Simulation of Implied Volatility Surfaces via Tangent Lévy Models

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Abstract

In this paper, we implement and test a market-based model for European-type options, based on the tangent Lévy models proposed in [4] and [3]. As a result, we obtain a method for generating Monte Carlo samples of future paths of implied volatility surfaces. These paths and the surfaces themselves are free of arbitrage, and are constructed in a way that is consistent with the past and present values of implied volatility. We use a real market data to estimate the parameters of this model and conduct an empirical study, to compare the performance of market-based models with the performance of classical stochastic volatility models. We choose the problem of minimal-variance portfolio choice as a measure of model performance and compare the tangent Lévy model to the SABR model. Our study demonstrates that the tangent Lévy model does a much better job at finding a portfolio with smallest variance. In addition, its prediction of the return variance is more reliable, and the portfolio weights are more stable. To the best of our knowledge, this is the first example of empirical analysis that provides a convincing evidence of the superior performance of the market-based models for European options using real market data.

1 Introduction

The existence of liquid markets for equity and volatility derivatives, as well as a well-developed over-the-counter market for exotic derivatives, generates a need for a modeling framework that is consistent across time and across financial instruments. Within this framework, once a model is chosen so that it matches both the present prices of liquid instruments and their past dynamics, it is expected to produce more realistic results for the problems of pricing and hedging of exotic instruments. In addition, such models can be used to quantify the risk embedded in portfolios of derivative contracts. Needless to say, evaluating and managing the risk of such portfolios is crucial for proper functioning of the financial markets: recall, for example, that VIX index, itself, is a portfolio of European options written on S&P 500.

In this paper we investigate an arbitrage-free modeling framework for multiple European-type options written on the same underlying, which is consistent across time and products. In particular, this framework allows to resolve one of the nagging challenges of quant groups supporting equity trading: i.e. how to generate realistic Monte Carlo scenarios of implied volatility surfaces which are consistent with present and historical observations? As mentioned above, such models can be used to address the problems of pricing, hedging and risk management. Herein, we implement several such models using real market data and conduct a numerical experiment which demonstrates clearly the advantages of this modeling approach.

The attempts to model the dynamics of implied volatility surface directly can be dated back as early as the “sticky smile model” and the “sticky delta model” (also known as “floating smile model”) (see Section 6.4 of [23] for the definitions). As an improvement of the two models, Cont et al. later proposed a multi-factor model of implied volatility surface in [7] and [8], where they applied a Karhunen-Loève decomposition on the daily variations of implied volatilities. It turns out that the first three eigenvectors could explain most of the daily variance, and a mean-reverting factor model based on the three eigenvectors is then constructed for future implied volatility surface. The major issue with these early attempts is that the proposed models for the dynamics of implied volatility are either too restrictive, not allowing to match the historical evolution of implied volatility, or too loose, so that they may contain arbitrage opportunities. While the importance of the first issue for any time-series analysis is very clear, the second one deserves a separate discussion. Indeed, what do we mean by arbitrage opportunities in a model for implied volatility and why do we need to avoid it? There are two types of arbitrage opportunities we refer to: static and dynamic. A given implied volatility surface contains static arbitrage if it is impossible to
obtain such a surface in any arbitrage-free model for the underlying. The fact that not every surface can be an arbitrage-free implied volatility simply follows from the well-known static no-arbitrage restrictions on the shape of a call price surface: e.g. monotonicity and convexity in strikes, etc (cf. [10] and [11]). Notice that a violation of any of these conditions leads to an obvious arbitrage opportunity which is very easy to implement, hence, it is natural to assume that every implied volatility surface is free of static arbitrage. This, in turn, implies that any realistic simulation algorithm for future implied volatility surfaces has to produce surfaces that are arbitrage-free: otherwise, the algorithm generates outcomes that are simply impossible. The static no-arbitrage conditions are rather difficult to state explicitly, in terms of the implied volatility surface itself (without mapping it to a call or put price surface first). Nevertheless, it is not hard to deduce from the existing necessary conditions (cf. [20]) that the set of arbitrage-free implied volatility surfaces forms a “thin” set in the space of all (regular enough) functions of two variables. Hence, it is a non-trivial task to construct a modeling framework that excludes static arbitrage in the implied volatility surfaces. The dynamic arbitrage adds to this problem, and it refers to a restriction on the evolution (i.e. the time increments) of implied volatility surface, rather than its values at a fixed moment in time. This restriction follows from the same arbitrage considerations for option prices. However, its associated arbitrage strategies are not as straightforward as in the case of static arbitrage. In addition, the simulated implied volatility surfaces that contain only dynamic arbitrage are, typically, very close to the ones that are arbitrage-free, when the time horizon is small (it is related to the fact that dynamic arbitrage only changes the drift term of the implied volatility, which is much smaller than the diffusion term, for small times). This is why, eliminating the dynamic arbitrage in a model for implied volatility surface is often viewed as a “second priority” for risk management. Nevertheless, we believe that a good model should exclude both types of arbitrage, in order to produce realistic dynamics of implied volatility surface (for risk management) and eliminate the possible arbitrage opportunities (for pricing).

We have already mentioned that it is not a trivial task to construct a model of implied volatility that excludes arbitrage opportunities. In fact, when trying to model the surface directly, the first challenge that one faces is: how to describe the space possible implied volatility surfaces? Note that, as discussed above, the existing characterizations of arbitrage-free implied volatility surfaces are rather implicit. In addition, if the resulting space is not an open subset of any linear space (which it is not), what kind of mathematical tools can be used to describe evolution in space? Recall, for example, that all statistical models of time-series are defined on linear spaces (or those that can be easily mapped in to a linear space). Hence, it appears natural to map the space of possible implied volatility surfaces to an open set in a linear space, and then proceed with the construction of arbitrage free models. Such mapping became known as a code-book mapping, and it turns out that it can be constructed by means of the so-called tangent models (cf. [2], [4], [3]). The concept of a tangent model is very close to the method of calibrating a model for underlying to the target derivatives’ prices (in the present case, European options calls). Consider a family of arbitrage-free models for the underlying, \( \mathcal{M}(\theta) \), parameterized by \( \theta \), taking values in a “convenient” set \( \Theta \) (an open set of a linear space). For any given surface of option prices (or, equivalently, any given implied volatility surface), we can try to calibrate a model for this family to a given surface of option prices (or, equivalently, to a given implied volatility surface). In, other words, we attempt to find \( \theta \in \Theta \) such that:

\[
C^0(T, K) = C(T, K),
\]

for all given maturities \( T \) and strikes \( K \), where \( C(T, K) \) is the given call price, and \( C^0(T, K) \) is the call price produced by the model \( \mathcal{M}(\theta) \). If the above calibration problem has a unique solution, we obtain a one-to-one correspondence between the call price surfaces and the models in a chosen family: \( \theta \leftrightarrow C^0 \). For every call price surface \( C = C^0 \), the associated (calibrated) model \( \mathcal{M}(\theta) \) is called a tangent model. Notice that \( C^0 \) is always arbitrage-free, hence, we obtain the desired code-book mapping \( C = C^0 \mapsto \theta \). Now, the problem of static arbitrage has been resolved, and one simply needs to prescribe the distribution of a stochastic process \( \{ \theta_t \} \), taking values in a convenient set \( \Theta \), in order to obtain a model for the dynamics of call prices \( \{ C_t = C^0_t \} \), and, in turn, the dynamics of implied volatility surface. Finally, one needs to characterize all possible dynamics of \( \theta_t \) that produce no dynamic arbitrage in the associated call prices \( \{ C^0_t \} \). An interested reader is referred to [3], for a more detailed description of this general algorithm, and, for example, to [2], [4], [13], [30], [19], [24], for the analysis of specific choices of the families of models \( \{ \mathcal{M}(\theta) \} \).

The idea of modeling prices of derivative contracts directly dates back to the work of Heath, Jarrow and Morton [14], who analyzed the dynamic of bond prices along with the short interest rate. Such models have become known as the market-based models (or simply market models), as opposed to the classical spot models, since the former are designed to capture the evolution of the entire market, including the liquid derivatives. This approach has been extended to more general mathematical settings, as well as to other derivatives’ markets. The list of relevant works includes [11], [25], [26], [28], [27], [12], in addition to those mentioned in the previous paragraph. Even though

\(^3\)It is important to remember that any such model serves only as a static description of option prices, and it does not describe their dynamics!
the notions of code-book and tangent models never appear in these papers, almost all of them follow the algorithm outlined in the previous paragraph (and described in more detail in [4]), in order to construct a market-based model.

Even though various code-books for implied volatility surface (or, equivalently, for call price surface) have been proposed and the corresponding arbitrage-free dynamics have been characterized, it was not until very recently that some of these models were implemented numerically. As is shown in the rest of the paper, the lack of such results is not a surprise given the complexity of the models. So far, the numerical implementations are mostly based on tangent Lévy models proposed in [4] and [3]: as the name suggests, this corresponds to a code-book which is constructed using non-homogeneous Lévy (or, additive) models as the tangent models. Karlsson [16] implements a class of tangent Lévy models with absolutely continuous Lévy densities and no continuous martingale component. Zhao [30] and Leclercq [19], on contrary, implemented the tangent Lévy models whose Lévy measure is purely atomic in the space variable. As opposed to [30], the work of Leclercq [19] allows for tangent models with continuous martingale component and includes options with multiple maturities, but it does require that the Lévy density possess certain symmetry, which may limit the ability of the model to capture the skew of the implied smile. All of the works [16], [30], [19] estimate the parameters of the model from real market data. In addition, [19] conducts a numerical experiment comparing the performance of a market-based model to a classical spot model. The actual results of this experiment, however, do not provide a convincing evidence in favor of the market-based approach. We believe that the later is simply due the choice of experiment and to the deficiency of the theory, and we intend to demonstrate it in the present work.

The purpose of this paper is to propose an implementation method for a class of tangent Lévy models and to test its performance using market data. These method provides an algorithm for simulating future arbitrage-free implied volatility surfaces, which are consistent with both present and past observations. Our method is similar to the one used in [16], but with a different “dynamic fitting” part. However, the most important original contribution of this paper is the numerical experiment which uses real market data to demonstrate clearly the advantages of market-based models for implied volatility (or, option prices), as compared to the classical spot models. To the best of our knowledge, this is the first convincing empirical analysis that justifies the use of market-based approach for modeling option prices (or, equivalently, for modeling the implied volatility surface).

The rest of the paper is organized as follows. Section 2 starts by reviewing the work on tangent Lévy models with continuous Lévy density and continuous martingale component, developed in [3]. We, then, proceed to describe the parametric estimation of the parameters of this model, which is partially based on double exponential jump processes. The estimated model is then tested against a popular classical model in a portfolio optimization problem in Section 3. Section 4 concludes the paper by highlighting the main contributions and the future work. Appendices A–B contain technical proofs and derivations, Appendix C contains all tables and graphs.

2 Double exponential tangent Lévy models

2.1 Model setup and consistency conditions

In this subsection, we review and update the results of [4], which serve as a foundation for the analysis in subsequent sections. Herein, we assume that the interest and dividend rates for the underlying asset are zero. In the implementation that follows, we discount the market data accordingly, to comply with this assumption. As in [3], we denote by \((S_t)_{t \geq 0}\) a stochastic process representing the underlying price, and assume that the true dynamics of \(S\) under the pricing measure \(Q\) are given by:

\[
S_t = S_0 + \int_0^t \int_{\mathbb{R}} S_u - (e^x - 1) [M(dx, du) - K_u(x) dx du].
\] (2.1)

Here, \(M\) is a general integer-valued random measure (not necessarily a Poisson measure!), whose compensator is \(K_u(x) dx du\), where \((K_t)_{t \geq 0}\) is a predictable stochastic process taking values in the function space \(\mathcal{S}_0\), defined in (2.1).

For any fixed time \(t \geq 0\) and a given value of \(S_t\), a stochastic process \((\bar{S}_T)_{T \geq t}\) is said to be tangent to the true model \((S_t)_{t \geq 0}\) if the time-\(t\) prices of all European call options written on \(S\) can be obtained by pretending the future risk-neutral evolution of the index value is instead given by \((\bar{S}_T)_{T \geq t}\) from \(t\) on. Throughout this section, for any fixed \(t \geq 0\), we assume that the tangent processes \(\bar{S}\) is in the form

\[
\bar{S}_T = S_t + \int_t^T \int_{\mathbb{R}} \bar{S}_u - (e^x - 1) [N_t(dx, du) - \kappa_t(u, x) dx du],
\] (2.2)
for \( T \in [0, T] \), where \( N_t(dx, du) \) is a Poisson random measure associated with the jumps of \( \tilde{S} \) whose compensator is given by a deterministic measure \( \kappa_t(u, x)dxdu \). Notice that the law of \( \tilde{S} \) is uniquely determined by \((S_t, \kappa_t)\). Let \( C^S_t, \kappa_t(T, x) \) denote the option prices generated by \((\tilde{S}_u)_{u \geq t}\), i.e

\[
C^S_t, \kappa_t(T, x) := \mathbb{E}[(\tilde{S}_T - e^x)^+|\tilde{S}_t = S_t], \quad \forall T \geq t, \; x \in \mathbb{R}. \tag{2.3}
\]

The concept of a tangent model, then, requires that, for each fixed \( t \in [0, T] \),

\[
C^S_t, \kappa_t(T, x) = \mathbb{E}[(S_T - e^x)^+|\mathcal{F}_t], \quad \forall T \geq t, \; \forall x \in \mathbb{R}. \tag{2.4}
\]

Thus, at each time \( t \), we obtain the code-book for call prices, given by \((S_t, \kappa_t)\). Of course, the value of the code-book may be different at a different time \( t \). Hence, we consider the dynamic tangent Lévy models characterized by a pair of stochastic processes \((S_t, \kappa_t)_{t \in [0, T]}\) that satisfies (2.4). Here, \( S \) is a positive martingale with dynamics given by (2.1); \( \kappa \) is progressively measurable positive stochastic process taking values in \( \mathcal{B} \) (cf. (5.2)). The dynamics of \( S_t \) and \( \kappa_t \) are given by

\[
S_t = S_0 + \int_0^t \int_{\mathbb{R}_+} S_u - (e^x - 1)|M(dx, du) - K_u(x)dxdu|, \tag{2.5}
\]

\[
\kappa_t(T, x) = \kappa_0(T, x) + \int_0^t \alpha_u(T, x)du + \sum_{n=1}^m \int_0^t \beta^n_u(T, x)dB^n_u,
\]

where \((\alpha_t)_{t \in [0, T]}\) is a progressively measurable integrable stochastic process taking values in \( \mathcal{B} \), and, for each \( n \in \{1, \ldots, m\}, (\beta^n_t)_{t \in [0, T]}\) is a progressively measurable square integrable stochastic process taking values in \( \mathcal{H} \) (cf. (5.2)).

Notice that (2.5) defines the dynamics of the code-book \((\tilde{S}_t, \kappa_t)_{t \in [0, T]}\), but it does not ensure that it does, indeed, produce tangent models at each time \( t \); in other words, there is no guarantee that (2.4) holds. Thus, additional “consistency” conditions have to be enforced to obtain models which are, indeed, tangent to the true underlying process. As shown in [4], this consistency is, in fact, equivalent to the fact that call prices generated by these tangent models are free of dynamic arbitrage. In order to present the main consistency result, we state the following regularity assumptions on \( \beta \).

**Assumption 1.** For each \( n \leq m \), almost surely, for almost every \( t \in [0, T] \), we have:

\[
\text{RA1} \quad \sup_{t \in [t, T]} \int_{-1}^1 |\beta^n_u(T, x)| dx < \infty
\]

\[
\text{RA2} \quad \text{For every } T \in [t, T], \text{ the function } \beta^n_u(T, \cdot) \text{ is absolutely continuous on } \mathbb{R} \setminus \{0\}.
\]

\[
\text{RA3} \quad \text{For any } T \in [t, T], \int_0^T (e^x - 1) \beta^n_u(T, x) = 0.
\]

Finally, we introduce some extra notation and formulate the consistency result, which is a simple corollary of Theorem 12 in [4].

\[
\beta^n_u(T, x) := \int_0^T \beta^n_u(u, x)du. \tag{2.6}
\]

**Theorem 1.** (Carmona-Nadtochiy 2012) Assume that \((S_t)_{t \in [0, T]}\) is a true martingale, \( \beta \) satisfies the above regularity assumptions RA1-RA4, and \( \kappa_t(T, x) \geq 0 \), almost surely for all \( t \in [0, T] \) and almost all \((T, x) \in [t, T] \times \mathbb{R} \). Then the processes \((S_t, \kappa_t)_{t \in [0, T]}\) satisfying (2.5) are consistent, in the sense that (2.4) holds, and if only if the following conditions hold almost surely for almost every \( x \in \mathbb{R} \) and \( t \in [0, T] \), and all \( T \in (t, T] \):

1. **Drift restriction:**

\[
\alpha_t(T, x) = - \sum_{n=1}^m \left\{ \int_{\mathbb{R}} \beta^n_u(T, y) \beta^n_u(T, x - y)dy - \beta^n_t(T, x) \cdot \int_{\mathbb{R}} \beta^n_u(T, z)dz \right\}.
\]

2. **Compensator specification:** \( K_t(x) = \kappa_t(t, x) \).
Theorem 1, along with equations (2.5), provide a general method for constructing a market-based model for call prices (i.e., an arbitrage-free dynamic model for implied volatility surface). Indeed, choosing \((\beta^1, \ldots, \beta^m)\) in \([0, T]\), we use the drift restriction in Theorem 1 and the second equation in (2.5) to generate the paths of \((\kappa_t)_{t \in [0, T]}\). Finally, to generate the paths of \((S_t)_{t \in [0, T]}\), one can use the compensator specification in Theorem 1 and the first equation in (2.5), after representing the random measure \(M\) through its compensator \(K\) and a Poisson random measure \(N\) (as shown in [4]). However, in the present paper we avoid simulating \((S_t)_{t \in [0, T]}\) at all, by simply noticing that

\[
\frac{1}{S_t} C^{S_t, \kappa_t}(T, x + \log S_t) = \mathbb{E} \left[ \left( \frac{S_T}{S_t} - e^x \right)^+ | S_t = S_t \right] = \mathbb{E} \left[ (\hat{S}_T - e^x)^+ | \hat{S}_t = 1 \right] = C^{1, \kappa_t}(T, x),
\]

\[
\frac{1}{S_t} C^{S_t, bs}(T, x + \log S_t; \sigma) = C^{1, bs}(T, x; \sigma),
\]

where \(C^{S_t, bs}(T, x)\) is the Black-Scholes price at time \(t\) of a call option with maturity \(T\) and strike \(e^x\) given that the level of underlying is at \(S_t\) and the volatility is \(\sigma\). At any time \(t\), regardless of the value of \(S_t\), if we find the level of \(\sigma\) that makes the right hand sides of the two equations above coincide, then the option prices in the left hand sides have to coincide as well. This means that we can obtain the implied volatility of \(C^{S_t, \kappa_t}\); in the maturity and log-moneyness variables, by computing the corresponding implied volatility of \(C^{1, \kappa_t}\), for which we do not need to generate \(S_t\).

2.2 Implied volatility simulation with tangent Lévy models

We first introduce the general framework of the simulation procedure. Our procedure has two stages, estimation and simulation. The estimation stage, where the additive density of the tangent process as well as its dynamics are fitted to market data, is performed in two steps:

- **Static fitting.** In static fitting, the additive density \(\kappa_t\) for each day \(t\) is obtained by least squares optimization which minimizes the squared difference between model prices and actual market prices. Notice that for any given day \(t\), \(\kappa_t\) is fixed and there is no dynamics involved, which explains the term ‘static’.

- **Dynamic fitting.** In dynamic fitting, we recover the dynamics of the time series \((\kappa_t)\). In view of the drift restriction in Theorem 1, this boils down to determining the volatility terms \(\{\beta^m\}_{m=1}^M\). This is done by applying the Principle Components Analysis to the time series of \((\kappa_t)\).

Once the estimation is completed, we generate the future paths of \((\kappa_t)\) using Euler scheme Monte Carlo applied to the second equation in (2.5). From the simulated additive densities, we compute call prices \(C^{1, \kappa_t}\) and, then, implied volatilities by inverting the Black-Scholes formula.

Within the general framework, the simulation stage is generic, but the static part of the estimation stage can be quite different depending on the specific subclass of tangent Lévy densities \(\kappa(u, x)\) that we fit to option price at any given time. In this section, we implement the procedure with the Lévy densities arising from the double exponential Lévy models proposed by Kou in [17]. The small number of parameters in double exponential models and the availability of an analytical pricing formula for call options make the resulting family of tangent Lévy models fairly easy to calibrate.

2.3 Market data

We use SPX (S&P 500) call option prices provided by OptionMetrics, an option database containing historical prices of options and their underlying instruments. Throughout the paper, we use the option data from two time periods: Jan. 2007 - Aug. 2008 and Jan. 2011 - Dec. 2012. Table 1 gives a quick summary of the two periods. We cut off the first period at August 2008 to reduce the impact of the financial crisis.

On each day of a period, we only keep the options with time to maturity less than one year, whose best closing bid price and best closing offer price are both available, and take the average of the two prices as the option price. To ensure the validity of all prices, the contracts with zero open interest are excluded. As a result, there are roughly 10 to 80 call contracts with valid prices available for each maturity. The log-moneyess (more precisely, the put log-moneyess, defined as \(\log(K/S_t)\)) of these call options ranges roughly from -0.3 to 0.1, varying for different \(t\) and \(T\). Our calibration also requires dividend and interest rate data available on OptionMetrics and the homepage of U.S. Department of Treasury, respectively. This dividend yield is recovered from option prices via put-call parity.
with the method proposed in [1]. On day \( t \), we denote the dividend yield by \( q_t \), and the risk-free rate between \( t \) and \( T \) by \( r_{t,T} \). To simplify our implementation, we perform a simple transformation on the market data so that we can assume that the interest and dividend rates are both zero from now on:

\[
C_{mkt}^{mt}(T, x) = e^{r(T-t)} C_{mkt}^{mt}(T, \bar{x}), \quad \text{with} \quad x = \bar{x} - (r_{t,T} - q_t)(T - t),
\]

where \( C_{mkt}^{mt}(T, \bar{x}) \) is the market price of a call option with maturity \( T \) and strike \( x \). The adjusted call prices \( C_{t}^{mt}(T, x) \), corresponding to maturity \( T \) and strike \( x \), are then consistent with the assumption of zero interest and dividend rates (i.e. they do not contain arbitrage under these assumptions). In a similar way, we define the adjusted bid and ask prices, \( C_{mkt,b}^{mt} \) and \( C_{mkt,a}^{mt} \).

In this section, we will perform the calibration of a tangent Lévy model on the time span from Jan. 3, 2007 to Dec. 31, 2007, denoted by \([t_0, T]\). In Section 3 data from both periods will be used to test the performance of the tangent Lévy model.

### 2.4 Static fitting

Before we proceed with the static fitting, let us first have a quick review of the double exponential model. In such a model, the logarithm of underlying follows a pure jump Lévy process whose jump sizes have a double exponential distribution. More specifically, assuming no diffusion term, the dynamics of the underlying are given by

\[
d\bar{S}_t = \mu \bar{S}_t \ dt + \bar{S}_t d \left( \sum_{i=1}^{N_t} (\exp(Y_i) - 1) \right),
\]

where \( \mu \) is the drift term, \( N_t \) is a Poisson process with rate \( \lambda \). \( \{Y_i\} \) is a sequence of i.i.d. random variables with asymmetric double exponential distribution, independent of \( N_t \). The density of an asymmetric double exponential distribution is given by

\[
f_Y(y) = p \cdot \lambda_1 e^{-\lambda_1 y} 1_{y \geq 0} + q \cdot \lambda_2 e^{\lambda_2 y} 1_{y < 0},
\]

where \( p, q \geq 0 \), \( p + q = 1 \) represent the probabilities of positive and negative jumps, and \( \lambda_1 > 1, \lambda_2 > 0 \) are the parameters of the two exponential distributions. In other words, a double exponential model is a martingale model for the underlying whose logarithm is a pure jump Lévy process, with the Lévy density

\[
\eta(x) = \lambda(p \cdot \lambda_1 e^{-\lambda_1 x} 1_{x \geq 0} + q \cdot \lambda_2 e^{\lambda_2 x} 1_{x < 0}).
\]

One of the advantages of double exponential models is the availability of analytical pricing formulas for European call options, which could greatly simplify the calibration. [17] gives the pricing formula for double exponential models with a diffusion term. A minor modification of the derivation in [17] gives us the pricing formula in absence of the diffusion term, as shown in the lemma below (its proof is given in Appendix B).

**Lemma 1.** Under the assumptions of zero interest and dividend rates, assume, in addition, that the underlying process \( S \) follows a double exponential process with Lévy density given by (2.11), under the risk-neutral probability measure. Then, the price of a European call option with strike \( K \) and maturity \( T \) is given by

\[
C_{t}^{\lambda_1, \lambda_2; \eta}(T, \log K) = S_t \Psi(-\lambda_1, \lambda_2, \lambda_1^*, \lambda_2^*, \log(K/S_t), T - t)
= K \Psi(-\lambda_1, \lambda, p, \lambda_1, \lambda_2, \log(K/S_t), T - t),
\]

where

\[
p^* = \frac{p}{1 + \zeta}, \quad \lambda_1^* = \frac{\lambda_1}{1 - 1}, \quad \lambda_2^* = \lambda_2 + 1,
\]

\[
\lambda^* = \lambda(\zeta + 1), \quad \zeta = \frac{p\lambda_1}{\lambda_1^* - 1} + \frac{q\lambda_2}{\lambda_2 + 1} - 1,
\]

and the function \( \Psi \) is given by:

\[
\Psi(\mu, \lambda, p, \lambda_1, \lambda_2; a, T)
= \pi_0 1_{a - \mu T \leq 0} + \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} P_{n,k} \left[ \sum_{i=0}^{k-1} \frac{(\lambda_1(a - \mu T))^i}{i!} e^{-\lambda_1(a - \mu T)} 1_{a - \mu T \geq 0} + 1_{a - \mu T < 0} \right] \\
+ \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} Q_{n,k} \left( 1 - \sum_{i=0}^{k-1} \frac{(-\lambda_2(a - \mu T))^i}{i!} e^{\lambda_2(a - \mu T)} \right) 1_{a - \mu T < 0},
\]

(2.13)
with
\[ \pi_n = \frac{e^{-\lambda T} (\lambda T)^n}{n!} \]
and
\[
P_{n,k} = \sum_{i=k}^{n-1} \binom{n-k}{i} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^{n-i-i} p^i q^{n-i}, \quad 1 \leq k \leq n - 1,
\]
\[
Q_{n,k} = \sum_{i=k}^{n-1} \binom{n-k}{i} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^{n-i-i} p^{-i} q^i, \quad 1 \leq k \leq n - 1,
\]
\[ P_{n,n} = p^n, \quad Q_{n,n} = q^n. \]

For each \( T_l \), with \( l = 1, \ldots, L \), we would like to find the set of parameters \( \{ \lambda, \lambda_1, \lambda_2, p \} \) that minimizes the difference between the market and the model prices. For practical reasons, we will work with time values instead of options prices. The market time value and the model time value are calculated as follows
\[
V_t^{mkt,j}(T_l) = C_t^{mkt}(T_l, e^{x_j}) - (S_t - e^{x_j})^+, \quad V_t^{\lambda_1, \lambda_2, p,j}(T_l) = C_t^{\lambda_1, \lambda_2, p}(T_l, e^{x_j}) - (S_t - e^{x_j})^+.
\]

There are two reasons for working with time values. Firstly, the time values go to zero for very large and very small log-moneyness, which allows us to truncate the \( x \)-space with negligible numerical errors. Secondly, time values and option prices are often of different magnitudes, especially for in-the-money options, with option prices much greater than time values, hence, working with time values is likely to result in smaller numerical errors. For fixed time \( t \) and fixed maturity \( T_l \), the optimization problem can be written as
\[
\min_{\lambda > 0, \lambda_1 > 0, \lambda_2 > 0, p \in (0,1)} \sum_{j=1}^{N} \omega_j |V_t^{\lambda_1, \lambda_2, p,j}(T_l) - V_t^{mkt,j}(T_l)|^2, \tag{2.14}
\]
where \( \omega_j = |C_t^{bid}(T_l, e^{x_j}) - C_t^{ask}(T_l, e^{x_j})|^2 \) are the weights we put on different options to take into account the difference in liquidity (measured by bid-ask spread). For every fixed maturity \( T_l \), the solution of the above optimization problem, \( (\lambda^l, \lambda_1^l, \lambda_2^l, p^l) \), yields the Lévy density \( \eta(T_l, x) \) via \( \eta(T_l, x) \). Then, we search for a function \( \kappa_t(\cdot, \cdot) \), such that
\[
\eta(T_l, x) = \frac{1}{T_l - t} \int_t^{T_l} \kappa_t(u, x) du, \tag{2.15}
\]
for every maturity \( T_l \) and all \( x \in \mathbb{R} \). The resulting tangent model on day \( t \) is defined as a martingale model for the underlying whose logarithm is a pure jump additive (non-homogeneous Lévy) process, with the Lévy density \( \kappa_t(\cdot, \cdot) \). It is easy to see that the call prices produced by this model, for every maturity \( T_l \) and strike \( e^{x_j} \), coincide with the prices produced by the double exponential model, \( C_t^{X^l, \lambda_1^l, \lambda_2^l, p^l}(T_l, e^{x_j}) \). Thus, for a given \( t \), the problem of static fitting is essentially a series of optimization problems \( (2.14) \), over all maturities \( T_l \), along with the fitting problem \( (2.15) \).

At the first glance, the optimization in (2.14) seems to have four parameters. However, the following constraints will reduce the number of parameters to two in our calibration:

- To improve the stability of small-jumps intensity over time, we would like the Lévy density \( \eta(T_l, x) \) to be continuous in \( x \). The continuity at \( x = 0 \) requires
  \[
p \cdot \lambda_1 = (1 - p) \cdot \lambda_2 \Leftrightarrow \lambda_2 = \frac{p}{1 - p} \lambda_1. \tag{2.16}
  \]
- In view of the results in Section 2.1, we have to impose the symmetry condition RA3 on \( \beta^u \)'s. A simple application of Itô's lemma shows that, for the symmetry condition RA3 to hold, it suffices to choose every \( \kappa_t \), so that
  \[
  \int_{\mathbb{R}} (e^x - 1) \kappa_t(T, x) dx
  \]
is a deterministic function of $T-t$, for all times $0 \leq t < T \leq \bar{T}$. To achieve this, in view of (2.15), we need to choose every $\eta_l(T,t)$ so that the symmetry index

$$\Xi(T-t) := \int_\mathbb{R} (e^x - 1) \eta_l(T,x) dx = \lambda \left( \frac{p}{\lambda_1 - 1} - \frac{1 - p}{\lambda_2 + 1} \right)$$

(2.17)

is a deterministic function of $T-t$. This yields:

$$p = \frac{-(1 + \Xi(T-t) / \lambda)(\lambda_1 - 1)}{\Xi(T-t) / \lambda \lambda_1 - 1)^2 - 2(\lambda_1 - 1) - 1},$$

(2.18)

where $\Xi$ is a fixed (estimated a priori) function.

With the two constraints, our calibration takes only two variables: $\lambda$ and $\lambda_1$. The condition $p \in (0, 1)$ transforms to the following condition on $\lambda_1$:

$$\lambda_1 \in \begin{cases} 
(1, \infty), & \text{if } \Xi(T-t) \leq 0, \\
(1, 1 + \frac{1}{\Xi(T-t)}) , & \text{if } \Xi(T-t) > 0. 
\end{cases}$$

(2.19)

As a result, the optimization problem (2.14) can be re-written as

$$\min_{\lambda > 0, \lambda_1 \in I_{\lambda_1}} \sum_{j=1}^{N} \omega_j |V_t^{\lambda,\lambda_1,j}(T_l) - V_t^{mkt,j}(T_l)|^2,$$

(2.20)

where $I_{\lambda_1}$ is the interval defined in (2.19). The symmetry index function $\Xi(\tau)$, for all $\tau \in \mathbb{R}_+$, can be obtained on the first calibration day $t = 0$, solving a three-variable optimization problem,

$$\min_{\lambda > 0, \lambda_1 > 1, \lambda_1 \in (0, 1)} \sum_{j=1}^{N} \omega_j |V_0^{\lambda,\lambda_1,p,j}(T_l) - V_0^{mkt,j}(T_l)|^2,$$

(2.21)

and setting

$$\Xi(T_l) = \lambda \left( \frac{p}{\lambda_1 - 1} - \frac{1 - p}{\lambda_2 + 1} \right),$$

(2.22)

for every maturity $T_l$, and, finally, interpolating linearly between every $T_{l-1}$ and $T_l$. We summarize the calibration procedure for \{$\eta_l(T_l, \cdot)$\} in the following algorithm:

**Algorithm 1:** Algorithm for calibrating \{$\eta_l(T_l, \cdot)$\}

1. Preprocess the market data according to (2.8).
2. For $t = 0$, run the three-variable optimization (2.21), without the symmetry condition, for all maturities, and compute $\Xi(\cdot)$ by (2.22) and linear interpolation;
3. For the subsequent days $t \in (0, T]$, run the two-variable optimization (2.20), with already estimated $\Xi$, to obtain the time series of Lévy densities \{$\eta_l(T_l, \cdot)$\} on $[0, T]$.

Below are the calibration results. The Lévy densities $\eta$ on Jan. 3, 2007 – the first day of calibration – is obtained by the three-variable optimization (2.21). From the calibrated parameters, we compute the symmetry index $\Xi$ via (2.22), which is shown in Figure 1. With the symmetry index $\Xi$, we run the two-variable optimization (2.20) on the following day, Jan. 4, 2007, and obtain the Lévy densities $\eta$ shown in Figure 2. The corresponding time values are shown in Figure 3. We can see that the calibration results are quite precise in the sense that the time value falls between the bid and the ask values most of the time. As for the calibrated Lévy densities $\eta$, its values tend to decrease as the time to maturity increases (cf. Figure 2). The magnitude of $\Xi$ (which measures the “asymmetry” of the Lévy measure) is decreasing with maturity as well. Both results are in line with empirical findings on jump intensities and volatility skews.

Next, for every day $t$, we need to find $\kappa_t$ that satisfies (2.15). Notice that, if $\eta_l(T, x)$ is differentiable in $T$, we obtain:

$$\eta_l(T, x) + (T - t) \frac{\partial \eta_l(T, x)}{\partial T} = \kappa_t(T, x),$$

(2.23)
for each $x \in \mathbb{R}$. The relationship (2.23) can be used to back out the additive densities $(\hat{\kappa}_t)_{t \in [0,T]}$ from the calibrated Lévy densities $(\hat{\eta}_t)_{t \in [0,T]}$. However, the calibrated densities $\hat{\eta}_t(T,\cdot)$ are only defined for $T = T_1$, hence, we need to interpolate them across maturities. An analysis of the calibrated Lévy densities shows that $\hat{\eta}_t(T, x)$ generally exhibits one of the following two patterns as a function of $T$.

- For small jump sizes $x$, $\eta_t(T, x)$ decreases rapidly as $T$ increases. To ensure that the recovered $\kappa$ is non-negative, we used a combination of exponential function and power function

$$\eta_t(T, x) = c_1(T - t)^{c_2} + c_3(T - t) \exp(-c_4(T - t)) + c_5$$

(2.24)

to fit $\eta$, for any fixed $x$. The corresponding Lévy density $\kappa$ can then be computed as

$$\kappa_t(T, x) = c_1(c_2 + 1)(T - t)^{c_2} + \exp(-c_4(T - t))(2c_3(T - t) - c_3c_4(T - t)^2) + c_5.$$  

(2.25)

- For large jump sizes $x$, $\eta_t(T, x)$ increases as $T$ increases. The function used to fit this scenario is a simple polynomial function

$$\eta_t(T, x) = c_1(T - t)^4 + c_2(T - t)^3 + c_3(T - t)^2 + c_4(T - t) + c_5.$$  

(2.26)

Then, $\kappa$ is computed as

$$\kappa_t(T, x) = 5c_1(T - t)^4 + 4c_2(T - t)^3 + 3c_3(T - t)^2 + 2c_4(T - t) + c_5.$$  

(2.27)

An illustration of the two scenarios together with an example of the reconstructed $\kappa$ is shown in Figure 4.

### 2.5 Dynamic fitting

Recall that, in view of (2.5), the Lévy density $\kappa$ has the following dynamics:

$$d\kappa_t(T, x) = \alpha_t(T, x)dt + \sum_{n=1}^{m} \beta^n_t(T, x)dB^n_t.$$  

(2.28)

In the dynamic fitting, we need to assume that the time increments of $\kappa$ are stationary, which is only natural if we work with the time to maturity $\tau = T - t$ instead of the maturity $T$. Namely, we define $\hat{\kappa}_t(\tau, x) = \kappa_t(t + \tau, x)$ and its dynamics

$$d\hat{\kappa}_t(\tau, x) = \hat{\alpha}_t(\tau, x)dt + \sum_{n=1}^{m} \hat{\beta}^n_t(\tau, x)dB^n_t.$$  

(2.29)

A simple application of Itô’s formula shows that

$$\hat{\alpha}_t(\tau, x) = \alpha_t(t + \tau, x) + \frac{\partial \kappa_t(t + \tau, x)}{\partial T}$$  

and

$$\hat{\beta}^n_t(\tau, x) = \beta^n_t(t + \tau, x).$$  

(2.30)

To simulate future implied volatility surfaces, all we need are the diffusion terms $\hat{\beta}^n$’s, because the drift term $\hat{\alpha}$ can be computed from $\hat{\beta}^n$’s. We assume that $\hat{\beta}_t^n(\tau, \cdot)$’s are deterministic and constant as functions of $t$, for any $(\tau, x)$ (from a finite family of points). Then, every increment $\Delta \hat{\kappa}_t = \hat{\kappa}_t - \hat{\kappa}_{t-1}$ is a sum of a Gaussian random vector, corresponding to the diffusion part, and a vector that corresponds to the drift term (we view every surface as a vector whose entries correspond to different values of $(\tau, x)$). Notice that the distribution of the Gaussian component is completely determined by its covariance matrix, hence, we will aim to choose $\hat{\beta}^n$’s to match the estimated covariance matrix. Assuming that the drift term is bounded, it is easy to notice that the standard estimate of the covariance of $\Delta \hat{\kappa}_t$ also provides a consistent estimate of the covariance of the aforementioned Gaussian vector, asymptotically, as the length of the time increments converges to zero. In the actual computations, we use daily increments – these are small compared to the time span of the entire sample, which is one year. To fit $\hat{\beta}^n$’s to the estimated covariance matrix, it is natural to use the Principal Components Analysis (PCA), which finds the directions that explain most of the variance in the increments $\Delta \hat{\kappa}_t$. However, the PCA can not be applied directly because the number of points on the surface is close to the sample size, which is 251: for each $t$, we have call prices for 10 maturities and 21 jump sizes, which gives us 210 points on the $\kappa$ surface after static fitting. To reduce
the number of points, we pick every other maturity and the 7 jump sizes whose intensities are larger than others across time \( t \). This gives us \( 5 \times 7 = 35 \) points on the reduced surface of \( \{ \Delta \hat{\kappa}_t \}_{t \in [0,T]} \).

Applying PCA on the reduced surface, we see that the first three eigenmodes \( \{ f^n(t, x) \}_{n=1}^3 \) explain over 93\% of the daily variance of \( \hat{\kappa} \), as shown in Figure 5. To extend the values of the eigenmodes to other points (i.e. other jump sizes and maturities), we simply perform a linear interpolation. The first three eigenmodes have very unique characteristics. The first eigenmode takes the most prominent feature of \( \hat{\kappa} \) - the densities are concentrated around small jumps at very short time to maturity. This eigenmode can be understood as a combination of the “level” factor and the “slope” factor (appearing in a typical PCA result for yield curve dynamics) along both the maturity and the jump size directions. The second eigenmode shows the curvature along the jump size direction, and the third eigenmode shows the curvature along the time to maturity direction. As the eigenmodes \( \{ f^n(t, x) \}_{n=1}^3 \) are normalized, to obtain the diffusion terms \( \hat{\beta}^n \)'s, we need to multiply the eigenmodes by the loading factors:

\[
\hat{\beta}^n(t, x) = \sqrt{\lambda_n} \cdot f^n(t, x), \quad n = 1, 2, 3.
\] (2.31)

Once we have \( \hat{\beta}^n \)'s, we change the variables to pass to \( \beta^u \)'s and calculate the drift term \( \alpha \) according to (2.7). Figure 6 shows the drift term \( \hat{\alpha} \) computed according to (2.7). Notice that \( \hat{\alpha} \) can then be computed as

\[
\hat{\alpha}_t(t, x) = \alpha_t(t + \tau, x) + \frac{\partial \hat{\kappa}_t(t + \tau, x)}{\partial \tau},
\] (2.32)

where we have no problem with evaluating the partial derivative, as, in the static fitting stage, \( \kappa_t \) was interpolated across maturities.

### 2.6 Monte Carlo simulation of implied volatility surfaces

Once all the terms in the right hand side of (2.29) are estimated, we can, for example, apply and explicit Euler scheme to simulate the future Lévy densities \( \hat{\kappa}_t \). However, we need to ensure that the simulated \( \hat{\kappa}_t \)'s stay nonnegative at all times. Inspired by [4], we incorporate a scaling factor in (2.29) as follows:

\[
d\hat{\kappa}_t(t, x) = \gamma_t^2 \hat{\alpha}_t(t, x)dt + \gamma_t \sum_{n=1}^m \hat{\beta}^n(t, x) dB^u_t,
\] (2.33)

where

\[
\gamma_t = \frac{1}{\epsilon} \left( \frac{1}{\inf_{\tau \in [0,\bar{\tau}], x \in \mathbb{R}} \hat{\kappa}_t(t, x) \wedge \epsilon} \right),
\] (2.34)

with \( \epsilon = 1 e^{-6} \) and \( \bar{\tau} = 1 \). Of course, this modification changes the diffusion term of \( \hat{\kappa}_t \), which was estimated from historical data. However, the value of \( \epsilon \) is chosen to be so small that, in the historical sample, \( \gamma_t \) is always equal to one. Hence, if we use the \( \hat{\beta}^n \)'s chosen in the previous subsection, the resulting dynamics are still consistent with the past observations. It is also easy to see that, since \( \gamma_t \) is a scalar, the drift restriction (2.7) is satisfied by the new drift and volatility of \( \kappa \). Finally, this modification ensures that \( \hat{\kappa}_t \) is almost surely nonnegative for any \( t \).

To simulate future values of \( \kappa \), we apply the explicit Euler scheme to (2.33), to obtain

\[
\hat{\kappa}_{t+\Delta t}(t, x) = \hat{\kappa}_t(t, x) + \gamma_t^2 \hat{\alpha}_t(t, x) \Delta t + \gamma_t \sum_{n=1}^m \hat{\beta}^n(t, x) \Delta B^u_t,
\] (2.35)

with \( \Delta t \) being one day. Having simulated \( \hat{\kappa}_t \), we compute \( \eta_t \) via (2.15). Then, for every fixed maturity \( T \), the option prices in the model given by the Lévy density \( \eta(T, \cdot) \) can then be computed, for example, using the methods proposed in [5] or [21]. These methods are based on Fourier transform and can be implemented efficiently via numerical integration.\(^3\) In particular, in our simulation, we use the following formula to calculate future option prices:

\[
C^1_{t,T}(T, x) = 1 - e^{x^2/2} \frac{1}{\pi} \int_0^{\infty} \frac{du}{u^2 + \frac{1}{4}} \text{Re} \left[ \exp (-iux) \phi_t \left( T, u - \frac{i}{2} \right) \right],
\] (2.36)

\(^3\)Please note that we cannot use (2.12) to calculate option prices, because, even though the calibrated Lévy densities \( \{ \eta(T, \cdot) \} \) are double exponential, there is no reason to believe that the simulated \( \eta \)'s remain double exponential.
where $\phi_t$ is the characteristic function of an exponential Lévy process with the Lévy density $\eta_t(T, \cdot)$, starting from one:
\[
\phi_t(T, u) = \exp \left[ -iu(T - t) \int_{\mathbb{R}} \eta_t(T, x)(e^x - 1)dx + (T - t) \int_{\mathbb{R}} (e^{iu x} - 1)\eta_t(T, x)dx \right].
\]

From the above option price, $C_t^{K_i}(T, x)$, we can easily calculate the implied volatility by inverting the Black Scholes formula, assuming that $S_t = 1$ and the interest and dividend rates are zero. As discussed at the very end of Subsection 2.1, this value is the same as the value of implied volatility of a call option for spot level $S_t$, strike $S_t e^x$, and maturity $T$, regardless of what the level of $S_t$ is (hence, we don’t need to simulate it). Using this method, we simulate the implied volatility surfaces five days into the future starting from Dec. 13, 2007, as shown in Figures 7 and 8.

3 Empirical analysis of the performance of tangent Lévy models

In this section, we discuss the importance of consistency in modeling derivatives prices. As we know, an investment manager’s portfolio or a trader’s trading book often contains multiple financial derivatives written on the same underlying. As a simple example, an equity trader might hold a calendar spread and a butterfly spread at the same time. To properly manage the risk, one needs to understand the joint dynamics of these derivatives, for which a consistent modeling framework is crucial. Tangent Lévy models (as any market-based model) are built to achieve this goal precisely. This is due to the fact that not only present but also historical information contained in the time series of options’ prices is used in the estimation of model dynamics. Classical stochastic volatility models cannot capture the historical evolution of options’ prices, hence there is a reason to believe that market-based models would lead to better performance in portfolio management. To show that tangent Lévy models do indeed work better, here, we test the model implemented in Section 2 using the portfolio choice problem described below. The results are, then, compared against one of the most popular volatility models in the industry – the Stochastic Alpha Beta Rho (SABR) model.

3.1 The variance-minimizing portfolio choice problem

This example is a simplified Markowitz-type portfolio optimization problem. Consider a portfolio manager who needs to decide how he/she should balance a portfolio of SPX options so that its risk is minimized. Among the many definitions of portfolio risk, we adopt the one used in the classic Markowitz problem (for example, see Section 6.6 of [22]) – namely, the standard deviation of the portfolio return over a given (future) time period. Notice that this is not a typical Markowitz portfolio problem, given we are not considering the trade-off between return and risk as a typical Markowitz problem would. As a matter of fact, we would assume that the portfolio manager lives in a risk-neutral world, so that the expected return is normalized. We admit that lacking excess return might make the example less exciting, but it helps us compare the model performance in an apples-to-apples fashion. With the normalized return, there is no need to worry about the impact of different market views portfolio managers might build into the investment decisions. Of course, without such a trade-off, there is a trivial solution to the portfolio choice problem – do not invest at all, reducing the risk to zero. To make the problem non-trivial, we require that the value of the portfolio at the time when it is constructed must be equal to a fixed number $M$. Such a restriction is relevant if the manager makes profits off the commission, proportional to the size of the investment portfolio he/she manages. For example, an option market maker might want to know the optimal inventory so that he/she can adjust the quoting strategy accordingly to reach the portfolio composition with minimal inventory risk. Or, a broker dealer might need to know her optimal position in options over the next several days to meet the risk and capital requirements.

We now formulate this problem mathematically. Let us assume that there are $n$ options with the same maturity $T$ but with different strikes $K_1, \ldots, K_n$ in the portfolio. Let $C_u(K)$ be the time-$u$ price of the $K$-struck option, and let $\omega_i$ be the quantity of this option in the portfolio, with a negative $\omega_i$ representing to short-selling. The weights $\omega_i$ have to be determined at the initial time $d$. The portfolio value at any future time $t$ is simply $V_t = \sum_{i=1}^n \omega_i C_t(K_i)$, and the return over a $u$-day period is $R_u = V_{t+u}/V_t$. For simplicity, we assume that the risk-free rate and the dividend yield are both zero, so the expectation of $R_u$ is simply 1. For a given $u \in (0, T)$, to determine the
portfolio weights, we need to solve the following convex optimization problem:

$$\min_{\omega \in \mathbb{R}^n} \mathbb{E}(R_u - 1)^2 = \frac{1}{M^2} \min_{\omega \in \mathbb{R}^n} \mathbb{E}(V_u - M)^2$$

s.t. \( V_d = \omega^T C_d = M, \)

where \( M \in \mathbb{R}, M > 0 \) is the initial value of the portfolio. This is equivalent to

$$\min_{\omega \in \mathbb{R}^n} \omega^T \Lambda_u \omega \quad \text{(3.1)}$$

s.t. \( V_d = \omega^T C_d = M, \)

where \( \Lambda_u = \mathbb{E}[(C_{d+u} - C_d)^2] \) is the covariance matrix of the time-\( d + u \) options’ prices. It is easy to see that the closed-form solution to the quadratic optimization (3.1) is

$$\omega = M \Lambda_u^{-1} C_d \quad \text{(3.2)}$$

Thus, as it is well-known, the key to solving this optimization problem is to estimate the covariance matrix \( \Lambda_u \). To do this, we compute the sample covariance matrix using the time-\( d + u \) option prices simulated under each model. Note that, to obtain a fair comparison, the parameters of each model are only estimated using the options data prior to day \( d \). Then, given \( N \) samples of the time-\( d + u \) options’ prices,

$$C(j) = \begin{bmatrix} C_{d+u}^{(j)}(K_1), \ldots, C_{d+u}^{(j)}(K_n) \end{bmatrix}^T, \; j = 1, \ldots, N,$$

the sample covariance matrix is estimated as

$$\Lambda_u = \frac{1}{N-1} \sum_{j=1}^{N} (C(j) - C_d)(C(j) - C_d)^T. \quad \text{(3.3)}$$

Different models generate different simulated paths of options’ prices, which then lead to different optimal weights. Naturally, how these optimal weights perform in the real world serves as an indicator of the model performance. To be more specific, a good model should be able to generate portfolios with smaller standard deviation in the returns. To estimate the standard deviation of portfolio returns, we define the figure of merit \( Q \) as the average realized deviation of the portfolio return in the testing period, i.e.

$$Q = \sqrt{\frac{1}{N_{\text{test}}} \sum_{k=1}^{N_{\text{test}}}(R_u^k - 1)^2}, \quad \text{(3.4)}$$

where \( N_{\text{test}} \) is the number of trials, and \( R_u^k \) is the actual portfolio return (given by market data) over a \( u \)-day period, with initial day \( d_k \) and with the optimal weights \( \omega^k \), obtained by (3.2) on day \( d_k \). Different trials correspond to different initial days \( d_k \), i.e.

$$R_u^k = \frac{1}{M} \sum_{i=1}^{n} \omega^k_i C_{d_k+u}(K_i). \quad \text{(3.5)}$$

Recall that, by assumption, the mean of \( R_u^k \) should always be 1. To make this assumption be consistent with the data, we choose a relatively small time horizon \( u \).

### 3.2 Simulation algorithms

As mentioned in the previous subsection, to find the optimal portfolio, we need to estimate the covariance matrix using simulated option prices. In this section, we describe the simulation algorithms for each model.

* **Double exponential tangent Lévy model.** For this experiment, we need to simulate both the underlying process \( S \) and the non-homogeneous Lévy density \( \kappa \). For the double exponential tangent Lévy model, in particular, we need to complete the following two steps to move one step ahead from \( t \) to \( t + \Delta t \):
3.3 Results of empirical analysis

• volatility can be simulated as follows: current volatility \( \alpha \) probably the most natural choice for equity market as it mimics a log-normal model most closely, and \( u \) methods, as described in Subsection 2.6, to calculate time-

\[ B \]

model, calculated with (3.7). With the parameters calibrated on the initial day by minimizing the sum of squared differences between the market call prices and those produced by the SABR model, calculated with (3.7). With the simulated spot \( F_t \) and the volatility \( \alpha_t \) as follows:

\[ F_t = \alpha_t F_t^\beta dB_t^1, \]

\[ d\alpha_t = \nu \alpha_t dB_t^2, \]

(3.6)

where \( F \) and \( \alpha \) are correlated through \( dB_t^1 dB_t^2 = \rho dt \). [13] provides the following asymptotic formula for the time-\( t \) implied volatility under the SABR model:

\[ \sigma_t(K, T, F_t, \alpha_t) \approx \frac{\alpha_t}{(F_t K)^{(1-\beta)/2}} \left\{ 1 + \frac{(1-\beta)^2}{24} \log^2 F_t / K + \frac{(1-\beta)^4}{1920} \log^4 F_t / K \right\}, \]

\[ \left\{ 1 + \left[ \frac{(1-\beta)^2}{24} \frac{\alpha_t^2}{(F_t K)^{(1-\beta)/2}} + 1 \frac{\rho \nu \alpha_t}{4 (F_t K)^{(1-\beta)/2}} + \frac{2 - 3 \rho^2 \nu^2}{24} \right] (T - t) \right\}, \]

(3.7)

where \( K \) is the strike value, \( T \) is the maturity, \( F_t \) is the current spot level, and \( z \) and \( x(z) \) are defined as

\[ z = \frac{\nu}{\alpha_t} (F_t K)^{(1-\beta)/2} \log \frac{F_t}{K}, \]

\[ x(z) = \log \left\{ \frac{\sqrt{1 - 2 \rho z + z^2} + z - \rho}{1 - \rho} \right\}. \]

(3.8)

As for the parameters’ values, [13] suggests that \( \beta \) can be fixed in advance and [29] verifies empirically that this is a reasonable assumption. In our example, we will use two values of \( \beta \): \( \beta = 1 \) and \( \beta = 0.7 \). \( \beta = 1 \) is probably the most natural choice for equity market as it mimics a log-normal model most closely, and \( \beta = 0.7 \) is widely used on trading desks as it provides better results for risk management. The other parameters – the current volatility \( \alpha_t \), the volatility of volatility \( \nu \) and the correlation \( \rho \) – will be calibrated to market prices by minimizing the sum of squared differences between the market call prices and those produced by the model, calculated with (3.7). With the parameters calibrated on the initial day \( d_k \), the forward price and the volatility can be simulated as follows:

\[ F_{t+\Delta t} = F_t e^{-0.5 \alpha_t^2 \Delta t + \alpha_t \Delta B_t^1}, \]

\[ \alpha_{t+\Delta t} = \alpha_t e^{-0.5 \nu^2 \Delta t + \nu \Delta B_t^1 + \sqrt{1-\rho^2} \Delta B_t^2}, \]

(3.9)

where \( B_t^1 \) and \( B_t^2 \) are independent. The time-\( u \) implied volatilities and option prices can then be computed via (3.7), with the simulated spot \( F_{t+u} \) and volatility \( \alpha_{t+u} \).

3.3 Results of empirical analysis

In this section, we will go through the test procedure in detail and present the test results for the following models:
• Double exponential tangent Lévy model (DETL).
• SABR model with $\beta = 1$.
• SABR model with $\beta = 0.7$.

Each model will be run in two periods: (I) Jan. 2007 - Aug. 2008 and (II) Jan. 2011 - Dec. 2012. For each period, we use the first year’s data as a training sample, to estimate the parameters of the tangent Lévy model, and we use the rest of the data as the testing sample, to compute the figure of merit $Q$ defined in (3.4). The division between training and testing samples is shown in Table 2. Please note that we cut off the first period at August 2008 to reduce the impact of the financial crisis. The tests will be run on a portfolio of call options and underlying – referred to as a “(C + S) portfolio” – with three, four and five strikes. In each case, we pick every other strike starting from the strike closest to the underlying spot value (in other words, closest to at-the-money) at the moment when the portfolio is constructed. We pick these options because their market prices are most accurate. Assuming the set of available strikes is $K_1 < K_2 < ... < K_n$ and the spot $S$ satisfies $K_{i-1} < S < K_i$, Table 3 illustrates the strikes used in each case.

For all portfolios, we use a simulation horizon of $u = 8$ days, and, at the time $d_k$ when the portfolio is constructed, the options have maturity of $T = d_k + u + 30$, so that their time-to-maturity becomes 30 days when the given simulation period ends. We also assume the budget constraint $M = 1$. In addition to the figure of merit $Q$, we also check the average predicted deviation defined as

$$P = \sqrt{\frac{1}{N_{\text{test}}} \sum_{k=1}^{N_{\text{test}}} (\omega^{k\ast})^T \Lambda d_k \omega^{k\ast}},$$

where $\omega^{k\ast}$ is the set of optimal weights obtained via (3.2) on day $d_k$. The difference between $Q$ and $P$ is another measure of the accuracy of a model’s prediction. Besides the predicted and realized deviation, one may be interested in how much the optimal portfolio weights fluctuate across the initial days $d_k$. To measure this fluctuation, we define the average quantity oscillation index $K$:

$$K = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{N_{\text{test}}} \sum_{k=1}^{N_{\text{test}}-1} |\omega_i^{(k+1)\ast} - \omega_i^{k\ast}| \right),$$

where $\omega_i^{k\ast}$ is the quantity of the $K_i$-struck option in the optimal portfolio constructed on day $d_k$.

3.3.1 Period I

For Period I, the estimation of the parameters of the tangent Lévy model is described in Section 2. Following the simulation algorithm outlined in Subsection 3.2 for every initial day $d_k$ in the testing sample, we simulating 500 sample paths for the underlying and the option prices, using each model, and starting with the actual prices observed on day $d_k$. In the simulation for the tangent Lévy model, we use the drift $\alpha$ and volatility $\beta$ estimated from the training sample, so that the testing is performed out-of-sample (this issue is irrelevant for SABR model, as it does not allow for any use of past option prices). Using the simulated prices, for each model, we calculate the average predicted deviation $P$, according to (3.10), and estimate the optimal portfolio weights $\omega^{k\ast}$ via (3.2). Using these weights, we construct the corresponding portfolio and record its value at time $d_k + u$ using the actual market prices. Collecting the results for all initial days $d_k$, we compute the average realized deviation $Q$ via (3.4).

The results are shown in Table 4. It is easy to see that, for a portfolio with 5 strikes, DETL model produces much smaller values of $Q$ than those produced by SABR model, indicating that the tangent Lévy models do a much better job at finding the minimal-variance portfolio. This can also be seen in Figure 10 which shows that the distribution of realized returns is much more concentrated around 1 under the DETL model than under the SABR model. Furthermore, if we look at the difference between $Q$ and $P$, we can see that it is much smaller for DETL than for SABR model. This suggests that the tangent Lévy models produce a more reliable prediction of the risk of an option portfolio (as measured by the standard deviation of its return) than the SABR model. Besides a small return deviation, another nice feature of tangent Lévy models is the stability of optimal option quantities across the initial days $d_k$. Figure 10 shows the optimal quantities of options and underlying index, in the portfolio with 5 strikes, across all initial days in the testing period, for every model. Similarly, Table 5 shows the average quantity oscillation $K$ (defined in (3.11)) for all portfolios and all models. It is easy to see that the...
portfolio weights constructed via DETL model are much more stable than those constructed using SABR model. This can be explained by the fact that the parameters of a tangent Lévy model are estimated from both the present and historical option prices, while a classical stochastic volatility model, such as SABR, can only be calibrated to the option prices available on day $d_k$. It is well known (and obvious intuitively) that an estimate based on a larger sample is more robust. Thus, the ability of tangent Lévy models to be fitted to the historical options prices makes their output (in this case, the optimal portfolio weights) more stable.

Tables 4 and 5 also show that the difference between the performances of DETL and SABR models shrinks as the number of strikes in the portfolio decreases. This is not a surprise: as the number of strikes decreases, the number of degrees of freedom in the dynamics of option prices, which have to be captured by the model, decreases as well. Eventually, for a very small number of strikes, the SABR model does relatively well. However, even in the case of 3 strikes, the tangent Lévy model does at least as good as SABR (although at a higher computational cost). Of course, the real benefit of using tangent Lévy models is only visible when the number of options in the portfolio is relatively large. Figure 11 provides a visual explanation for DETL’s outperformance. It shows the simulated call option prices, as functions of strike, at the end of the simulation period in 500 sample paths under DETL model and under SABR model with $\beta = 1$. It is easy to see that the SABR model only allows for very limited shapes of the simulated call price curves, while the tangent Lévy model is able to generate a much wider variety of shapes. It is the lack of variety of different scenarios for the joint evolution of call prices (not merely the lack of parameters in the model) that prohibits the classical stochastic volatility models, such as SABR, from capturing the true dynamics of option prices (or, of implied volatility surface) contained in the historical data.

3.3.2 Period II

Herein, we repeat the same analysis for Period II. The main purpose of this analysis is to show that the outperformance of tangent Lévy models is not due to our choice of a testing period, but that it is a persistent property. First, we need to estimate the parameters of DETL model using the data of year 2011. The estimation procedure is exactly the same as the one described in Section 2, so we only present the main results here.

In particular, the PCA shows that the first three eigenmodes explain over 93% of the variance. The eigenvalues and the eigenmodes are shown in Figure 12, and the corresponding drift term $\alpha$ is shown in Figure 13.a. Comparing to Figures 5 and 6, we see that these results are almost the same as for the year 2007, suggesting that this model is very robust.

Once the estimation is completed, we can repeat the same simulation and testing procedures as in Subsection 3.3.1, to obtain the results shown in Tables 6 and 7, as well as in Figures 14 and 15. These results confirm the finding of Subsection 3.3.1: for sufficiently many strikes in the portfolio, the tangent Lévy model does a much better job at finding a portfolio with smallest variance, their predictions for the variance are more reliable, and the portfolio weights are more stable.

4 Conclusion

In this paper, we implement and test a market-based model for European-type options. This model is a numerically tractable specifications of the family of tangent Lévy models proposed in [4] and [3]. Such models, in particular, provide a method for generating Monte Carlo samples of future implied volatility surfaces, in a way that is consistent with their past and present values. We estimate the parameters of this model using real market data, for two periods: 2007-2008 and 2011-2012. The estimation procedure is described in detail, so that it can be reproduced by any interested reader.

In addition, we use the estimated model and the real market data to conduct an empirical study, whose goal is to compare the performance of market-based models with the performance of classical stochastic volatility models. We choose the problem of minimal-variance portfolio choice to compare the performance of the tangent Lévy model with the SABR model. Our study demonstrates that the tangent Lévy model does a much better job at finding a portfolio with smallest variance. In addition, its prediction of the future return variance is more reliable, and the portfolio weights are more stable. To the best of our knowledge, this is the first example of empirical analysis that provides a convincing evidence of the outperformance of the market-based models for European options using real market data.

Our work is also subject to certain limitations, which suggest directions of future research. One of the most serious problem with implementing a market-based model is numerical complexity and, potentially, instability, of the static fitting stage. To mitigate this issue, in Subsection 2.4, we chose to work with a parametric family of
Lévy densities. Although this increases the stability of computations, we still have to rely on the convergence of a generic optimization algorithm, which is applied to a non-convex problem. In addition, the restriction to a parametric form of the density also implies that we may not be able to fit option prices with a required precision. To address these two issues, one has to come up with a family of tangent Lévy densities that is rich enough — so that it can approximate well the option prices observed in the market — and, at the same time, not too large — so that the calibration procedure is numerically tractable and more stable (ideally, we would like to be able to propose an algorithm for the associated optimization problem that is guaranteed to converge). This is a balance that seems to be hard to find. One can go even further along these lines and search for other families of tangent models — not necessarily based on Lévy processes, which can always fit an arbitrary family of arbitrage-free option prices. This, in turn, motivates the search for other families of models, which can always fit an arbitrary family of arbitrage-free option prices. An example of such a family is provided in [6], but the existence and description of consistent dynamics within this family of tangent models remains an open question.

5 Appendix A

Here, we define the Banach spaces associated with tangent Lévy processes.

- $B_0$ is a Banach space of Borel measurable functions satisfying
  \[
  \|f\|_{B_0} := \int_{\mathbb{R}} (|x| \wedge 1)|x|(1 + e^x)|f(x)|dx < \infty.
  \]  
  (5.1)

- $\mathcal{B}$ is a Banach space of absolutely continuous functions $f : [0, \bar{T}] \rightarrow \mathcal{B}_0$ satisfying
  \[
  \|f\|_{\mathcal{B}} := \|f(0)\|_{\mathcal{B}_0} + \int_0^\bar{T} \|d_u f(u)\|_{\mathcal{B}_0} du < \infty.
  \]  
  (5.2)

- $\mathcal{H}_0$ is a Hilbert space of Borel measurable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying
  \[
  \|f\|_{\mathcal{H}_0}^2 := \int_{\mathbb{R}} |x|^4(1 + e^x)^2|f(x)|^2dx < \infty.
  \]  
  (5.3)

- $\mathcal{H}$ is a Hilbert space of absolutely continuous functions $f : [0, \bar{T}] \rightarrow \mathcal{H}_0$ satisfying
  \[
  \|f\|_{\mathcal{H}}^2 := \|f(0)\|_{\mathcal{H}_0}^2 + \int_0^{\bar{T}} \|d_u f(u)\|_{\mathcal{H}_0}^2 du < \infty.
  \]  
  (5.4)

- $C([0, \bar{T}])$ is a Banach space of continuous functions $f : [0, \bar{T}] \rightarrow \mathbb{R}$ satisfying
  \[
  \sup_{x \in \mathbb{R}} |f(x)| < \infty.
  \]  
  (5.5)

- $W^{1,2}([0, \bar{T}])$ is a Hilbert space of absolutely continuous functions $f : [0, \bar{T}] \rightarrow \mathbb{R}$ satisfying
  \[
  |f(0)|^2 + \int_0^{\bar{T}} \left| \frac{d}{du} f(u) \right|^2 < \infty.
  \]  
  (5.6)

- $\mathcal{B}_d$ is a Banach space of absolutely continuous functions $f : [0, \bar{T}] \rightarrow \mathbb{R}$ satisfying
  \[
  \|f\|_{\mathcal{B}_d} := |f(0)| + \int_0^{\bar{T}} \left| \frac{d}{du} f(u) \right| du < \infty.
  \]  
  (5.7)

Here the subscript $d$ is used to indicate the “discrete” models.

\[3\] In fact, one has to go beyond Lévy processes to do this. For example, it is not hard to find a combination of arbitrage-free prices of three call options, with the same maturity and different strikes, which cannot be approximated with an arbitrary precision (simultaneously) by any exponential Lévy model. This means that, in principle, if there are more than two strikes traded in the market, the associated call prices (even with the bid-ask spreads) may be such that there is no tangent Lévy model that can match them.
• $H_d$ is the Hilbert space of absolutely continuous functions $f : [0, T] \to \mathbb{R}$ satisfying

$$\|f\|_{H_d}^2 := |f(0)|^2 + \int_0^T |\frac{d}{du} f(u)|^2 du < \infty. \quad (5.8)$$

We know that $H_0 \subset B_0$, $B \subset H \subset W^{1,2}([0, T]) \subset C([0, T])$ and $H_d \subset B_d$. In addition, it is not hard to see that the completion of $H_0$ is $B_0$ with respect to the norm $\| \cdot \|_{B_0}$. Similarly, the completion of $H$ is $B$ with respect to $\| \cdot \|_B$, the completion of $W^{1,2}([0, T])$ is $C([0, T])$ with respect to the “sup” norm, and the completion of $H_d$ is $B_d$ with respect to the $\| \cdot \|_{B_d}$ norm. Hence, we conclude that the couples $(H, B)$, $(W^{1,2}([0, T]), C([0, T]))$, and $(H_d, B_d)$ are all conditional Banach spaces (see III 5.3 in [18] for definition).

6 Appendix B

Proof of Lemma 1. The proof is similar to the one given in [17] except that $Z(T) = \mu T + \sum_{i=1}^{N_T} Y_i$ now follows a gamma distribution in the absence of a diffusion term. The tail probability is given by

$$P\{Z(T) \geq a\} = \Psi(\mu, \lambda, \rho, \lambda_1; a, T),$$

with $\Psi$ given in (2.13). If we set $V_i = \exp(Y_i)$ for $i = 1, \cdots, N$, the drift term has to satisfy $\mu = -\lambda E[V - 1]$ for $S_t$ to be a martingale, so the dynamics become

$$dS_t = -\lambda E[V - 1] S_t \, dt + S_t \cdot \left[ \sum_{i=1}^{N_t} (V_i - 1) \right].$$

Let $\zeta = E[V_i - 1] = \frac{p \lambda_1}{\lambda_1 - 1} + \frac{(1-p) \lambda_2}{\lambda_2 + 1} - 1$. Using results on equivalence of measures for compound Poisson processes (see Proposition 9.6 in [9] for example), we can see the time-$t$ price of a call option with maturity $T$ and strike $K$ is

$$C_t(T, K) = E[(S_T - K)^+] | F_t] = E[S_T 1_{S_T > K} | F_t] - E[K 1_{S_T > K} | F_t],$$

$$= S_t \cdot \Psi\left(-\lambda \zeta, \lambda^*, \rho^*, \lambda_1^*, \lambda_2^*; \log\left(\frac{K}{S_t}\right), T - t\right) - K \cdot \Psi\left(-\lambda \zeta, \lambda, p, \lambda_1, \lambda_2; \log\left(\frac{K}{S_t}\right), T - t\right),$$

where $\lambda^* = \lambda(\zeta + 1)$, $p^* = \frac{p \lambda_1}{\lambda_1 - 1}$, $\lambda_1^* = \lambda_1 - 1$ and $\lambda_2^* = \lambda_2 + 1$.

7 Appendix C

<table>
<thead>
<tr>
<th>Table 1: Time periods</th>
</tr>
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<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>-----------------------</td>
</tr>
<tr>
<td>419</td>
</tr>
<tr>
<td>Range of SPX spot price</td>
</tr>
</tbody>
</table>
Table 2: Testing periods

<table>
<thead>
<tr>
<th>Period</th>
<th>Training period</th>
<th>Testing period</th>
</tr>
</thead>
</table>

Table 3: Strikes used in each portfolio

<table>
<thead>
<tr>
<th># of strikes</th>
<th>Strikes used</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$K_{i-3}$(call), $K_{i-1}$(call), $K_{i+1}$(call), $K_{i+3}$(call), $K_{i+5}$(call)</td>
</tr>
<tr>
<td>4</td>
<td>$K_{i-3}$(call), $K_{i-1}$(call), $K_{i+1}$(call), $K_{i+3}$(call)</td>
</tr>
<tr>
<td>3</td>
<td>$K_{i-1}$(call), $K_{i+1}$(call), $K_{i+3}$(call)</td>
</tr>
</tbody>
</table>

Table 4: Average deviation of $(C + S)$ portfolio in period I

<table>
<thead>
<tr>
<th># of strikes</th>
<th>DETL</th>
<th>SABR ($\beta = 1$)</th>
<th>SABR ($\beta = 0.7$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average realized deviation $Q$</td>
<td>5</td>
<td>0.55%</td>
<td>84.97%</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.54%</td>
<td>4.69%</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.64%</td>
<td>2.18%</td>
</tr>
<tr>
<td>Average predicted deviation $P$</td>
<td>5</td>
<td>0.87%</td>
<td>0.19%</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.88%</td>
<td>0.30%</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.05%</td>
<td>0.53%</td>
</tr>
</tbody>
</table>

Table 5: Average quantity oscillation $K$ (as defined in (3.11)) in $(C + S)$ portfolio in Period I

<table>
<thead>
<tr>
<th># of strikes</th>
<th>DETL</th>
<th>SABR ($\beta = 1$)</th>
<th>SABR ($\beta = 0.7$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.0039</td>
<td>1.1747</td>
<td>2.3846</td>
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<tr>
<td>4</td>
<td>0.0038</td>
<td>0.1339</td>
<td>0.4629</td>
</tr>
<tr>
<td>3</td>
<td>0.0027</td>
<td>0.0263</td>
<td>0.0807</td>
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Table 6: Average deviation of $(C + S)$ portfolio in Period II

<table>
<thead>
<tr>
<th># of strikes</th>
<th>DETL</th>
<th>SABR ($\beta = 1$)</th>
<th>SABR ($\beta = 0.7$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average realized deviation $Q$</td>
<td>5</td>
<td>0.41%</td>
<td>9.07%</td>
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<tr>
<td></td>
<td>4</td>
<td>0.42%</td>
<td>3.51%</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.42%</td>
<td>0.90%</td>
</tr>
<tr>
<td>Average predicted deviation $P$</td>
<td>5</td>
<td>0.79%</td>
<td>0.36%</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.79%</td>
<td>0.43%</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.94%</td>
<td>0.62%</td>
</tr>
</tbody>
</table>

Table 7: Average quantity oscillation $K$ (as defined in (3.11)) of $(C + S)$ portfolio with 5 strikes in Period II

<table>
<thead>
<tr>
<th># of strikes</th>
<th>DETL</th>
<th>SABR ($\beta = 1$)</th>
<th>SABR ($\beta = 0.7$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.0011</td>
<td>0.1410</td>
<td>0.6642</td>
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<tr>
<td>4</td>
<td>0.0012</td>
<td>0.0537</td>
<td>0.2736</td>
</tr>
<tr>
<td>3</td>
<td>0.0011</td>
<td>0.0145</td>
<td>0.0474</td>
</tr>
</tbody>
</table>
Figure 1: Symmetry index $\Xi$ as a function of time to maturity, in DETL model
Figure 2: Calibrated densities $\eta$ for DETL model on the second day, Jan. 4, 2007
Figure 3: Calibrated time values for DETL model on the second day, Jan. 4, 2007
Figure 4: Calculating Lévy density $\kappa$ from $\eta$
(a) Percentage of variance explained by the eigenmodes  
(b) The first eigenmode scaled by $\sqrt{\lambda_1}$
(c) The second eigenmode scaled by $\sqrt{\lambda_2}$
(d) The third eigenmode scaled by $\sqrt{\lambda_3}$

Figure 5: Eigenvalues and eigenmodes of $\Delta \kappa$ for DETL model

Figure 6: The drift term $\alpha$ for DETL model
Figure 7: Simulated $\kappa$’s and implied volatility surfaces using DETL model (1)
Figure 8: Simulated $\kappa$’s and implied volatility surfaces using DETL model (2)
Figure 9: Distribution of the 8-day returns of (C + S) portfolio with 5 strikes in Period I. Different scales are used to show more details.
The testing period

(a) Under SABR model with $\beta = 1$
(b) Under SABR model with $\beta = 0.7$.

(c) Under double exponential tangent Lévy model

Figure 10: Option quantities in (C + S) portfolio with 5 strikes in Period 1. Different scales are used to show more details

(a) Under double exponential tangent Lévy model
(b) Under SABR model with $\beta = 1$

Figure 11: Terminal option prices in (C + S) portfolio, as functions of strike, simulated using 500 sample paths
Figure 12: Eigenvalues and eigenmodes of $\Delta \hat{\kappa}$ under DETL, estimated using 2011 data

Figure 13: The drift term $\alpha$ in DETL model, estimated using 2011 data
Figure 14: Distribution of the 8-day returns of (C + S) portfolio with 5 strikes in Period II
Figure 15: Option quantities in (C + S) portfolio with 5 strikes in Period II. Different scales are used to show more details.
References


