nu) \]

\[ = \text{esssup}_{p, q, r} \mathbb{E}_m^\alpha \left( J^{(p, q, r)}(m+1, S_{m+1}^{m+1, s, (p, q, r)}, \alpha, \nu) - g_m^{\nu}(p_m, q_m, r_m, D_{m+1}) \right) \]

\[ \leq \text{esssup}_{p, q, r} \mathbb{E}_m^\alpha \left( V_{m+1}^{\nu}(S_{m+1}^{m+1, s, (p, q, r)}, \alpha) - g_m^{\nu}(p_m, q_m, r_m, D_{m+1}) \right) \]

Next, we show that the right hand side of (36) is less than its left hand side. For any \((p, q, r)\), we have, \( \mathbb{P} \)-a.s.:

\[ \mathbb{E}_m^\alpha \left( J^{(p, q, r)}(m+1, S_{m+1}^{m+1, s, (p, q, r)}, \alpha) - g_m^{\nu}(p_m, q_m, r_m, D_{m+1}) \right) \]

\[ = \mathbb{E}_m^\alpha \left( J^{(\tilde{p}, \tilde{q}, \tilde{r})}(m+1, S_{m+1}^{m+1, s, (p, q, r)}, \alpha, \nu) - g_m^{\nu}(p_m, q_m, r_m, D_{m+1}) \right) \]

\[ = J^{(\tilde{p}, \tilde{q}, \tilde{r})}(m, s, \alpha, \nu) \leq V_m^{\nu}(s, \alpha), \]

where \((\tilde{p}_n, \tilde{q}_n, \tilde{r}_n)\) coincide with \((\hat{p}_n, \hat{q}_n, \hat{r}_n)\), for \( n \geq m+1 \), while they are equal to \((p_m, q_m, r_m)\), for \( n = m \).

The proof is completed easily by plugging the dynamics of the state process, (1), into (36).

The following corollary provides a more explicit recursive formula for the value function and optimal control. In particular, it states that the value function of an agent at any time remains linear in \( s \), in both positive and negative half lines (with possibly different slopes).

**Corollary 2.** Assume that an admissible LOB \( \nu \) has an optimal control \((\hat{p}, \hat{q}, \hat{r})\). Then, for any \((s, \alpha) \in \mathbb{S}\), the following holds \( \mathbb{P} \)-a.s., for all \( n = 0, \ldots, N - 1 \):

\[ \cdots \left( 1 \{ q_n \geq 0, D_n^+(p_n) > \nu^+((\infty, p_n)) \} + 1 \{ q_n < 0, D_n^-(p_n) > \nu^-((p_n, \infty)) \} \right) \]

\[ + 1 \{ r_n = 1 \} \left[ \hat{q}_n p_n - \hat{q}_n p_n^* + \mathbb{E}_n \nu^{\nu}(s - q_n, \alpha) \right]. \]

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1. \( V_n^\nu(s, \alpha) = s^+ \lambda_n^a(\alpha) - s^- \lambda_n^b(\alpha) \), with some adapted processes \( \lambda_n^a(\alpha) \) and \( \lambda_n^b(\alpha) \), such that \( \lambda_n^a(\alpha) = p_n^a \) and \( \lambda_n^b(\alpha) = p_n^b \).

2. \( p_n^a \geq \mathbb{E}_n^a(\lambda_{n+1}^a(\alpha)) \) and \( p_n^b \leq \mathbb{E}_n^b(\lambda_{n+1}^b(\alpha)) \); 

3. if, for some \( p \in \mathbb{R} \), \( \mathbb{P}_n^a(D_{n+1}^+(p) > \nu_n^+((-\infty, p))) > 0 \), then

\[
 p \leq \mathbb{E}_n^a(\lambda_{n+1}^a(\alpha) \mid D_{n+1}^+(p) > \nu_n^+((-\infty, p))) ;
\]

4. if, for some \( p \in \mathbb{R} \), \( \mathbb{P}_n^a(D_{n+1}^-(p) > \nu_n^-((p, \infty))) > 0 \), then

\[
 p \geq \mathbb{E}_n^a(\lambda_{n+1}^a(\alpha) \mid D_{n+1}^-(p) > \nu_n^-((p, \infty))) ;
\]

5. for all \( s > 0 \),

- \( \lambda_n^a(\alpha) = \max \left\{ p_n^a, \mathbb{E}_n^a \lambda_{n+1}^a(\alpha) + \left( \sup_{p \in \mathbb{R}} \mathbb{E}_n^a \left[ (p - \lambda_{n+1}^a(\alpha)) \mathbf{1}_{\{D_{n+1}^+(p) > \nu_n^+((-\infty, p))\}} \right] \right\}^+ ,
\]

- if \( \hat{q}_n(s, \alpha) \neq 0 \) and \( \hat{r}_n(s, \alpha) = 0 \), then

\[
 \lambda_n^a(\alpha) = \mathbb{E}_n^a \lambda_{n+1}^a(\alpha) + \sup_{p \in \mathbb{R}} \mathbb{E}_n^a \left[ (p - \lambda_{n+1}^a(\alpha)) \mathbf{1}_{\{D_{n+1}^+(p) > \nu_n^+((-\infty, p))\}} \right] ,
\]

and \( p = \hat{p}_n(s, \alpha) \) attains the above supremum,

- if \( \hat{q}_n(s, \alpha) = 0 \) and \( \hat{r}_n(s, \alpha) = 0 \), then \( \lambda_n^a(\alpha) = \mathbb{E}_n^a \lambda_{n+1}^a(\alpha) \),

- if \( \hat{r}_n(s, \alpha) = 1 \), then \( \lambda_n^a(\alpha) = p_n^a \).

6. for all \( s < 0 \),

- \( \lambda_n^b(\alpha) = \min \left\{ p_n^b, \mathbb{E}_n^b \lambda_{n+1}^b(\alpha) - \left( \sup_{p \in \mathbb{R}} \mathbb{E}_n^b \left[ (\lambda_{n+1}^b(\alpha) - p) \mathbf{1}_{\{D_{n+1}^-(p) > \nu_n^-((p, \infty))\}} \right] \right\}^+ ,
\]

- if \( \hat{q}_n(s, \alpha) \neq 0 \) and \( \hat{r}_n(s, \alpha) = 0 \), then

\[
 \lambda_n^b(\alpha) = \mathbb{E}_n^b \lambda_{n+1}^b(\alpha) - \sup_{p \in \mathbb{R}} \mathbb{E}_n^b \left[ (\lambda_{n+1}^b(\alpha) - p) \mathbf{1}_{\{D_{n+1}^-(p) > \nu_n^-((p, \infty))\}} \right] ,
\]

and \( p = \hat{p}_n(s, \alpha) \) attains the above supremum,

- if \( \hat{q}_n(s, \alpha) = 0 \) and \( \hat{r}_n(s, \alpha) = 0 \), then \( \lambda_n^b(\alpha) = \mathbb{E}_n^b \lambda_{n+1}^b(\alpha) \),

- if \( \hat{r}_n(s, \alpha) = 1 \), then \( \lambda_n^b(\alpha) = p_n^b \).

**Proof:**

Let us plug the piecewise-linear form of the value function into (35):

\[
 V_n^\nu(s, \alpha) = \esssup_{p_n,q_n} \left\{ \mathbf{1}_{\{r_n=0\}} \left[ s^+ \mathbb{E}_n^a \lambda_{n+1}^a(\alpha) - s^- \mathbb{E}_n^b \lambda_{n+1}^b(\alpha) \right. \right. \\
 + \mathbb{E}_n^a \left[ (q_n p_n + (s - q_n)^+ \lambda_{n+1}^a(\alpha) - (s - q_n)^- \lambda_{n+1}^b(\alpha) - s^+ \lambda_{n+1}^a(\alpha) + s^- \lambda_{n+1}^b(\alpha)) \cdot \right. \left. \mathbf{1}_{\{q_n \geq 0, D_{n+1}^+(p_n) > \nu_n^+((-\infty, p_n))\}} + \mathbf{1}_{\{q_n < 0, D_{n+1}^-(p_n) > \nu_n^-((p_n, \infty))\}} \right] \\
 \left. \right\}
\]
Proof: Assume that, for any simple $\epsilon_V$, the conditional $L$-norm and the proximity of expectations of certain functions of the random variables $\epsilon_n$ do not attain the supremum, they can be improved, so that $(p_n, q_n)$ increase by no more than a fixed constant.

Moreover, the essential supremum can be replaced by the supremum over all deterministic $(p_n, q_n, r_n) \in \mathbb{R}^2 \times \{0, 1\}$. To see the latter, it suffices to assume that the supremum is not attained by the optimal strategy (with positive probability), and construct a superior strategy via the standard measurable selection argument (cf. Corollary 18.27 and Theorem 18.26 in [2]), which results in a contradiction. It is easy to see that, for any fixed $(p_n, r_n, s)$, the above function is piece-wise linear in $q_n$, with the slope changing at $q_n = 0$ and $q_n = s$. Hence, for a finite maximum to exists, the slope of this function must be nonnegative, at $q_n \to -\infty$, and non-positive, at $q_n \to \infty$. This must hold for any $(p_n, r_n, s)$, to ensure that the value function of an agent is finite: otherwise, an agent can scale up her position to increase the value function arbitrarily. Considering $r_n = 1$, we obtain condition 2 of the corollary. The case $r_n = 0$ yields conditions 3 and 4. Notice also that the maximum of the aforementioned function is always attained at $q_n = 0$ or $q_n = s$. Considering all possible cases: $r_n = 0, 1, q_n = 0, s$, then we obtain the recursive formulas for $\lambda_n^a$ and $\lambda_n^b$ (i.e. conditions 5 and 6 of the corollary). In addition, as the optimal $q_n$ takes values 0 and s, it is easy to see that the piece-wise linear structure of the value function in $s$ is propagated backwards, and, hence, condition 1 of the corollary holds.

It is also useful to have a converse statement.

Corollary 3. Consider an admissible LOB $\nu$ and admissible control $(\hat{p}, \hat{q}, \hat{r})$, such that $\hat{q}_n(s, \alpha) \in \{0, s\}$. Assume that, for any $\alpha \in \Lambda$ and any $n = 0, \ldots, N$, there exists a progressively measurable random function $V^*(\cdot, \alpha)$, such that, for any $s \in \mathbb{R}$, $\hat{p}, \hat{q}, \hat{r}, V^*$ satisfy the conditions 1–6 of Corollary 2. Then, $(\hat{p}, \hat{q}, \hat{r})$ is an optimal control for the LOB $\nu$.

Proof: It suffices to revert the arguments in the proof of Corollary 2, and recall that $\hat{q}$ can always be chosen to be equal to 0 or $s$, without compromising the optimality.

11 Appendix B

Proof of Lemma 1. The following lemma shows that the normalized price increments are close to Gaussian in the conditional $L^2$ norm.

Lemma 10. Let Assumptions 1, 2, 3, 4 hold. Then, there exists a deterministic function $\epsilon(\cdot) \geq 0$, such that $\epsilon(\Delta t) \to 0$, as $\Delta t \to 0$, and, $\mathbb{P}$-a.s., for all $\alpha \in \Lambda$ and all $n = 1, \ldots, N$, we have:

$$\mathbb{E}_{n-1}^\alpha\left[\left(\frac{\xi_n}{\sqrt{\Delta t}} - \sigma_{t_{n-1}}(W_{t_n}^\alpha - W_{t_{n-1}}^\alpha)/\sqrt{\Delta t}\right)^2\right] \leq \epsilon(\Delta t).$$

Proof:

$$\xi_n/\sqrt{\Delta t} - \sigma_{t_{n-1}}(W_{t_n}^\alpha - W_{t_{n-1}}^\alpha)/\sqrt{\Delta t} = \frac{1}{\sqrt{\Delta t}} \int_{t_{n-1}}^{t_n} \mu_s^\alpha ds + \frac{1}{\sqrt{\Delta t}} \int_{t_{n-1}}^{t_n} (\sigma_s - \sigma_{t_{n-1}})dW^\alpha_s.$$ 

Then, using Assumptions 2, 4 and Itô’s isometry, we obtain the statement of the lemma.

The next lemma connects the proximity in terms of $L^2$ norm and the proximity of expectations of certain functions of the random variables. This result would follow trivially from the classical theory, but, in the present case, we require additional uniformity – hence, a separate lemma is needed (whose proof is, nevertheless, quite simple).

14The admissibility constraint does not cause any difficulties here, as, in the case where $(p_n, q_n, r_n)$ do not attain the supremum, they can be improved, so that $(p_n, q_n)$ increase by no more than a fixed constant.
Lemma 11. For any constant $C > 1$, there exists a deterministic function $\gamma(\cdot) \geq 0$, s.t. $\gamma(\varepsilon) \to 0$, as $\varepsilon \to 0$, and, for any $\varepsilon > 0$, $\sigma \in [1/C, C]$, and any random variables $\xi$ and $\eta \sim N(0, \sigma^2)$, satisfying $\mathbb{E}(\xi - \eta)^2 \leq \varepsilon$, the following holds for all $p \in \mathbb{R}$:

(i) $|(p \lor 1)| \mathbb{P}[\xi > p] - \mathbb{P}[\eta > p]| \leq \gamma(\varepsilon)$

(ii) $|\mathbb{E}[\chi_{\{\xi > p\}}] - \mathbb{E}[\chi_{\{\eta > p\}}]| \leq \gamma(\varepsilon)$

Proof: (i) Note that
\[
|\mathbb{E}[\chi_{\{\xi > p\}}] - \mathbb{E}[\chi_{\{\eta > p\}}]| \leq \left| \mathbb{E}\left[ (\xi - \eta) \chi_{\{\xi > p\}} \right] \right| + \left| \mathbb{E}\left[ \eta \chi_{\{\xi > p\}} - \chi_{\{\eta > p\}} \right] \right|
\]
\[
\leq \sqrt{\varepsilon} + \|\eta\|_2 \sqrt{\mathbb{P}[\xi > p, \eta \leq p] + \mathbb{P}[\xi \leq p, \eta > p]},
\]
and
\[
\mathbb{P}[\xi > p, \eta \leq p] \leq \mathbb{P}[p \geq \eta \geq p - \sqrt{\varepsilon}] + \mathbb{P}[|\xi - \eta| > \sqrt{\varepsilon}] \leq M \sqrt{\varepsilon} + \frac{\mathbb{E}(\xi - \eta)^2}{(\sqrt{\varepsilon})^2} \leq (M + 1) \sqrt{\varepsilon},
\]
where we used the fact that $\eta$ has a density bounded by a fixed constant $M$. We can similarly show that $\mathbb{P}[\xi \leq p, \eta > p] \leq (M + 1) \sqrt{\varepsilon}$. The resulting estimates yield the statement of the lemma.

Taking $\varepsilon(\Delta t) = \gamma(\varepsilon(\Delta t))$ and applying the above lemmas, we get the statement of Lemma 1, with $(W^\alpha_{t\leftarrow} - W^\alpha_{t\leftarrow})/\sqrt{\Delta t}$ in place of $\eta_0$. Finally, we note that the laws of the two random variables coincide under $\mathbb{P}_{\alpha n}$, and the statement depends only on these laws. The last statement of Lemma 1 follows from the fact that Lemma 11 is stable under analogous substitution.

References


