MEAN FIELD SYSTEMS ON NETWORKS, WITH SINGULAR INTERACTION THROUGH HITTING TIMES

SERGEY NADTOCHIY AND MYKHAYLO SHKOLNIKOV

Abstract. Building on the line of work [DIRT15a, DIRT15b, NS17a, DT17, HLS18, HS18] we continue the study of particle systems with singular interaction through hitting times. In contrast to the previous research, we (i) consider very general driving processes, (ii) allow for inhomogeneous connection structures, and (iii) analyze a game in which the particles determine their connections strategically. Hereby, we uncover two completely new phenomena. First, we characterize the “times of fragility” of such systems (e.g., the times when a macroscopic part of the population defaults or gets infected simultaneously, or when the neuron cells “synchronize”) explicitly in terms of the dynamics of the driving process, the current distribution of the particles’ values, and the topology of the underlying network (represented by its Perron-Frobenius eigenvalue). Second, we use such systems to describe a dynamic credit-network game and show that, in equilibrium, the system regularizes: i.e., the times of fragility never occur, as the particles avoid them by adjusting their connections strategically. Two auxiliary mathematical results, useful in their own right, are uncovered during our investigation: a generalization of Schauder’s fixed-point theorem for the Skorokhod space with M1 topology, and the application of max-plus algebra to the equilibrium problem for credit networks.

1. Introduction

In this paper, we continue our investigation of the mean field particle systems with singular interaction through hitting times. The finite particle systems of such type are characterized by the following feature. Whenever a particle (represented by a stochastic process on a real line) hits a given barrier, it shifts the positions of all other particles instantaneously in the direction of the barrier. As a result, a single initial hitting time may cause an instantaneous “cascade” of jumps. It turns out that, unlike in classical mean field systems, these cascades do not disappear as the population size increases to infinity, and they manifest themselves in the jumps of a representative particle. Such jumps are, often, connected to important real-world phenomena (e.g., phase transitions), hence, their presence in a model is necessary in order to analyze these phenomena. On the other hand, such discontinuities are the main source of the associated mathematical challenges.

The particle systems of such type arise in a variety of contexts. In [DIRT15a, DIRT15b], they are used as neuron firing models with the purpose of capturing the synchronization phenomenon, namely, a macroscopic number of neurons in a part of the brain firing simultaneously. The latter, in turn, has been found to be correlated with many cognitive functions. Another particle system interacting through the hitting times is introduced in [DT17], motivated by the earlier research [FP81, FP83, FPH089, FPH090] on supercooled liquids. Thereby, the goal is to describe the rapid advancement of the liquid-solid boundary when the supercooled liquid is located
next to the warmer corresponding ice. In our article [NS17a] and the papers [HLS18], [HS18] that followed, the interaction through hitting times represents the losses banks incur due to the defaults of other banks, which may cause further defaults and potentially trigger a default cascade of macroscopic size. The common feature of the three settings lies in the self-excitation on the microscopic (particle) level that, under appropriate circumstances, leads to discontinuities on the macroscopic level.

In addition to considering very general driving processes, the main contribution of the present work is two-fold. First and foremost, we allow for an inhomogeneous connection structure across particles. Recall that, in the mean field particle systems described in the previous paragraphs, the hitting time of any particle has a direct and equal effect on the positions of all other particles. This implies that any two particles are directly connected, and the strength of each connection is the same. Herein, on contrary, we model the connections via an infinite directed weighted graph, where the weight of an edge indicates the strength of a connection. Specifying the weights of the graph, we can produce systems with various connection structures. This generalization allows one to produce more realistic models: e.g., it is clear that not all neurons are directly connected to each other, and that not all banks have equal exposures to each other. To the best of our knowledge, the particle systems of such form have not been considered in the existing literature. It is worth mentioning that the connection structures used herein appear as the limits of finite random graphs in [BJR07], and the particle systems with such connection structures are analyzed, e.g., in [NTS7], [BFFT12], [BBW17], [BW16]. However, the latter systems are of a classical, smooth, type, without the singular interaction through hitting times. In Section 2 we discuss the proposed particle systems in detail and show their well-posedness (Theorem 2.2).

As an auxiliary result, we establish a generalization of Schauder’s fixed-point theorem (Theorem 2.3) for the mappings defined on the Skorokhod space with M1 topology. Moreover, we investigate how the topology of a network affects the occurrence of jumps of a representative particle. As such jumps may often be interpreted as the events of crises (e.g., a significant number of defaults or infections occurring instantly), we refer to them as the times of fragility and analyze them in detail in Section 3 (Theorems 3.1 and 3.4). Remarkably, the fragility times turn out to be closely connected to the Perron-Frobenius eigenvalue of the corresponding graph (Theorems 3.4).

The second contribution of this work is in the analysis of the associated mean field games. The literature on mean field games is vast (see, e.g., [CD18], [LL07], [Car10], and the references therein). However, to date, only [Del17] has considered mean field games of graphs, using exogenously given homogeneous (random) connection structure. Herein, we allow the particles to control the weights of their connections, thus, obtaining the connection structure endogenously, in equilibrium. In Section 4 we present the general form of the proposed “resource-sharing” games and focus on a more specific “credit network” game. There exists a fairly large body of work on modeling equilibrium in credit networks, or, more generally, in asset-liability networks: see, e.g., [AEW15], [EV17], [AOTS14], [BH17], [EHE15], [Far15], [NS17b], [Wan16], and the references therein. However, most of the existing models are either static or do not allow for the connections to be determined endogenously in equilibrium. The model proposed herein does possess both of these features. The output of such a model includes the equilibrium dynamics of the interest rate and of the credit connections in the system, which are easy to compute numerically. It turns out that the equilibrium

\[\text{1} \quad \text{The exceptions are [NS17b], [Wan16]. Nevertheless, the latter models do not include default risk, which is in the center of the present analysis.}\]
connections are such that the system never exhibits a jump: each particle decreases
the strength of its connections to the particles that are close to the barrier, eliminating
the cascades. Thus, allowing the particles to act strategically has a regularizing effect
on the system. It is worth mentioning that our construction of equilibrium relies upon
the existing results on linear systems in “max-plus” algebra (Proposition 4.7), making
an interesting connection to a very different part of the Mathematical theory.

The rest of the paper is organized as follows. Section 2 introduces the mean field
systems with exogenously given connection structures and proves their existence (The-
orem 2.2). It also contains Theorem 2.3 which is a generalization of Schauder’s fixed-
point theorem for the mappings defined on the Skorokhod space with M1 topology.
Section 3 is concerned with the times of fragility of such particle systems. Namely,
Theorems 3.1 and 3.4 provide sufficient conditions for fragility in terms of the mar-
ginal distribution of the solution, the topology of the network, and the dynamics of the
driving process. In Section 4 we consider a game in which every particle is allowed to
control the strength of its connections dynamically, aiming to optimize its objective.
The individual optimization problems and the notion of equilibrium are introduced in
Subsection 4.1 The approach for constructing an equilibrium is described in Subsec-
tion 4.2 The construction is split into two steps: the dynamic problem (Subsection
4.3) and the static problem (Subsection 4.4). The main theorem, showing the existence
of an equilibrium and describing its structure, is given in Subsection 4.4 (Theorem 4.8).
The appendix contains technical proofs and auxiliary statements.

2. Definition and existence of mean field dynamics

Herein, we deal with a system of particles characterized by their “level of healthi-
ness” $Y_x^x$, taking values in $[0, \infty)$, and by their “type” $x \in \mathcal{X}$, with an abstract finite
set $\mathcal{X}$. We assume that the type of each particle remains constant, whereas the health-
iness level changes stochastically over time. When the healthiness level drops to zero,
the particles “die”. In addition, we consider a directed weighted graph, whose
nodes are the elements of $\mathcal{X}$. The connection structure of this graph is encoded by
a nonnegative function $C$ and a probability kernel $\kappa$, on $\mathcal{X}$. The weight of the edge
from $x$ to $x'$ is given by $C^{x}_\kappa(x, \{x'\})$. Zero weight is interpreted as the absence of a
connection. The strength of the connection from a particles of type $x$ to the particles
of type $x'$ is determined by the weight of the associated edge of the graph. If a particle
dies, it shifts down the healthiness levels of all particles that are connected to it. The
size of the shift is given by the strength of these connections. In the absence of such
shifts, the healthiness level of a particle of type $x$ evolves according to a continuous
stochastic process $Z^x$, with $Z^x_0 \geq 0$. Thus, $\{Y^x\}_{x \in \mathcal{X}}$ are defined as a family of câdlâg
processes, satisfying, for each $x \in \mathcal{X}$,

$$Y^x_t = Z^x_t + C^{x} \int_{\mathcal{X}} \log \mathbb{P}(\tau^x > t) \kappa(x, dx), \quad t \in [0, \tau^x \wedge T],$$
$$Y^x_{\tau^x} = Y^x_{\tau^x}, \quad t \in (\tau^x \wedge T, T],$$
$$\tau^x = \inf\{t \in [0, T] : Y^x_t \leq 0\}. \tag{2.1}$$

We point out that the dependence between the driving processes $Z^x$ across $x \in \mathcal{X}$
is immaterial, as each pair $(Y^x, Z^x)$ can be defined on its own probability space.
Note that, although $(\mathcal{X}, \{C^{x}_\kappa(x, \{x'\})\}_{x, x' \in \mathcal{X}})$ is a finite graph, the actual network on
which the particles evolve is an infinite (directed, weighted) “dense” graph. Namely, if
$C^{x}_\kappa(x, \{x'\}) > 0$, then, every particle of type $x$ is connected to all particles of type $x'$
that are still alive (and there is an infinite number of them), with the weight of each connection being the same across all particles of type \( x' \).

The interpretation of the proposed system is as follows. The particles may represent cells, individuals, or organizations, having certain “exposures” to each other. For example, a neuron in a human brain is physically connected to surrounding neurons, a computer in a network allows other computers to upload data to it, and banks may lend funds to other banks and households. The strength of each connection indicates the size of this exposure. When a particle dies, it causes instantaneous shifts in the healthiness levels of the particles connected to it. For example, when the electric charge in a neuron exceeds a certain threshold, it spreads this charge to the neurons connected to it. Once a computer gets infected with a virus, the latter initiates attempts to spread to the connected computers. Similarly, if a company defaults, it causes instantaneous losses to the companies that have lent funds to it.

The aforementioned applications to neural and lending networks have been discussed, respectively, in [DIRT15a], [DIRT15b], [CCP11], [CP14], and in [NS17a], [HLS18]. However, all of the existing models have been limited to fully connected systems, with Brownian dynamics, homogeneous across particles. In the present notation, it corresponds to

\[
X := \{0\}, \quad Z^0_t := Z^0_0 + \alpha t + \sigma B_t, \quad t \in [0, T], \quad \kappa(0, \cdot) := \delta_0,
\]

with \( \alpha \in \mathbb{R} \), \( \sigma > 0 \) and \( B \) being a standard Brownian motion. In [NS17a], \( Y^0 \) stands for the logarithmic monetary reserve process of a representative bank in a large banking system. The profits and losses that the bank makes are due to external investments (i.e., the Brownian noise) and due to the defaults of other banks, who have borrowed funds from it (i.e., the terms containing \( \mathbb{P}(x' > t) \)). For a more detailed explanation, we refer the reader to [NS17a, theorem 2.3], and the paragraph preceding it, as well as the discussion in the first part of Section 4. However, in reality, the borrowing and lending exposures are not distributed uniformly over all members of a credit network. For examples, the types \( x \in X \) may represent banks’ key characteristics such as the geographic region and the size. It is, then, natural to choose the processes \( Z^x \), of returns from external investments, as well as the credit exposures, given by \( (C^x, \kappa(x, \cdot)) \), to depend upon the type. Similarly, the connections between neurons are established according to their physical proximity, or according to the common role played by each group of the cells. This explains the need for models of the form (2.1), with general \( X \), \( \{Z^x\} \), \( \{C^x\} \), and \( \kappa \).

The main goal of this section is to establish the existence of a solution to (2.1). Note that, herein, we do not address two important questions: the uniqueness of the solution and the convergence of the associated finite particle systems (i.e., the “propagation of chaos”). The question of uniqueness appears to be a very challenging one. It is currently under investigation by the authors and other researchers in the simpler setting of (2.2), with a single type and Brownian dynamics. Similarly, the propagation of chaos has been established (see [DIRT15b] and [NS17a]) only in such a simple setting, and no results are available for the general case. It is worth mentioning that [NT87], [BBW17], [BW16] prove the propagation of chaos for the mean field particle systems whose connection structures are given by a “dense” graph structure, as the one proposed herein.\(^2\) However, their systems are of a classical, smooth, type.

\(^2\)Strictly speaking, [BBW17] studies a sequence of finite particle systems on inhomogeneous random graphs, which converges to a system whose connection structure is similar to the one proposed herein.
without the singular interaction through hitting times. The latter feature makes the problem significantly more complicated, and, at the same time, its presence is necessary to be able to capture certain real world phenomena, as discussed in the next paragraph.

Before we proceed to the existence result, it is important to emphasize the role of singular interaction through hitting times between the particles. Already in the special case \( (2.2) \) with \( \alpha \leq 0 \) and \( C^0 > \mathbb{E}[Z_0^0] \), the paths of the solutions \( Y^0 \) to \( (2.1) \) cannot be continuous, as an adaptation of [HLS18, proof of theorem 1.1] reveals.

Since the driving process \( Z^0 \) is continuous, the discontinuity of \( Y^0 \) is very surprising at the first glance and, clearly, must be due to the “self-excitation” term \( \mathbb{P}(\tau^0 > t) \) in \( (2.1) \). Indeed, as a particle dies, it causes instantaneous shifts in the healthiness levels of the particles connected to it, potentially, causing them to die as well. Thus, in a finite particle system (cf. [DIRT15b], [NS17a]), one may observe a cascade, when several particles die at the same time. Even though, as the number of particles grows, the effect of each single particle on the rest of the population does vanish, these cascades do not disappear in the limit, and the resulting solution \( Y^0 \) does jump (cf. [DIRT15b], [NS17a]). Note that the times of such jumps have a special meaning in the associated applications. These are the times when, in a single instance, the neurons “synchronize”, a macroscopic fraction of computers get infected, or a macroscopic fraction of firms default. As such times are often the subject of practical interest (e.g., as the times of “crises”), it is important that (i) the model does possess this feature, and (ii) that we are able to analyze it.

The above discussion makes it clear that, in the full generality of \( (2.1) \), we need to look for solutions \( \{Y^x \}_{x \in \mathcal{X}} \) with values in \( D([0, T], \mathbb{R})^\mathcal{X} \), the product of the spaces of càdlàg real-valued paths on \( [0, T] \). We equip \( D([0, T], \mathbb{R})^\mathcal{X} \) with the product topology and each factor \( D([0, T], \mathbb{R}) \) with the topology induced by the inclusion

\[
(2.3) \quad D([0, T], \mathbb{R}) \hookrightarrow D([0, T + 1], \mathbb{R}), \quad y_t \mapsto \hat{y}_t := \begin{cases} y_t & \text{if } t \in [0, T], \\ y_T & \text{if } t \in (T, T + 1], \end{cases}
\]

where \( D([0, T + 1], \mathbb{R}) \) is the space of càdlàg real-valued paths on \( [0, T + 1] \) left-continuous at \( T + 1 \), endowed with Skorokhod’s M1 topology (see any of [DIRT15b, section 4.1], [Sko56], [Whi02, chapter 12] for a detailed discussion of the latter). Our first main result establishes the existence of solutions \( \{Y^x \}_{x \in \mathcal{X}} \) to \( (2.1) \) with values in \( D([0, T], \mathbb{R})^\mathcal{X} \) under the following assumption.

**Assumption 2.1.** (a) There exists a family \( \{ (\eta_n^x)_{n \in \mathbb{N}} \}_{x \in \mathcal{X}} \) of \( (0, T] \)-valued sequences decreasing to 0 that satisfies

\[
(2.4) \quad \int_{\mathcal{X}} \log \mathbb{P}\left(Z_0^x > \eta_n^x \right) \kappa(x, dx') > -\frac{\eta_n^x}{C_x}, \quad x \in \mathcal{X}, \quad n \in \mathbb{N}.
\]

(b) For all \( x \in \mathcal{X} \) and \( t \in [0, T] \), the law of \( Z_t^x \) possesses no atoms.

(c) For all \( x \in \mathcal{X} \), \( \varepsilon \in (0, T) \) and \( Z^x \)-stopping times \( \theta^x \), it holds

\[
(2.5) \quad \mathbb{P}\left( \min_{t \in [\theta^x, \theta^x + \varepsilon]} (Z_t^x - Z_0^x) < 0 \middle| \theta^x \leq T - \varepsilon \right) = 1
\]

whenever \( \mathbb{P}(\theta^x \leq T - \varepsilon) > 0 \).

(d) For all \( x \in \mathcal{X} \) and non-increasing \( f \in D([0, T], \mathbb{R}) \) with \( f_0 = 0 \), one has

\[
(2.6) \quad \mathbb{P}(Z_t^x + f_t > 0, \ t \in [0, T]) > 0.
\]

The condition \( (2.4) \) ensures that, initially, not too many particles are about to die: i.e. that the initial distribution of their healthiness does not have too much mass...
around zero. This assumption is needed in order to ensure that \( Y^x \) does not jump at \( t = 0 \). Indeed, it is clear intuitively that the jumps of \( Y^x \) occur when too many particles die and drag others with them. The latter, in turn, occurs when too many particles are close to zero (this connection is analyzed rigorously in the next section). And, in order to apply the appropriate fixed-point theorem and to establish existence of a solution, one needs to ensure that \( Y^x \) does not jump at zero (and, hence, \( Y^x_0 = Z^x_0 \)).

To obtain an even clearer picture, let us consider the special case of (2.2). Indeed, inserting (2.2) into (2.4) and rearranging the resulting inequality one arrives at

\[
(2.7) \quad \mathbb{P}(Z^0_0 \leq \eta_n^0) < 1 - e^{-\eta_n^0/C^0}, \quad n \in \mathbb{N},
\]

provided \((\eta_n^0)_{n \in \mathbb{N}}\) is chosen to take values in the set of the continuity points for the cumulative distribution function of \( Z^0_0 \). At the same time, if the slightly weaker condition (note that \( 1 - e^{-\eta/C^0} = \eta/C^0 + o(\eta) \) as \( \eta \downarrow 0 \))

\[
(2.8) \quad \mathbb{P}(Z^0_0 \leq \eta_n^0) < \frac{\eta_n^0}{C^0}, \quad n \in \mathbb{N}
\]

is violated, we know from [NS17a, proposition 2.5] that a solution \( Y^0 \) right-continuous at 0 cannot exist. The condition (2.4) is the network version of (2.7).

The parts (c) and (d) of the assumption 2.1 are fulfilled, for example, in the continuous semimartingale framework,

\[
(2.9) \quad Z^x_t = Z^x_0 + \int_0^t \alpha^x_s \, ds + \int_0^t \sigma^x_s \, dB_s, \quad t \in [0, T], \quad x \in \mathcal{X},
\]

with \( B \) being a Brownian motion (w.r.t. a filtration supporting \((\alpha^x, \sigma^x))\), if every \( \alpha^x, \sigma^x \) lie between two constants (possibly depending on \( x \)) in \( \mathbb{R} \) and \((0, \infty)\), respectively. This can be deduced by means of the Dambis-Dubins-Schwarz theorem (see e.g. [KS91, chapter 3, theorem 4.6 and problem 4.7]) and elementary properties of standard Brownian motion (see e.g. [KS91, chapter 2, theorem 9.23] for the part (c) and [KS91] chapter 2, proposition 8.1 and theorem 5.12] for the part (d)).

**Theorem 2.2.** Under Assumption 2.1, the problem (2.1) admits a \( D([0, T], \mathbb{R})^\mathcal{X} \)-valued solution \( \{Y^x\}_{x \in \mathcal{X}} \), with \( Y^x_0 = Z^x_0 \).

**Remark 1.** Under Assumption 2.1, for any \( t > 0 \), Theorem 2.2 yields the existence of a solution \( Y \) on the time interval \([0, t] \). If, in addition, \( Y \) is such that Assumption 2.1.a is satisfied with \( \{Z^0_n\} \) replaced by \( \{Y^0_n\} \), then, Theorem 2.3 allows us to extend the solution to a right neighborhood of \( t \), so that the resulting \( \{Y^x\} \) are continuous at \( t \). This observation becomes important in the next section.

The proof of this theorem is given in Appendix A. It is based on a version of the Schauder-Tychonoff fixed-point theorem for the space \( D([0, T], \mathbb{R})^\mathcal{X} \). We emphasize that, even for \( \mathcal{X} = \{0\} \), the addition operation in \( D([0, T], \mathbb{R})^\mathcal{X} \) is not continuous (see e.g. [Whi02, section 12.7]), so that \( D([0, T], \mathbb{R})^\mathcal{X} \) is not a topological vector space and does not fall under the scope of the classical Schauder-Tychonoff fixed-point theorem in [Tyc35]. We believe that this auxiliary result, established herein, is interesting in its own right.

**Theorem 2.3.** Let \( S([0, T], \mathbb{R}) \subset D([0, T], \mathbb{R}) \) be the subspace of piecewise constant functions with finitely many discontinuity points and \( \mathcal{D} \subset D([0, T], \mathbb{R})^\mathcal{X} \) be convex, closed and such that \( S([0, T], \mathbb{R})^\mathcal{X} \cap \mathcal{D} \) is dense in \( \mathcal{D} \). Then, every continuous map \( F : \mathcal{D} \to \mathcal{D} \) whose range has compact closure possesses a fixed-point.
Proof. The following is an adaptation of the proof of the Schauder-Tychonoff fixed-point theorem for locally convex Hausdorff topological vector spaces in [PT00] to the space $D([0,T], \mathbb{R})^X$. For $k \in \mathbb{N}$, we denote by $U^{1/k}(f)$ the open ball around an $f \in D([0,T], \mathbb{R})^X$ of radius $1/k$ in the sup product metric associated with the strong M1-metric on $D([0,T], \mathbb{R})$ (see e.g. [Whi02 Section 12.3]). The open cover

$$\text{ran} \overline{F} \subset \bigcup_{f \in S([0,T], \mathbb{R})^X \cap \mathcal{D}} U^{1/k}(f)$$

(2.10)

has a finite subcover $\text{ran} \overline{F} \subset \bigcup_{j=1}^{J} U^{1/k}(f_{j,k})$ due to the assumed compactness of $\text{ran} \overline{F}$. Hence, Theorem 2.2 applies and yields the appropriately bounded characteristics, then, Assumption 2.1 is satisfied, as follows.

Next, we assign to $f^{j,k}$, $j = 1, 2, \ldots, J$ the sets

$$\mathcal{A}^{j,k} := \{ f \in \mathcal{D} : F(f) \notin U^{1/k}(f_{j,k}) \} \cap \text{conv}\{f^{1,k}, f^{2,k}, \ldots, f^{J,k}\}, \quad j = 1, 2, \ldots, J,$$

respectively, where conv stands for the convex hull of a set. We observe that each $\mathcal{A}^{j,k}$ is closed in the Euclidean topology on $\text{conv}\{f^{1,k}, f^{2,k}, \ldots, f^{J,k}\}$, since a sequence in $\mathcal{A}^{j,k}$ converging in the Euclidean topology admits a common finite set of discontinuity points, so that its componentwise convergence in the M1 sense is a consequence of the componentwise pointwise convergence, and the set $\{ f \in \mathcal{D} : F(f) \notin U^{1/k}(f_{j,k}) \}$ is closed in the product M1 sense. Moreover, $\cap_{j=1}^{J} \mathcal{A}^{j,k} = \emptyset$ in view of $\text{ran} F \subset \bigcup_{j=1}^{J} U^{1/k}(f_{j,k})$. Thus, the Knaster-Kuratowski-Mazurkiewicz lemma (see [KKM29]) demonstrates that, for some $1 \leq j_1 < j_2 < \cdots < j_L \leq J$ and $f^{0,k} \in \text{conv}\{f^{j_1,k}, f^{j_2,k}, \ldots, f^{j_L,k}\}$, it holds $f^{0,k} \notin \bigcup_{j=1}^{L} \mathcal{A}^{j,k}$. In other words, $F(f^{0,k}) \in \bigcap_{j=1}^{L} U^{1/k}(f_{j,k})$, and, by the triangle inequality, $F(f^{\ell',k}) \in \bigcap_{j=1}^{L} U^{2/k}(f_{j,k})$, $\ell' = 1, 2, \ldots, L$.

Lastly, one can use $f^{k,k} \in S([0,T], \mathbb{R})^X$, $\ell = 1, 2, \ldots, L$ and the definition of the strong M1-metric in terms of graph parameterizations to deduce that $U^{2/k}(f_{j,k})$, $\ell = 1, 2, \ldots, L$ are convex, and therefore $\cap_{\ell=1}^{L} U^{2/k}(f_{j,k})$ is convex as well. Putting this together with $f^{0,k} \in \text{conv}\{f^{j_1,k}, f^{j_2,k}, \ldots, f^{j_L,k}\}$ we end up with $f^{0,k} \in \cap_{\ell=1}^{L} U^{2/k}(f_{j,k})$, hence, $F(f^{0,k}) \in U^{4/k}(f^{0,k})$ thanks to the triangle inequality. Taking $k \to \infty$ and appealing to the compactness of $\text{ran} \overline{F}$ we infer the existence of a subsequence of $F(f^{0,k})$, $k \in \mathbb{N}$ converging to an $f^{0,\infty} \in \mathcal{D}$. But $f^{0,k}, k \in \mathbb{N}$ must tend to $f^{0,\infty}$ along the same subsequence, and the continuity of $F$ yields $f^{0,\infty} = F(f^{0,\infty})$. □

We conclude this section with a discussion of uniqueness of the solutions to (2.1), which also clarifies the definition of the times of fragility, analyzed in the next section. The solutions to (2.1) are not unique in general. For instance, consider the situation of (2.2) with $P(Z_0 \in [1,3]) = P(Z_0 \in [z_0 - 1, z_0 + 1]) = 1/2$, where $z_0 > 1 + C^0 \log 2$ and $C^0 \geq 3/\log 2$. If, in addition, $Z^0$ is a continuous semimartingale with appropriately bounded characteristics, then, Assumption 2.1 is satisfied, as follows from the discussion after the assumption. Hence, Theorem 2.2 applies and yields the existence of a $D([0,T], \mathbb{R})$-valued solution to (2.1), with $P(Y_0^0 > 0) = 1$ and $Y_0^0 = Z_0^0$. On the other hand, if $Z^0 - Z_0^0$ is independent of $Z_0^0$, we can obtain a $D([0,T], \mathbb{R})$-valued solution of (2.1) with $P(Y_0^0 > 0) = 1/2$ as follows. First, we construct a solution $Y^0$ of (2.1) with the initial condition being independent of $Z_0 - Z_0^0$ and distributed as $\mathcal{L}(Z_0^0 | Z_0 \in [z_0 - 1, z_0 + 1])$, via Theorem 2.2 (note that Assumption 2.1 is, again,
satisfied and that $\mathbb{P}(\tilde{Y}_0 > 0) = 1$). Finally, we define
\begin{align}
Y_t^0 &= Z_0^0 + \alpha t + \sigma B_t + C^0 \log \mathbb{P}(\tilde{\tau}_0 > t) - C^0 \log 2, \quad t \in [0, \tau^0 \wedge T], \\
Y_t^0 &= Y_0^0, \quad t \in (\tau^0 \wedge T, T], \\
\tau^0 &= \inf \{ t \in [0, T] : Y_t^0 \leq 0 \}.
\end{align}
(2.12)

Note that the second solution constructed above jumps at $t = 0$ even though the distribution of $Z_0^0$ has no mass around zero. Heuristically speaking, a solution may “jump when it does not have to”. In particular, the jumps of $\{Y^x\}_{x \in X}$ are not determined intrinsically by (2.1), and one can only hope for the uniqueness of the solution under an extra condition that pins down the jump sizes. In [DIRT15b] and [NS17a], which study systems of the simpler form (2.2), the authors choose to define every jump size as follows:
\begin{align}
Y_t^0 &= Y_{t_c}^0 + C^0 \log \left( 1 - \frac{\mathbb{P}(\tilde{\tau}_0 \geq t, Y_{t_c}^0 \in (0, D_t))}{\mathbb{P}(\tilde{\tau}_0 \geq t)} \right), \quad t \in [0, T],
\end{align}
(2.13)

where
\begin{align}
D_t := \inf \left\{ z > 0 : z + C^0 \log \left( 1 - \frac{\mathbb{P}(\tilde{\tau}_0 \geq t, Y_{t_c}^0 \in (0, z)))}{\mathbb{P}(\tilde{\tau}_0 \geq t)} \right) > 0 \right\}, \quad t \in [0, T].
\end{align}
(2.14)

This definition corresponds to choosing the \textit{minimal} magnitude of a jump at any time $t$: this is the smallest size of a jump that a solution to (2.1) may have at the given time. In the simpler setting of (2.2), it is shown in [DIRT15b] and [NS17a] that such a choice corresponds to a natural definition of a cascade in the associated finite particle systems, and that these particle systems converge to a solution of (2.1), with the jump sizes given by (2.13)–(2.14). A direct analogue of the above jump size specification, in the case of general $X$, is the following:
\begin{align}
Y_t^x &= Y_{t_c}^x + C^x \int_X \log \left( 1 - \frac{\mathbb{P}(\tilde{\tau}_0 \geq t, Y_{t_c}^x \in (0, D_t)))}{\mathbb{P}(\tilde{\tau}_0 \geq t)} \right) \kappa(x, dx'), \quad t \in [0, T], \quad x \in X,
\end{align}
(2.15)

\begin{align}
D_t := \inf \left\{ z > 0 : z + C^x \int_X \log \left( 1 - \frac{\mathbb{P}(\tilde{\tau}_0 \geq t, Y_{t_c}^x \in (0, z)))}{\mathbb{P}(\tilde{\tau}_0 \geq t)} \right) \kappa(x, dx') > 0, \quad x \in X \right\}, \quad t \in [0, T].
\end{align}
(2.16)

However, unlike the case of $X = \{0\}$, for a general $X$, a heuristic analysis of the associated finite particle systems reveals that other jump size specifications may be relevant. To wit, one may treat the particles of different types asymmetrically when resolving a cascade. For this reason, herein, we do not choose a particular specification of the jump sizes. Instead, for any given $\{Z^x, C^x\}$ and $\kappa$, we analyze the times when any solution to (2.1) has to jump, which we call the \textbf{times of fragility} (as the jump time is often interpreted as the time of a crisis)\footnote{It is worth mentioning that the uniqueness of a solution to (2.1), even with the specified jump sizes, as in (2.13)–(2.13), is an open question. Nevertheless, we conjecture that such uniqueness does hold, at least, in the simpler setting of (2.2).}.
Motivated by the discussion at the end of the previous section, herein, we aim to characterize the times of fragility of a given system \((Z, C, \kappa)\). The first question we need to address is: in which form do we want to describe these times? Recalling the relevant applications, we notice that it is reasonable to assume that the distribution of every \(Y_t^x\) is observable at time \(t\): indeed, it can be identified with the empirical distribution of the particles “right before” time \(t\). In addition, the results of \cite{DIRT15b} and \cite{NS17a}, as well as the discussion at the end of the previous section (recall (2.13)–(2.14)), indicate that the jumps of \(Y^x\) at time \(t\) should be determined by the distribution of the survived particles, \(\mathcal{L}(Y_t^x \mid \tau^x \geq t)\), for all \(x' \in \mathcal{X}\). Thus, for a given \((Z, C, \kappa)\), we say that a time \(t\) and a collection of probability distributions \(\{\mathcal{L}(Y_t^x \mid \tau^x \geq t)\}\) on \([0, \infty)\) form a \bf{fragile state}, if any solution to (2.1), with the given conditional distributions at time \(t\), has a nonzero jump at time \(t\). In fact, it is easy to guess that only the asymptotic properties of the conditional distributions determine whether the jump occurs or not. Hence, for any \(\{\mathcal{L}(Y_t^x \mid \tau^x \geq t)\}\), we introduce the associated \(\{c^x \geq 0\}_{x \in \mathcal{X}}\) and \(\{z^x > 0\}_{x \in \mathcal{X}}\), s.t.

\[
\mathbb{P}(Y_t^x \in (0, z) \mid \tau^x \geq t) = \frac{\mathbb{P}(\tau^x \geq t, Y_t^x \in (0, z))}{\mathbb{P}(\tau^x \geq t)} \geq c^x z, \quad z \in [0, z^x], \quad x \in \mathcal{X}.
\]

Clearly, such \(\{c^x, z^x\}\) always exist, and there are typically infinitely many choices of such constants. The larger are \(c^x\) and \(z^x\), the more particles of type \(x\) are concentrated around zero (i.e. the more fragile the state is).

In this section, for general \((Z, C, \kappa)\), we formulate two sufficient conditions for a state to be fragile, expressed as the existence of an associated family \(\{c^x, z^x\}\) satisfying certain additional properties. We show how these conditions can be used and that they are fairly sharp. The proofs of all theorems stated in this section are given in Appendix B. It is worth mentioning that all of them are based on the key Lemma 6.1 which is a generalization of Proposition 2.7 in \cite{DIRT15b}.

First, we make a somewhat strong assumption on \(Z\), in order to obtain a sharper sufficient condition in terms of \((C, \kappa, c, z)\).

\bf{Theorem 3.1.} Fix an arbitrary \(t \in [0, T]\). Suppose that

\[
Z_t^x = Z_0^x + \int_0^t \alpha_r^x \, dr + \int_0^t \sigma_r^x \, dB_r, \quad s \in [0, T], \quad x \in \mathcal{X},
\]

where \(\alpha^x \leq \alpha, x \in \mathcal{X}\) and \(\sigma^x \leq \sigma \leq \sigma^x, x \in \mathcal{X}\) with suitable \(\alpha \in \mathbb{R}\) and \(\sigma, \sigma^x \in (0, \infty)\). Assume that a solution \(\{Y_t^x\}_{x \in \mathcal{X}}\) of (2.1) admits \(\{c^x, z^x\}\), satisfying (3.1) and such that

\[
\sum_{n=1}^{\infty} \left( \sqrt{s_n - s_{n+1}} + \hat{\alpha}(s_{n+1} - t) \right) \cdot \int_{\mathcal{X}^n} \prod_{m=0}^{n-1} (C_m^x e^{x_m+1}) \prod_{m=1}^{n-1} \left( 1 - 4 \Phi \left( \frac{z^{x_m} - e^{x_m} + \alpha(s_m - t)}{\sqrt{\sigma^2(s_m - s_{m+1}) + \sigma^2(s_{m+1} - t)}} \right) \right) 
\]

\[
\kappa(x_0, dx_1) \kappa(x_1, dx_2) \ldots \kappa(x_{n-1}, dx_n) = \infty
\]

for some \(\{s_n\}_{n \in \mathbb{N}}\) decreasing to \(t\) as \(n \to \infty\), \(\hat{\alpha} < -\sqrt{\pi/2} \alpha / \sigma, x_0 \in \mathcal{X}\), and \(c^x \in (0, z^x)\). Then \(Y_t^x > Y_t^{x'}, \) for at least one \(x \in \mathcal{X}\). Hereby, \(\Phi\) is the standard normal tail cumulative distribution function.
Let us analyze the meaning behind the main condition of Theorem 3.1 and show that, in many cases of interest, it simplifies significantly. Consider an arbitrary solution $Y$ of (2.1), satisfying the assumptions of the theorem. Notice that the choice of $s_n := t + \iota/n$, $n \in \mathbb{N}$, with sufficiently small $\iota \in (0, 1)$, and any choice of $\{\varepsilon^x \in (0, z^x)\}$, ensure that
\begin{equation}
\prod_{m=1}^{\infty} \left( 1 - 4 \mathbb{P} \left( \frac{z^x - \varepsilon^x + \alpha(s_m - t)}{\sqrt{\alpha^2(s_m - s_{m+1}) + \alpha^2(s_{m+1} - t)}} \right) \right) > 0, \quad x \in \mathcal{X}.
\end{equation}

Then, the condition $C^x \varepsilon^x > 1$, for all $x, x' \in \mathcal{X}$, is sufficient for the state to be fragile. The same sufficient condition for a jump is derived [DIRT15b] and [NS17a], in the single-type setting (2.2). In fact, the model in [DIRT15b] does not have the logarithm in the dynamics (2.1), which allows the authors to show that a jump occurs if and only if the density of particles around zero exceeds a given threshold, for each type, a jump has to occur (in fact, for each type $x$) at the time. On the other hand, Theorem 2.2 (cf. Remark 1) yields the existence of a solution $Y^0$ continuous at $t$ under the following condition: for every $x \in \mathcal{X}$, $\alpha^x$ is bounded from below by a constant, and there exist $\{\eta^x_n \downarrow 0\}$, s.t.
\begin{equation}
\mathbb{P}(x^x \geq t, Y^x_{x^x} \in (0, \eta^x_n)) \leq 1 - e^{-\eta^x_n/C^x}, \quad n \in \mathbb{N}, \quad x \in \mathcal{X}.
\end{equation}

(The validity of Assumption 2.1(b) is shown in the proof of Theorem 3.1.) Note that the above assumption is slightly stronger than the condition: $C^x \varepsilon^x < 1$, for all $x, x' \in \mathcal{X}$ and all $\{c^x, z^x\}$ satisfying (3.1). This shows the (relative) sharpness of the result of Theorem 3.1. Nevertheless, for a general $\mathcal{X}$, the range of cases “in between” the two extremal ones considered above (i.e., when neither “$C^x \varepsilon^x > 1$, for all $x, x' \in \mathcal{X}$” nor “$C^x \varepsilon^x < 1$, for all $x, x' \in \mathcal{X}$” hold) is very large. It is exactly in this intermediate regime when the topology of the network, given by $\kappa$, becomes important for characterizing the fragility.

In the next result, we make weaker assumptions on $Z$ but slightly stronger assumptions on $(X, \kappa, c, z)$ (this statement is explained in more detail further in this section). As a preparation for the upcoming Theorem 3.4, we supply the following assumption.

**Assumption 3.2.** Fix $t \in [0, T)$. For any $\{z^x > 0\}_{x \in \mathcal{X}}$, there exist a $\Theta \in (0, T - t)$ and a random field $\xi^x_{t_1, t_2}$, $t \leq t_1 < t_2 \leq t + \theta$, $x \in \mathcal{X}$, with
\begin{equation}
\mathbb{P} \left( - \min_{t_1 \leq s \leq t_2} (Z^x_s - Z^x_t) \geq z \bigg| Z^x_s, s \in [0, t] \right) \geq \mathbb{P}(\xi^x_{t_1, t_2} \geq z)
\end{equation}
for every $z \in (-\infty, z^x]$, $t \leq t_1 < t_2 \leq t + \theta$, and $x \in \mathcal{X}$.

The above assumption states that the increments of $Z^x$ can be dominated stochastically, uniformly over all initial times. It is shown in the proof of Theorem 3.1 that any $Z$ in the form (3.2), with appropriately bounded characteristics (as in Theorem 3.1), does satisfy Assumption 3.2. On the other hand, there exist many processes that satisfy Assumption 3.2 and are not semimartingales with bounded characteristics (see the example at the end of this section).

**Definition 3.3.** Let the matrix $(C^x \kappa(x, \{x\})c^x)_{x, x' \in \mathcal{X}}$ be irreducible. The limit
\begin{equation}
\rho := \lim_{n \to \infty} \frac{1}{n} \log \int_{\mathcal{X}^n} \prod_{m=0}^{n-1} (C^{x_m} \kappa(x_{m+1}) \kappa(x_0, dx_1) \kappa(x_1, dx_2) \ldots \kappa(x_{n-1}, dx_n))
\end{equation}

is referred to as its logarithmic Perron-Frobenius eigenvalue (see the Perron-Frobenius theorem in the form of [DZ10] theorem 3.1(e)).

The above definition is a key to understanding the next result. Notice that “$C^x \epsilon^x > 1$, for all $x, x' \in \mathcal{X}$” implies “$\varrho > 0$”, and “$C^x \epsilon^x < 1$, for all $x, x' \in \mathcal{X}$” implies “$\varrho < 0$”. However, “$\varrho > 0$” does not imply “$C^x \epsilon^x > 1$, for all $x, x' \in \mathcal{X}$”. Thus, the sign of $\varrho$ provides the descriptin of the “in between” cases omitted in the discussion following Theorem 3.1.

**Theorem 3.4.** Fix an arbitrary $t \in [0,T)$ and assume that $\{Z^x\}_{x \in \mathcal{X}}$ satisfy Assumptions 2.1(b) and 3.2, with the associated $\{\xi^x_{s,s'}\}$. Suppose that a solution $\{Y^x\}_{x \in \mathcal{X}}$ of (2.1) admits $\{c^z, z^x\}$, satisfying (3.1), such that there exist $s_n \downarrow t$ fulfilling

$$\lim_{n \to \infty} \frac{1}{n} \log \prod_{m=1}^{n-1} \mathbb{P}(\xi^x_{s_{m+1},s_m} \leq \varepsilon) = 0, \quad \forall \varepsilon > 0, \ x \in \mathcal{X},$$

(3.8)

$$\lim_{n \to \infty} \frac{1}{n} \log \left( \min_{x \in \mathcal{X}} \mathbb{E}[\xi^x_{s_{m+1},s_m} \land z^x] \right) \geq 0,$$

(3.9)

and that $(C^x \kappa(x, \{x'\}) \epsilon^x)_{x,x' \in \mathcal{X}}$ is irreducible, with a strictly positive logarithmic Perron-Frobenius eigenvalue. Then, $Y^x_t > Y^x_t$ for at least one $x \in \mathcal{X}$.

**Remark 3.5.** If the matrix $(C^x \kappa(x, \{x'\}) \epsilon^x)_{x,x' \in \mathcal{X}}$ is not irreducible, the latter assumption can be replaced by the condition that at least one of its non-empty closed irreducible components has a strictly positive logarithmic Perron-Frobenius eigenvalue.

If every $Z^x$ is a “nice” semimartingale (as in Theorem 3.1), then, the conditions (3.8) and (3.9) are satisfied. Indeed, using $\{\xi^x_{s,s'}\}$ constructed in the proof of Theorem 3.1 (cf. Appendix B), we obtain:

$$\lim_{n \to \infty} \frac{1}{n} \log \prod_{m=1}^{n-1} \mathbb{P}(\xi^x_{s_{m+1},s_m} \leq z^x - \varepsilon^x) = 0,$$

(3.10)

$$\lim_{n \to \infty} \frac{1}{n} \log \left( \min_{x \in \mathcal{X}} \mathbb{E}[\xi^x_{s_{m+1},s_m} \land z^x] \right) \geq \frac{z^x - \varepsilon^x + \alpha(s_m - t)}{\sqrt{2}(s_m - s_{m+1}) + \alpha^2(s_{m+1} - t)} = 0,$$

(3.11)

for $s_n := t + \iota/n$, $n \in \mathbb{N}$, with any small enough $\iota \in (0,1)$. Thus, if $\{Z^x\}$ are “nice” semimartingales, a sufficient condition for fragility can be formulated solely in terms of the associated Perron-Frobenius eigenvalue. Such an explicit formulation comes at a slight loss of generality: the case of $\varrho = 0$ is not covered by Theorem 3.4, whereas, in general, it is within the scope of Theorem 3.1.

Theorem 3.4 and the above discussion, allow us to ignore the exact dynamics of $Z$ (provided it is a “nice” continuous semimartingale) and describe the states of fragility only in terms of the current conditional distribution and the topology of the network. This leads to very interesting observations, such as the following example. Consider two particle systems, $(C, \kappa^1, Z)$ and $(C, \kappa^2, Z)$, with arbitrary $\{C^x > 0\}$ (the same for both models), with $\kappa^1(x, dx') = \delta_x(dx')$, and with $\kappa^2(x, dx') = dx'/|\mathcal{X}|$ (on $\mathcal{X}$, we
denote by ‘\(dx\)’ the counting measure). We refer to the first system as a “completely
classified” network, and to the second one as the “fully connected” one. The rationale
behind these terms is clear: the particles in the first system are only connected within
the same type, while the particles in the second system are connected to every other
particle in the network, uniformly. Note that the total strength of the out-coming
connections from each particle is the same in both networks – only the distribution of
the connections varies. Let us also assume that \(\{Z^x\}\) are the same for both systems
and are “nice” semimartingales, so that (3.8) and (3.9) are satisfied. Thus, with
everything else being equal, we can compare the fragility of the two systems at time
\(t=0\) by computing the associated Perron-Frobenius eigenvalues. For the first network,
the associated matrix \((C^x \kappa^1(x, \{x'\}) \kappa^2_{x',x} x, x' \in X)\) is, clearly, reducible, hence, we analyze
every closed irreducible component \(\{x\}\) separately.\(^4\)

\[
\varrho^1 = \lim_{n \to \infty} \frac{1}{n} \log(C^x e^x)^n = \log(C^x e^x).
\]

Thus, the first system is fragile whenever we can find \(\{c^x, z^x\}\), satisfying (3.1), s.t.
\(\max_{x \in X} C^x e^x > 1\). For any closed irreducible component \(X'\) of the second system, we have:

\[
\varrho^2 = \lim_{n \to \infty} \frac{1}{n} \log \int_{X'} \prod_{m=0}^{n-1} (C^x m \ k^x m+1) \ k(dx_1) \ k(dx_2) \ldots k(dx_n) = \log \left( \frac{1}{|X|} \sum_{x \in X} C^x e^x \right).
\]

Thus, the second system is fragile whenever we can find \(\{c^x, z^x\}\), satisfying (3.1), s.t.
\(\frac{1}{|X|} \sum_{x \in X} C^x e^x > 1\). Since the average never exceeds the maximum, the above
analysis indicates that the fully connected system is more stable. Of course, it does
not constitute a rigorous proof of this statement, as Theorem 3.4 only provides a
sufficient condition for fragility. Nevertheless, one can use Theorem 2.2 to show that
the fragility of the second system implies the existence of \(\{c^x, z^x\}\), satisfying (3.1), s.t.
\(\frac{1}{|X|} \sum_{x \in X} C^x e^x \geq 1\) and, in turn, \(\max_{x \in X} C^x e^x \geq 1\). Assuming that every \(Z^x\) is given
by the sum of a Brownian motion and a random initial condition, with continuous
density, we can use the results of [NS17a] to conclude that \(\max_{x \in X} C^x e^x > 1\) implies
the fragility of the first system. Thus, up to the special case where the density of \(Z^x_0\)
at zero is exactly \(1/C^x\), for all \(x\), the topology of the fully connected network is at
least as stable as the topology of the completely clustered one. Using the existence
result in Theorem 2.2, one can easily show that the fully connected network is strictly
more stable: i.e., there exist states at which the first one is fragile, while the second
one is not.

We conclude this section by noting that, while the condition (3.8) is expected to
hold for all cases of interest, the condition (3.9) is not a mere technicality. Although
it is satisfied by all “nice” continuous semimartingales \(Z\), it may fail for many other
processes \(Z\) (e.g., for continuous semimartingales with unbounded characteristics). In
the latter case, the positivity of the logarithmic Perron-Frobenius eigenvalue no longer
implies that the state is fragile. Consider, e.g., \(X := \{0\}\) and \(Z^0_t := Z^0_0 + \int_0^t B_t, t \in [0, T]\) with a continuous function \(f : [0, T] \to [0, \infty)\) satisfying \(\theta := \lim \inf_{t \to 0} (f(t) \sqrt{T}) > 0\). We claim that (3.9) does not hold at \(t = 0\), since for any \(z^0 > 0\), any \((s_n)_{n \in \mathbb{N}}\)

\(^4\)Strictly speaking, \(\{x\}\) may not be irreducible if \(C^x e^x = 0\), but it is easy to see that our conclusion
about fragility remains valid even when such cases are taken into account.
and $\epsilon > 0$ independent of $B$ (3.14)

$E$ exponentially and connected by a single edge of the graph (unless stated otherwise).

Moreover, with such a choice of $Z$, even if $g > 0$, the state may not be fragile. Indeed, consider $f_t = -C^0 \log P(Z^0_0 + \min_{s \in [0,t]} B_s > 0)$, $t \in [0,T]$, and assume that $Z^0_0$ is independent of $B$ and satisfying $P(Z^0_0 \in (0, z)) \geq h z$, $z \in [0,\varepsilon]$, for some $h > 1/C^0$ and $\varepsilon > 0$. It is easy to see that $\lim_{t \downarrow 0} (f_t / \sqrt{t}) > 0$. Choosing $t = 0$, $c^0 = h$, and $z^0 = \varepsilon$, we notice that (3.13) is satisfied and that $g = \log C^0 c^0 > 0$. Nevertheless, (2.1) does have a continuous solution

$$Y^0_t := Z^0_0 + B_t = Z^0_0 + f_t + B_t + C^0 \log P(Z^0_0 + \min_{s \in [0,t]} B_s > 0), \quad t \in [0,T].$$

Intuitively, the solution $Y^0$ does not jump at $t = 0$ because the driving process $Z^0$ has an “infinitely” large positive drift at that time, which pushes the particles up very fast. Hence, even though a sufficiently large fraction of the particles is concentrated around zero (which would be enough to generate a jump if $Z^0$ was a “nice” continuous semimartingale), it does not cause a non-zero fraction of them to die simultaneously.

4. CONTROLLED MEAN FIELD DYNAMICS AND EQUILIBRIUM

4.1. Model setup. In this section, we consider particle systems of the form (2.1), in which the strength of connections across types, given by the kernel $\kappa(x, dx)$, is controlled dynamically by the particles. We choose natural objectives for the individual particles and analyze the equilibrium connections that arise in the associated mean field game. Remarkably, we show that, in equilibrium, the particles adjust their connections in such a way that a cascade never occurs.

As in the previous sections, we consider an abstract (nonempty) finite set $\mathcal{X}$. We endow $\mathcal{X}$ with a probability measure $\mu$, s.t. $\mu(\{x\}) > 0$ for all $x \in \mathcal{X}$. We fix an arbitrary function $O : \mathcal{X}^2 \to \{0,1\}$, which indicates the types that are “physically” connected to each other: i.e., $x$ is physically (directly) connected to $y$ if and only if $O(x,y) = 1$ 5. The pair $(\mathcal{X}, O)$ defines a directed graph, whose edges indicate the physical connections. As before, we consider a system of particles characterized by their type $x \in \mathcal{X}$ and their level of healthiness $Y^x$ (the latter changes stochastically over time). We assume that there exist infinitely many particles of each type. The value

5In what follows, we drop the adverb “directly”, and say that two nodes are connected if they are connected by a single edge of the graph (unless stated otherwise).
of $\mu(\{x\})$ represents the total mass of all particles of type $x$, and the initial distribution of particles across their characteristics, $(x, y)$, is given by $\mathbb{P}(Y^x \in dy) \mu(dx)$.

In this section, we consider a **resource-sharing dynamic network game** with a continuum of players. More specifically, the proposed model falls within the class of mean-field games (see Section 1 for more on such games). In this game, the edges of $(\mathcal{X}, O)$ indicate the opportunities for sharing a given resource: i.e., if $x$ is connected to $x'$, then, a particle of type $x$ is allowed to give access to a certain resource for the particles of type $x'$. In addition, the edges of $(\mathcal{X}, O)$ have weights, which indicate the amount of resource shared between the particles of two types. The particles are allowed to change these weights dynamically, in order to optimize their objective. In such a game, the dynamics of a representative particle are given by

\begin{equation}
Y^x_t = Z^x_t + \int_0^t \int_0^L C^b_{s, \nu_s(x')} d\log \theta_s(x') \mu(dx'), \quad t \geq 0, \quad x \in \mathcal{X},
\end{equation}

where

\begin{equation}
Z^x_t := Y^x_0 + \sigma W_t + \int_0^t L \left[ x, P(x, C^b_s, C^d_s) - C^b_s r_s(x) + \int_{\mathcal{X}} C^d_s r_s(x') \nu_s(x') \mu(dx') \right] ds,
\end{equation}

$Y^x_0$ is independent of the Brownian motion $W$, and the following consistency requirements hold:

\begin{equation}
\tau^x = \inf \{ t \geq 0 : Y^x_t \leq 0 \}, \quad \theta_t(x) := \mathbb{P}(\tau^x > t) = \mathbb{P}\left( \inf_{s \in [0, t]} Y^x_s > 0 \right), \quad O(x, x') = 0 \Rightarrow \nu(x') = 0 \quad \forall \ x' \in \mathcal{X}.
\end{equation}

If the input elements $(C^b, \nu(\cdot))$ are deterministic and constant in time (but, possibly, different across $x$), the equations (4.1) and (4.3) coincide with (2.1). The main difference of this part of the paper is that the quantities $(C^b, C^d, \nu(\cdot), r(\cdot))$ are determined endogenously.

Let us discuss the general interpretation of the above dynamics. Recall that each particle can decide on how much of the resource it is willing to share with other types, thus, creating the weights of the graph edges\(^7\). The number $C^d_t$ denotes the overall amount of resource that the given particle is willing to share at time $t$. The probability measure $\kappa_t(dx') = \nu_t(x') \mu(dx')$ denotes the distribution of the total amount $C^d_t$ among various neighbors: i.e., the contribution of this particle to the weight of the edge from $x$ to $x'$ is $C^d_t \nu_t(x') \mu(dx')$. Similarly, $C^b_t$ denotes the overall amount of resource it is willing to receive at time $t$.\(^6\) Gaining access to the resource is costly: i.e., a particle of type $x'$ has to pay $r(x')$ units of currency for having access to one unit of resource per unit of time. Similarly, the particles sharing the resource with a particle of type $x'$ receive $r(x')$. This explains the last two terms in the right hand side of (4.2). Having received $C^b_t$ and shared $C^d_t$ units of resource at time $t$ yields the profit or loss of $P(x, C^b_t, C^d_t)$ per unit of time, through an internal mechanism of each particle (e.g., the business profits.

\(^6\)If a particle shares a unit of resource with a certain type, it is distributed uniformly across all particles of this type.

\(^7\)More precisely, as we consider a continuum player game (cf. [Aum64]), $C^d_t$ denotes the amount of resource per unit mass of agents lent by the particle. The same applies to other amounts (of resource or currency) related to an individual particle.
or losses).\footnote{In the simplest case, $P(x, C^b_t, C^l_t)$ is proportional to $C^b_t - C^l_t$, but one can easily come up with other relevant examples where the dependence is non-linear.} We assume that the profits and losses affect the healthiness of a particle through the function $L$, which explains the third term in the right hand side of (4.2). The Brownian noise in (4.2) represents the changes in the healthiness of a particle that are due to external factors. Finally, the death of each particle (occurring when $Y$ hits zero) causes immediate damage to the healthiness of the particles that share the resource with it. The magnitude of this damage is determined by the strength of the connection, which explains the last term in the right hand side of (4.1) (see more on the interpretation of this term in the discussion following equation (2.1)).

There are several concrete examples where such games arise naturally. For example, consider a cybersecurity problem, where the particles represent companies. Each company may allow others to gain access to its computer resources (e.g., servers, advertisement platforms, etc.), collecting a fee for this service. The other companies agree to pay the fee as this service may generate additional profits for their business (e.g. increase the sales volume). The process $Y$, in this case, represents the level of protection of a given company against cyber-attacks. When $Y$ falls to zero, the company gets infected with a malicious software, which spreads to other companies connected to it, instantaneously decreasing their level of protection. Clearly, there exist many other examples. However, in order to obtain stronger results, one needs to choose a specific interpretation of such a game, leading to a specific choice of the functions $L$ and $P$. In this section, we focus on the credit network game, in which

$$L(x, z) = z, \quad P(x, c^b_t, c^l_t) = \overline{\alpha}(x) + \alpha(x)(c^b_t - c^l_t),$$

with some $\overline{\alpha}, \alpha \in [0, \infty)^{|X|}$, fixed throughout the rest of this section.

In this game, the particles represent individuals or organizations engaged in short-term lending and borrowing of funds, and $Y^x$ represents the logarithm of the capital of a representative particle of type $x$. The value of $C^l_t$ represents the amount of funds lent by the particle within the network at time $t$ (i.e. the particle’s contribution to the total weight of out-coming edges of node $x$, where $x$ is the particle’s type), measured as a multiple of the particle’s capital. Similarly, $C^b_t$ represents the amount of funds borrowed by the particle (i.e. the particle’s contribution to the total weight of in-coming edges of $x$). The probability measure $\kappa_t(dx') = \nu(x') \mu(dx')$ represents the distribution of the funds lent by a particle across the borrowers’ types. The capital lent to each type is distributed uniformly among the particles of this type. We assume that the net balance of borrowing less lending, for each particle, is invested externally (e.g. invested in real economy), producing the return rate $\alpha(x)$ for the particles of type $x$. In the absence of borrowing and lending, a particle of type $x$ generates returns $\overline{\pi}(x)$ (one may assume $\overline{\pi} \equiv \alpha$). The value of $\overline{r}_t(x)$ represents the interest rate paid by a particle of type $x$ for borrowing funds within the network. A particle dies, or defaults, when $Y$ becomes non-positive, causing instantaneous losses to other particles, equal to the strength of the connection from the latter to the former (i.e., the amount borrowed by the defaulted particle). A defaulted particle leaves the system, in the sense that it can no longer be a lender or a borrower. The value of $\theta_t(x)$ indicates the fraction of particles of type $x$ that have not died by time $t$. The processes $(C^b_t, C^l_t, \nu)$ are controlled by the particle. We consider only short-term lending and borrowing, which is reflected by the ability of each particle to adjust $(C^b_t, C^l_t, \nu)$ at any time.

\footnote{Strictly speaking, we normalize the capital of each particle so that its default barrier is equal to one.}
Finally, we assume that a regulator has imposed a constraint on the maximum size of non-core liabilities: i.e., the maximum size of a particle’s own funds lent within the system, measured as a multiple of its capital, $C^l - C^h$, cannot exceed a given constant $\bar{C} \in (0, \infty)$. Note that such a constraint, effectively, limits the strengths of connections within the system. However, unlike the formulations in [NS17a] and in Sections 2 and 3 of this paper, where the strength of every connection was prescribed exogenously, the present formulation allows for a strategic choice of the connections, within the limits imposed by the cap on the non-core liabilities $\bar{C}$ and by the “physical connections” matrix $O$.

Remark 2. There are several reasons for decomposing $\kappa_i$ into $\kappa_i = \nu_i \mu$. The first one is purely technical: ex ante, $\theta_i(x)$ may exhibit jumps, hence, without such a decomposition, it is not clear how to define the second term in the expression for $Y_t^x$ in (4.1). In the equilibrium we construct, in fact, $\theta_i(x)$ is absolutely continuous w.r.t. Lebesgue measure, so, ex post, we do not have this problem, nevertheless, we do not want to a priori exclude the possibility of jumps in equilibrium. Another reason is to be able to formulate the market clearing condition, stated later in this section.

Let us introduce the mathematical conventions we use throughout the rest of the paper. We denote $\mathbb{R}_+:=[0, \infty)$ and $\mathbb{R}_-:=(-\infty, 0]$. For any $p \in [0, \infty]$ and any subsets $A$ and $B$ of the Euclidean space, we denote by $L^p(A \rightarrow B)$ the space of all measurable functions, satisfying the associated integrability or essential boundedness conditions. By $L^p(A \rightarrow B)$ we understand the space of equivalence classes of the elements of $L^p(A \rightarrow B)$. We also use the short-hand notation, $L^p(A)$ and $L^p(A)$, if the associated functions take values in $\mathbb{R}$, as well as $L^p_+(A)$, $L^p_+(A)$, $L^p_+(A)$, and $L^p_+(A)$, if the associated functions take values in $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. Finally, we assume that a supremum or maximum over an empty set is equal to “$-\infty$”.

We assume that $C^h, C^l \in L^1_+(\mathbb{R}_+)$; $\nu \in (L^1_+(\mathbb{R}_+))^{|\mathcal{X}|}$ is such that $\int_{\mathcal{X}} \nu(x) \mu(dx) \in \{0, 1\}$, for all $t \geq 0$; $r \in (L^0_+(\mathbb{R}_+))^{|\mathcal{X}|}$, and $\theta \in (L^0_+(\mathbb{R}_+))^{|\mathcal{X}|}$ is such that, for all $x \in \mathcal{X}$, $\theta(x)$ is a non-increasing function with values in $(0, 1]$, starting from one. For convenience, we introduce a special notation for the spaces of possible $\theta$ and $\nu$:

$$\Theta = \{\theta \in L^0_+(\mathbb{R}_+ \rightarrow (0, 1]^{|\mathcal{X}|}) : \forall x \in \mathcal{X}, \theta(x) \text{ is non-increasing, } \theta_0(x) = 1\},$$

$$\mathcal{V}_0 = \left\{\nu \in \mathbb{R}^{||\mathcal{X}||} : \int_{\mathcal{X}} \nu(x') \mu(dx') = 1\right\},$$

$$\mathcal{V} = \left\{\nu \in L^0_+(\mathbb{R}_+ \rightarrow \mathbb{R}^{||\mathcal{X}||}) : \nu_t \in \mathcal{V}_0, \text{ a.e. } t \geq 0\right\}.$$

Remark 3. Note that we have restricted $\theta_t(x)$ to be strictly positive. We could easily lift this restriction, and allow for $\theta_t(x) = 0$, by making the notation slightly more complicated. We chose not to do it, because, in the equilibrium we construct, $\theta_t(x)$ does remain strictly positive at all times.

Similarly, we could allow the controls of the particles to be random (e.g., depending on $Y$). However, even then, there would still exist an equilibrium with deterministic optimal controls. Hence, we a priori restrict the setting to deterministic controls, to simplify the notation.

Definition 4.1. For a given pair $(r, \theta) \in (L^0_+(\mathbb{R}_+))^{|\mathcal{X}|} \times \Theta$, the control $(C^h, C^l, \nu) \in (L^0_+(\mathbb{R}_+))^2 \times \mathcal{V}$ is admissible for a particle of type $x$, if

1. the integrals in the right hand sides of (4.1) and (4.3) are finite for every $t \geq 0$;
2. $C^l_t \leq C^h + \bar{C}$, for a.e. $t \geq 0$;
(3) for a.e. \( t \geq 0 \) and all \( y \in \mathcal{X} \), \( O(x, y) = 0 \) implies \( \nu_t(y) = 0 \).

The second condition in the above definition represents the constraint on non-core exposure. The third condition ensures that the lending and borrowing occurs only between the particles that are physically connected (according to the adjacency matrix \( O \)).

**Remark 4.** Note that there is a certain asymmetry between lending and borrowing reflected in the dynamics (4.1)–(4.2) and in the definition of admissibility of the controls. Namely, we assume that the particles control both the size and the **distribution** of their lending, but they only control the size of their borrowing. Nevertheless, in equilibrium, the lending and borrowing distributions will be matched, so that “market clears”.

The asymmetry between lending and borrowing is an intentional modeling choice we make, which persists throughout the remainder of the paper. Intuitively, this choice corresponds to “dominant lenders” (equivalently, “competitive borrowers”): i.e., in any borrowing-lending deal, the lender decides whom to lend (more precisely, to which type to lend), and the borrower accepts the funds from whoever is willing to lend to it. We believe that this is a natural assumption, and it allows us to narrow down the set of possible equilibria, as will be shown later in the paper.

To emphasize the dependence of the state process \( Y \) on the market and on the controls, we will often write \( Y^x(r, \theta; \overline{C^b}, \overline{C^l}, \nu) \). A particle with (initial) characteristics \((x, y)\) aims to maximize the following objective

\[
J(x, y; r, \theta; \overline{C^b}, \overline{C^l}, \nu) := \mathbb{E} \left( \int_0^{\tau^x} e^{-\gamma t} Y^x_t(r, \theta; \overline{C^b}, \overline{C^l}, \nu) \, dt \mid Y^x_0 = y \right),
\]

with a constant \( \gamma > 0 \).

**Remark 5.** It is easy to see that the dynamics of \( Y \) and the objective are such that any particle will simply aim to maximize the drift of \( Y \). The same would be true for almost any other (reasonable) objective function. We view this (almost) independence of a specific choice of objective as a strength of the proposed model.

In order to define the notion of equilibrium in this model, we introduce \((\overline{C^b}, \overline{C^l}, \bar{\nu})\), representing the collective controls of all particles. Following the standard methodology of continuum-player games, we shall view the collective controls as a mapping from the set of particles’ characteristics, \((x, y)\), to the space of individual controls. However, it turns out that, in the present case, the optimal controls of each particle depend only on its type, \( x \), hence, to simplify the notation, we define: \( \overline{C^b}, \overline{C^l} \in (L^0_+(\mathbb{R}^+))^{|\mathcal{X}|} \), and \( \bar{\nu} \in \bar{\mathcal{V}} \), where we introduce

\[
\bar{\mathcal{V}}_0 := \left\{ \bar{\nu} : \mathcal{X} \to \mathbb{R}^{|\mathcal{X}|}_+ : \bar{\nu}(x, \cdot) \in \mathcal{V}_0, \forall x \in \mathcal{X} \right\}, \quad \mathcal{V} := L^0_+(\mathbb{R}^+ \to \bar{\mathcal{V}}_0).
\]

The functions \((r, \theta, \overline{C^b}, \overline{C^l}, \bar{\nu})\) are determined endogenously in equilibrium.

**Definition 4.2.** For a given pair \((r, \theta) \in (L^0_+(\mathbb{R}^+))^{|\mathcal{X}|} \times \Theta \) and (initial) characteristics \((x, y) \in \mathcal{X} \times (0, \infty)\), the control \((\overline{C^b}, \overline{C^l}, \nu)\) is optimal if it is admissible, and

\[
J(x, y; r, \theta; \overline{C^b}, \overline{C^l}, \nu) \geq J(x, y; r, \theta; \tilde{C^b}, \tilde{C^l}, \tilde{\nu}),
\]

for any other admissible control \((\tilde{C^b}, \tilde{C^l}, \tilde{\nu})\).

**Definition 4.3.** \((r, \theta, \overline{C^b}, \overline{C^l}, \bar{\nu}) \in (L^0_+(\mathbb{R}^+))^{|\mathcal{X}|} \times \Theta \times \{(L^0_+(\mathbb{R}^+))^{|\mathcal{X}|}\}^2 \times \bar{\mathcal{V}}\) form an equilibrium if
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(1) for every \((x, y) \in \mathcal{X} \times (0, \infty)\), \((\bar{C}^b(x), \bar{C}^l(x), \bar{v}(x, \cdot))\) is optimal;
(2) market clears: for every \(x \in \mathcal{X}\) and a.e. \(t \geq 0\),
\[
\bar{C}^b_t(x) = \int_{\mathcal{X}} \bar{v}_t(x', x) \bar{C}^l_t(x') \mu(dx');
\]
(3) and the model is consistent: for every \(x \in \mathcal{X}\) and every \(t \geq 0\),
\[
\theta_t(x) = \mathbb{P} \left( \inf_{s \in [0,t]} Y^x_s(r, \theta; \bar{C}^b(x), \bar{C}^l(x), \bar{v}(x, \cdot)) > 0 \right).
\]

In the remainder fo this section, we show how to construct such an equilibrium.

4.2. Construction of equilibrium: general approach. We will construct an equilibrium satisfying the following ansatz.

Ansatz 1. There exists a default rate \(\lambda \in (L^0_\mathcal{F}([\mathbb{R}_+])^{|\mathcal{X}|}, such that, for every \(x \in \mathcal{X}\),
\[
\theta_t(x) = 1 + \int_0^t \lambda_s(x) ds > 0, \quad \forall t \geq 0.
\]

We denote
\[
\bar{\lambda}_t(x) := \frac{d}{dt} \log \theta_t(x) = \frac{\lambda_t(x)}{\theta_t(x)}.
\]

The above ansatz implies that no cascades are possible in the equilibrium we construct. The economic explanation of this is as follows. If the distribution of particles’ health levels, within a given type, approaches a critical level, at which a cascade would be triggered by this type, the creditors will simply stop lending to the particles of this type (i.e. the total weight of in-coming edges for this node will vanish), thus, eliminating the propagation of contagion. A side effect of such action, of course, is that the particles in distress are more likely to die (as their drifts are reduced due to the lack of borrowing opportunities). In the equilibrium constructed herein, the above ansatz holds, and the firms act according to the principle described in above.

Remark 6. The above conclusions, and the entire model proposed herein, rely on the following assumptions: (i) the particles are well informed about the default intensities at each type, and (iii) they have a perfect control over their connections. In reality, of course, none of these assumptions are satisfied perfectly, and one expects to observe systems whose behavior lies somewhere in between the perfectly controlled and completely uncontrolled systems.

Under the above ansatz, the admissible dynamics of \(Y^x_t\), prior to default, can be rewritten as follows:
\[
dY^x_t = b(r_t(x), \bar{\lambda}_t, C^b_t, C^l_t, \nu_t; x) dt + \sigma W_t,
\]
where
\[
b(r_t(x), \bar{\lambda}_t, C^b_t, C^l_t, \nu_t; x) := \bar{\pi}(x) + (C^b_t - C^l_t) \alpha(x) - C^b_t r_t(x)
\]
\[
+C^l_t \int_{\mathcal{X}} \nu_t(x') \left( r_t(x') + \bar{\lambda}_t(x') \right) \mu(dx').
\]

Our strategy for constructing an equilibrium is based on the following observation. Note that the dynamics of \(\{Y^x_t\}\) are coupled across \(x \in \mathcal{X}\) via the interest rates \(\{r(x)\}\), the collective controls \(\{C^b(x), C^l(x), \bar{v}(x, \cdot)\}\), and the default rates \(\{\bar{\lambda}(x)\}\).

\[\text{It is natural to christen our construction a “controlled-fire equilibrium”, by analogy with the common practice in forest management. We thank R. Sircar for suggesting this name.}\]
Lemma 4.4. Consider any \( x \in X \), \((r, \theta) \in (L_0^0(\mathbb{R}_+))^{|X|} \times \Theta \) and \((C^b, C^l, \nu) \in (L_+^0(\mathbb{R}_+))^2 \times \mathcal{V} \). Assume that \( \theta \) satisfies Ansatz 4 with the associated \( \lambda \). Assume also that, for a.e. \( t \geq 0 \), we have:

\[
\begin{align*}
(4.7) & \quad \nu_t 1_{\{x': O(x,x') = 0\}} = 0, \\
(4.8) & \quad \int_X \nu_t(x') (r_t(x') + \bar{\lambda}_t(x')) \mu(dx') = R_t(x), \\
(4.9) & \quad r_t(x) = \alpha(x) \lor R_t(x), \\
(4.10) & \quad C^b_t \geq 0, \quad C^l_t = (\bar{C} + C^b_t) 1_{\{R_t(x) > \alpha(x)\}}, \\
(4.11) & \quad R_t(x) := \left( \max_{x': O(x,x') = 1} (r_t(x') + \bar{\lambda}_t(x')) \right)^+. 
\end{align*}
\]

If, in addition, \( b(r_t(x), \bar{\lambda}_t, C^b_t, C^l_t, \nu_t; x) \) is locally integrable in \( t \in \mathbb{R}_+ \), then, \((C^b, C^l, \nu)\) is optimal for any particle with (initial) characteristics \((x,y)\), for any \( y > 0 \).

Remark 7. The conditions (4.8) and (4.11) can be interpreted as follows. The lenders lend everything to the neighbors that offer the best profit, i.e. have the largest \("r_t + \bar{\lambda}_t"\) (provided it exceeds their profits from external investment), and borrowing generates zero profits. This, in particular, implies that \( \nu_t \) must be supported on the set on which this maximum is attained, and it is consistent with the implicit assumption of "dominant lenders" (or "competitive borrowers"), as discussed in Remark 4. Of course, this is an extreme case – in practice, borrowing should result in some profit – but, as explained below, this assumption allows for a tractable equilibrium, and more realistic equilibria can be constructed as perturbations of the proposed one.

Proof. It is easy to see that \((C^b, C^l, \nu)\) is admissible. Consider any other admissible control \((\bar{C}^b, \bar{C}^l, \bar{\nu})\) and denote by \( b \) and \( \bar{b} \) the associated drifts. Notice that (4.8) and (4.11) imply:

\[
\int_X \bar{\nu}_t(x') (r_t(x') + \bar{\lambda}_t(x')) \mu(dx') \leq R_t(x) = \int_X \nu_t(x') (r_t(x') + \bar{\lambda}_t(x')) \mu(dx').
\]

In addition, (4.9) and (4.10) yield

\[
(C^b_t, C^l_t) \in \arg\max_{(y,z): y \geq 0, 0 \leq z \leq \bar{C} + y} (y(\alpha(x) - r_t(x)) + z(R_t(x) - \alpha(x))).
\]

Recalling (4.6) we conclude that \( b \geq \bar{b} \) and, hence, \( J(x,y; r, \theta; C^b, C^l, \nu) \geq J(x,y; r, \theta; \bar{C}^b, \bar{C}^l, \bar{\nu}) \). \( \square \)
Extending the above result in an obvious way, we obtain a sufficient condition for equilibrium.

**Corollary 1.** Consider any \( (r, \theta, \bar{C}^b, \bar{C}^l, \bar{\nu}) \in (L^0_+([0, +\infty))) \times \mathcal{X} \times ((L^0_+([0, +\infty)))^2 \times \bar{\nu}. \)
Assume that, for every \( x \in \mathcal{X} \), the quadruple \( (r, \theta, \bar{C}^l, \bar{\nu}) \) and the control \( (C^b := \bar{C}^b(x), C^l := \bar{C}^l(x, \nu := \bar{\nu}(x, \cdot)) \) satisfy the assumptions of Lemma 4.4. If, in addition, \( (\bar{C}^b(x), \bar{C}^l(x), \bar{\nu}(x)) \) holds for a.e. \( t \) and every \( x \), and \( (\bar{C}^b, \bar{C}^l, \bar{\nu}) \) holds for every \( t \) and \( x \), then, \( (r, \theta, \bar{C}^b, \bar{C}^l, \bar{\nu}) \) form an equilibrium.

To illustrate our solution approach, we notice that, for a given \( \lambda_t(\cdot) \), the conditions \( (4.4) \) and \( (4.7) - (4.11) \) can be viewed as a system of equations for \( \{r_t(x), \bar{C}^b_t(x), \bar{C}^l_t(x), \bar{\nu}_t(x)\}_{x \in \mathcal{X}} \). Assume that we can solve the static problem: i.e., we can find a function that maps any \( \lambda_t(\cdot) \) into a solution to this system. Then, to construct an equilibrium, we only need to solve the dynamic problem: i.e. find a fixed point of a mapping from \( \lambda \) into the time-derivative of a logarithm of the right hand side of \( (4.5) \), computed via the solution to the static problem \( (r(\lambda), C^b(\lambda), C^l(\lambda), \bar{\nu}(\lambda)) \).

The solutions to the two problems are described in the remainder of this section.

### 4.3. Solving the dynamic problem

In this subsection, we assume that the static problem has been solved (in the next section, we construct such a solution).

**Assumption 1.** Assume that there exists a measurable mapping
\[
S : \mathbb{R}^{|\mathcal{X}|} \ni \lambda_t \mapsto \left( \hat{r}(\lambda_t), \hat{C}^b(\lambda_t), \hat{C}^l(\lambda_t), \bar{\nu}(\lambda_t) \right) \in \left( \mathbb{R}^{|\mathcal{X}|} \right)^3 \times \bar{\nu}_0,
\]
such that,

1. for every \( \lambda_t \in \mathbb{R}^{|\mathcal{X}|} \) and every \( x \in \mathcal{X} \),
\[
\begin{align*}
  r_t := \hat{r}(\lambda_t), \quad C^b_t := \hat{C}^b(\lambda_t), \quad C^l_t := \hat{C}^l(\lambda_t), \quad \bar{\nu}_t := \bar{\nu}(\lambda_t),
\end{align*}
\]
\( C^b_t := \bar{C}^b_t(x), \quad C^l_t := \bar{C}^l_t(x), \quad \nu_t := \bar{\nu}_t(x, \cdot) \)
\( \) satisfy \( (4.4) \) and \( (4.7) - (4.11) \);

2. the mapping
\[
\mathbb{R}^{|\mathcal{X}|} \ni \lambda_t \mapsto \left( x \mapsto \hat{b}(\lambda_t; x) := b \left( \hat{r}(\lambda_t; x), \lambda_t, \hat{C}^b(\lambda_t; x), \hat{C}^l(\lambda_t; x), \bar{\nu}(\lambda_t; x, \cdot; x) \right) \right) \in \mathbb{R}^{|\mathcal{X}|},
\]
\( \) with \( b \) given by \( (4.6) \), is Lipschitz and absolutely bounded by a constant.

In the next subsection, we construct \( S \) satisfying the above assumption. It is worth mentioning that the construction of such \( S \) is the main mathematical challenge resolved in this part of the paper. In this subsection, we assume that the desired \( S \) is given.

Notice that, for any measurable \( [0, T] \ni t \mapsto \lambda_t \in \mathbb{R}^{|\mathcal{X}|} \), the mappings
\[
(t, x, x') \mapsto \hat{r}(\lambda_t; x), \hat{C}^b(\lambda_t; x), \hat{C}^l(\lambda_t; x), \bar{\nu}(\lambda_t; x, x'; x')
\]
are measurable, and \( (r, \bar{C}^b, \bar{C}^l, \bar{\nu}) \), defined in part 1 of Assumption 1, satisfy all assumptions of Corollary 1 except, possibly, \( (4.5) \).

The main goal of this subsection is to construct \( \lambda \) such that the associated \( (r, \theta, \bar{C}^b, \bar{C}^l, \bar{\nu}) \) satisfy \( (4.5) \). To this end, we fix an arbitrary \( T \in (0, +\infty) \) and analyze the fixed-point problem of the mapping \( D \) from \( L^2 \left( [0, T] \to \mathbb{R}^{|\mathcal{X}|} \right) \) to \( L^0 \left( [0, T] \to \mathbb{R}^{|\mathcal{X}|} \right) \), given by
\[
D : \lambda \mapsto \left( (t, x) \mapsto \frac{d}{dt} \log \mathbb{P} \left( \inf_{s \in [0,t]} Y^x_s(\lambda) > 0 \right) \right),
\]
where
\[(4.13)\] 
\[Y_t^x(\hat{\lambda}) := Y_0^x + \int_0^t \hat{b}(\hat{\lambda}_s; x) ds + \bar{\sigma}(x)t + \sigma W_t.\]

Note that the mapping \(D\) is well defined due to Assumption \[1\]. Indeed, for any measurable \([0, T] \ni t \mapsto \hat{\lambda}_t \in \mathbb{R}^{\mathcal{X}_1}\), the mapping \((t, x) \mapsto \hat{b}(\hat{\lambda}_t; x)\) is measurable. The latter and the boundedness of \(\hat{b}\) imply that \(Y^x(\hat{\lambda})\) is well defined. In addition, for any two modifications \(\hat{\lambda}_1\) and \(\hat{\lambda}_2\), for every \(x\), we have: \(\hat{b}(\hat{\lambda}_1; x) = \hat{b}(\hat{\lambda}_2; x)\) for a.e. \(t\), and, in turn, \(Y^x(\hat{\lambda}_1)_t = Y^x(\hat{\lambda}_2)_t\) for all \(t\).

Our analysis of the fixed-point problem for \(D\) is based on the associated system of PDEs for \(p = \{p(t, y; x)\}\), where \(p(t, \cdot; x)\) is the density of the restriction of the law of \(Y^x_t\) to \((0, \infty)\). In accordance with the existing results (cf. [NS17a, DIRT15a]), we expect \(p\) to satisfy the following system of equations:
\[(4.14)\]
\[p_t(t, y; x) = -\hat{b}_t(\hat{\lambda}_t; x) p_y(t, y; x) + \frac{1}{2} \sigma^2 p_{yy}(t, y; x), \quad p(0, y; x) = p_0(y; x), \quad p(t, 0; x) = 0,\]
\[(4.15)\]
\[\hat{\lambda}_t(x) = -\frac{\sigma^2}{2} \frac{p_y(t, 0; x)}{\int_0^\infty p(t, y; x) dy}.\]

The following proposition proves that there exists a unique weak solution to the above system, in turn, proving the existence and uniqueness of the fixed point of \(D\). It is based on the results of [NS17a]. In this proposition, and in its proof, we denote by \(W^n_2(\mathbb{R}_+)^n\) and \(W^{n,m}_2([0, T] \times \mathbb{R}_+)\) the standard Sobolev spaces.

**Proposition 4.5.** Consider a mapping \(S\), satisfying Assumption \[7\] and assume that, for every \(x\), \(Y_0^x \sim p_0(y; x) dy\), with \(p_0(\cdot; x) \in W^1_2(\mathbb{R}_+)\) and \(p_0(0; x) = 0\). Then, there exists a unique fixed point of the associated mapping \(D\).

**Proof.** Let us fix an arbitrary \(\bar{\lambda} \in L^2([0, T] \to \mathbb{R}_{\mathcal{X}_1})\). Then, repeating steps 1 and 2 in the proof of Lemma 3.1 of [NS17a], we prove the following statement. For every \(x \in \mathcal{X}\), we have that, for all \(t \in [0, T]\), the distribution of
\[Y_t^x(\bar{\lambda}) 1_{\{\inf_{s \in [0, t]} Y_s^x(\bar{\lambda}) > 0\}},\]
restricted to \((0, \infty)\), has a density \(p(t, \cdot; x)\) (recall that \(Y^x(\bar{\lambda})\) is defined in \(4.13\)).

Moreover, \(p(\cdot, \cdot; x)\) is the unique solution of \(4.14\) in \(W^{1,2}_2([0, T] \times \mathbb{R}_+)\). Lemma 3.2 of [NS17a] shows that, for every \(x\) and a.e. \(t\), \(D[S\bar{\lambda}]_t(t, x)\) is given by the right hand side of \(4.15\). In addition, the last paragraph of step 1 in the proof of Proposition 4.1, in [NS17a], shows that, for every \(x\), \(D[S\bar{\lambda}](t, x) \in L^2([0, T])\), hence, the range of \(D\) is contained in \(L^2([0, T] \to \mathbb{R}_{\mathcal{X}_1})\).

Next, step 3 in the proof of Proposition 4.1, in [NS17a], shows that there exists an increasing function \(C_1 : [0, T] \to \mathbb{R}_+\), converging to zero at zero, such that, for any \(\hat{\lambda}_1, \hat{\lambda}_2 \in L^2([0, T] \to \mathbb{R}_{\mathcal{X}_1})\), we have, for every \(x\):
\[\int_0^s (D[\hat{\lambda}_1](t, x) - D[\hat{\lambda}_2](t, x))^2 dt \leq C_1^2(s) \int_0^s (\hat{\lambda}_1(t, x) - \hat{\lambda}_2(t, x))^2 dt, \quad \forall s \in [0, T],\]
which yields the contraction property of \(D\) for sufficiently small \(T > 0\). Then, Banach theorem yields the existence and uniqueness of the fixed point of \(D\), for sufficiently small \(T > 0\). Step 4 in the proof of Proposition 4.1, in [NS17a], shows how to extend this result to arbitrary \(T \in (0, \infty)\). \(\Box\)
Note that the domain of \( D \) can be extended in a straight-forward way to the space of locally square integrable functions \( \mathbb{R}_+ \to \mathbb{R}^{\vert X \vert} \). Proposition 4.5 then, implies that there exists a unique fixed point of \( D \) in this space. Whenever we refer to a fixed point \( \hat{\lambda} \) of \( D \), we understand it as an element of this space.

For any \( \bar{\lambda} \in L^0(\mathbb{R}_+ \to \mathbb{R}^{\vert X \vert}) \), we denote

\[
\hat{\theta}_t(\bar{\lambda}; x) := \exp \left( \int_0^t \tilde{\lambda}_s(x) ds \right).
\]

Notice that, if \( \bar{\lambda} \) is (a modification of) the fixed point of \( D \), then \( \theta : (t, x) \mapsto \theta_t(x) := \hat{\theta}_t(\bar{\lambda}; x) \) belongs to \( \Theta \), and \((r, \theta, \tilde{C}^b, \tilde{C}^d, \tilde{\nu})\), with \((r, \tilde{C}^b, \tilde{C}^d, \tilde{\nu})\) defined in part 2 of Assumption \( \text{[1]} \) satisfy all assumptions of Corollary \( \text{[1]} \) including (4.5). Thus, we have proved the following result.

**Corollary 2.** Consider a mapping \( S \), satisfying Assumption \( \text{[4]} \) with the associated \((\hat{r}, \tilde{C}^b, \tilde{C}^d, \tilde{\nu})\). Assume that, for every \( x \in X \), \( Y_0^x \sim p_0(y; x) dy \), with \( p_0(\cdot; x) \in W^1_2(\mathbb{R}_+) \).

Let \( \bar{\lambda} \) be the fixed point of the associated mapping \( D \), and denote

\[
\begin{align*}
\hat{r}_t(x) &:= \hat{r}(\bar{\lambda}; x), \\
\theta_t(x) &:= \hat{\theta}_t(\bar{\lambda}; x), \\
\tilde{C}^b_t(x) &:= \tilde{C}^b(\bar{\lambda}; x), \\
\tilde{C}^d_t(x) &:= \tilde{C}^d(\bar{\lambda}; x), \\
\tilde{\nu}_t(x, x') &:= \tilde{\nu}(\bar{\lambda}; x, x').
\end{align*}
\]

Then, \((r, \theta, \tilde{C}^b, \tilde{C}^d, \tilde{\nu})\) form an equilibrium.

4.4. **Solving the static problem.** Herein, we present a solution to the static fixed-point problem: i.e., we construct a mapping \( S \) satisfying Assumption \( \text{[1]} \). Consistent with the initial formulation of the problem and with the equations (4.8)–(4.9) (cf. Remarks \( \text{[4]} \) and \( \text{[7]} \), the specific choice of construction presented herein can be interpreted as the assumption of “dominant lenders” (or “competitive borrowers”).

**Remark 8.** Even under the implicit assumption of dominant lenders, the uniqueness of equilibria is not clear. Nevertheless, certain characteristics of equilibria (such as the interest rate) may be unique, as discussed in Remark \( \text{[9]} \). In any case, the (constructive) proof of Proposition \( \text{[4.7]} \) shows that there exists a natural choice of such equilibrium.

As the derivations in this subsection are performed for a fixed time \( t \), for convenience, we drop this subscript. Below we formulate a sufficient condition for \( S : \lambda \mapsto (\hat{r}, \tilde{C}^b, \tilde{C}^d, \tilde{\nu}) \) to satisfy Assumption \( \text{[1]} \) (i.e., to be a solution to the static problem). Assume that we are given \((r, \bar{\lambda}, \tilde{C}^b, \tilde{C}^d, \tilde{\nu}) \in (L^0_+(\mathbb{R}_+))^{|X|} \times (L^0_+(\mathbb{R}_+))^{|X|} \times (L^0_+(\mathbb{R}_+))^{|X|} \times \tilde{\nu} \times \tilde{V} \) and consider the following conditions.

1. \( r \) satisfies

\[
(4.16) \quad r(x) = \alpha(x) \lor \max_{y : O(x, y) = 1} (r(y) + \bar{\lambda}(y)), \quad \forall x \in X.
\]

2. For every \( x \in X \):

\[
(4.17) \quad r(x) > \alpha(x) \Rightarrow \tilde{\nu}(x, x') = 1_{A(x)}(x')/\mu(A(x)), \quad \forall x',
\]

where \( A(x) \) is a nonempty subset of

\[
(4.18) \quad \tilde{A}(x) := \{ z \in X : O(x, z) = 1, r(z) + \bar{\lambda}(z) = \max_{y : O(x, y) = 1} (r(y) + \bar{\lambda}(y)) \},
\]

if \( r(x) \leq \alpha(x) \), then \( \tilde{\nu}(x, \cdot) \equiv 0 \).
Lemma 4.6. Let (4.11). This observation is formalized in the following lemma. (4.19) \( \bar{r} \) satisfies (4.16)–(4.19). To explain the rationale behind equations (4.16)–(4.19), we recall the principle discussed in Remarks 4 and 7 (compare (4.16)–(4.20) to (4.7)–(4.11)). Namely, in the equilibrium we construct, the particles lend everything to the neighbors that offer the best profit, i.e. have the largest “equilibrium we construct, the particles lend everything to the neighbors that offer the best profit, i.e. have the largest “

Proposition 4.7. There exists a mapping \( S : \mathbb{R}^{[X]} \ni \bar{\lambda} \mapsto \left( \hat{r}(\bar{\lambda}), \hat{C}^b(\bar{\lambda}), \hat{C}^l(\bar{\lambda}), \hat{\nu}(\bar{\lambda}) \right) \in \left( \mathbb{R}_+^{[X]} \right)^3 \times \mathcal{V}_0 \), such that the associated \( \left( r := \hat{r}(\bar{\lambda}), \bar{\lambda}, \hat{C}^b(\bar{\lambda}), \hat{C}^l(\bar{\lambda}), \hat{\nu}(\bar{\lambda}) \right) \) satisfy (4.16), (4.17), (4.19), for every \( \bar{\lambda} \). Moreover, for every \( x \in X \), the associated \( R(x) \), given by (4.20), is Lipschitz and bounded in \( \bar{\lambda} \).
Proof. For any given $\lambda$, we need to construct $(r, \tilde{C}, \tilde{C'}, \tilde{\nu})$ satisfying (4.16), (4.17), (4.19). W.l.o.g. we assume $X = \{1, \ldots, n\}$.

**Step 1: solving (4.16).** Notice that equation (4.16) is nothing else but a system of linear equations in the max-plus algebra. Recall that max-plus algebra is a semiring on $\mathbb{R} \cup \{-\infty\}$, in which the addition is replaced by the maximum operator, denoted $\oplus$, and the multiplication is replaced by the addition, $\odot$:

$$x \oplus y = \max(x, y), \quad x \odot y = x + y, \quad \forall x, y \in \mathbb{R} \cup \{-\infty\}.$$  

Note that the matrix addition and multiplication in the max-plus algebra can be defined in the usual way, via the above operations, and they satisfy the usual associativity and distributivity rules. Then, (4.16) reads as

$$r = A \odot r \oplus \alpha, \quad r, \alpha \in (\mathbb{R} \cup \{-\infty\})^n, \quad A \in (\mathbb{R} \cup \{-\infty\})^{n \times n},$$

where $\alpha_i = \alpha(i)$, for $i = 1, \ldots, n$, and $A$ is a square matrix, with

$$A_{ij} = \bar{\lambda}(j) - \infty \cdot 1_{\{O(i,j) = 0\}} - \infty \cdot 1_{\{O(i,j) = 1\}}, \quad i, j = 1, \ldots, n,$$

where $\infty \cdot 0 = 0$. Since the entries of $A$ are nonpositive, Theorems 3.20 and 4.75, along with Remark 4.80, in [BCOQ93], imply that

(4.22)  
$$r := A^* \odot \alpha,$$

with the Kleene star matrix

$$A^* := I \oplus A \oplus A^2 \oplus \cdots \oplus A^{n-1}, \quad I_{ij} = -\infty \cdot 1_{\{i \neq j\}},$$

is the smallest solution to (4.16), in the sense that it solves (4.16), and $r_i \leq \tilde{r}_i$, for all $i$, for any solution $\tilde{r}$. Note that the associated vector $R$, with $R_i := R(i)$, is given by

$$R = A \odot r \oplus e = (A \odot A^2 \odot \cdots \odot A^n) \odot \alpha \oplus e,$$

where $e$ is the vector of zeros. It is clear that $R_i \leq \max_j \alpha_j$, for all $i$. In addition, it is easy to see that, if $v$ is a Lipschitz function of $\lambda$, with values in $\mathbb{R}^n$, then, $A \odot v$ is also Lipschitz in $\lambda$. The operator $\oplus$ also preserves Lipschitz property. Thus, by induction, $R$ is Lipschitz in $\lambda$.

**Step 2: eliminating cycles.** Let us show that there exists an optimal lending allocation, denoted $\tilde{\nu}^m$, that satisfies (4.17) and produces no lending cycles. To this end, we define $\tilde{\nu} \in \tilde{V}$:

$$\tilde{\nu}(i, j) = \begin{cases} 1_{\tilde{A}(i)}(j)/\mu(\tilde{A}(i)), & r(i) > \alpha(i), \\ 0, & r(i) = \alpha(i), \end{cases}$$

where $\tilde{A}(i)$ is given by (4.18). Note that $\tilde{\nu}$ satisfies (4.17). Next, consider the new adjacency matrix $\tilde{O}$ induced by $\tilde{\nu}$,

$$\tilde{O}(i, j) = 1_{\tilde{\nu}(i, j) \neq 0},$$

and the associated directed graph. In the new graph $(\tilde{X}, \tilde{O})$, let us pick a node $x_0 \in X$ and denote by $C$ the union of all cycles containing this node. We assume that there exists $x_0 \in X$, s.t. $C$ is nonempty (otherwise, $(\tilde{X}, \tilde{O})$ contains no cycles, and we can skip this step). We denote $C = (x_0, x_1, \ldots, x_{k-1})$, with $k \geq 1$. The set $C$ has the following properties: (i) $C$ is connected, (ii) any two $x, y \in C$ are contained in at least one cycle, and (iii) all cycles containing at least one $x \in C$ are included in $C$.

Note that, for every cycle $(x_{i_0}, x_{i_1}, \ldots, x_{i_{l-1}}) \subset C$, we have

$$R(x_{i_j}) = r(x_{i_j}) = \max_{y:O(x_{i_j}, y) = 1}(r(y) + \bar{\lambda}(y)) = r(x_{i_{(j+1) \mod l}}) + \bar{\lambda}(x_{i_{(j+1) \mod l}}).$$
Since $\bar{\lambda} \leq 0$ an since any two points in $C$ are contained in a cycle in $C$, the above implies that
\[
\bar{\lambda}(x_0) = \cdots = \bar{\lambda}(x_{k-1}) = 0, \quad r(x_0) = \cdots = r(x_{k-1}) =: r^c > \max_{x_i \in C} \alpha(x_i) \geq 0.
\]
Next, for every $x_i \in C$, we define the “alternative profit”
\[
R^a(x_i) := \max_{y : O(x_i, y) = 1, y \notin C} \{ r(y) + \bar{\lambda}(y) \} \leq r^c.
\]
Our next goal is to show that there exists $x_i \in C$, s.t. $R^a(x_i) = r^c$. Assume the contrary: $R^a(x_i) < r^c$, for every $i = 0, \ldots, k-1$. Consider a new graph $(\tilde{X}, \tilde{\alpha})$, which only differs from $(X, \alpha)$ in that the set $C$ is “collapsed into” a single state, denoted $\tilde{x}$. Setting $\alpha(\tilde{x}) := r^c$, $\bar{\lambda}(\tilde{x}) = 0$, and $r(\tilde{x}) = r^c$, we note that $r$ solves (4.16) for the network $(\tilde{X}, \tilde{\alpha})$. On the other hand, choosing a constant
\[
\tilde{r}^c \in \big( \max_{i=0, \ldots, k-1} R^c(x_i) \big) \vee \alpha(x_i), \ r^c,
\]
and setting $\tilde{\alpha}(\tilde{x}) := \tilde{r}^c$ (with the other values of $\tilde{\alpha}$ being the same as those of $\alpha$), we can find the minimal solution $\tilde{r}$ to (4.16) for the network $(\tilde{X}, \tilde{\alpha}, \tilde{\lambda})$. It is clear from (4.16) that $\tilde{r}(\tilde{x}) \geq \tilde{\alpha}(\tilde{x}) = \tilde{r}^c$. In addition, from the minimality of $\tilde{r}$ and the monotonicity of the right hand side of (4.22) w.r.t. $\alpha$, we conclude that $\tilde{r}(x) \leq r(x)$, for every $x \in \tilde{O}$. This observation, along with our assumption and the choice of $\tilde{r}^c$, imply that,
\[
\max_{y : O(x_i, y) = 1} \{ \tilde{r}(y) + \bar{\lambda}(y) \} < \tilde{r}^c = \tilde{\alpha}(\tilde{x}).
\]
This, in turn, yields that $\tilde{r}(\tilde{x}) = \tilde{\alpha}(\tilde{x}) = \tilde{r}^c < r^c$. Finally, we turn back to the network $(X, \tilde{O}, \alpha)$ and, expanding $\tilde{r}$ in an obvious way (i.e. setting $\tilde{r}(x) = \tilde{r}^c$ for all $x \in C$), we obtain $\tilde{r}$ that solves (4.16) for $(X, \tilde{O}, \alpha)$. Since $\tilde{r} = \tilde{r}^c < r^c = r$ on $C$, we obtain a contradiction with the minimality of $r$.

Thus, there must exist $x^1 \in C$, s.t. $R^a(x^1) = r^c$. The latter equality implies that $\tilde{A}(x^1)$ contains at least one element outside of $C$. Then, we can modify $\tilde{\nu}$ by creating
\[
\tilde{\nu}^1(x, y) := \begin{cases} 
0, & x = x^1, y \in C, \\
1_{\tilde{A}(x) \setminus C}(y)/\mu \left( \tilde{A}(x) \setminus C \right), & x = x^1, y \notin C, \\
\tilde{\nu}(x, y), & \text{else}.
\end{cases}
\]
Note that $\tilde{\nu}^1$ satisfies (4.17). Thus, we define $\tilde{O}^1$ via $\tilde{\nu}^1$, and construct the new set $C^1$ in the same way as $C$ was constructed above, but in the new graph $(X, \tilde{O}^1)$. Note that any cycle in $(X, \tilde{O}^1)$ is also a cycle in $(X, \tilde{O})$. In addition, the node $x^1$ is contained in at least one cycle in $(X, \tilde{O})$ and is not contained in any cycle in $(X, \tilde{O}^1)$. These observations imply that: $x^1 \notin C^1$, $\tilde{\nu}^1(x, \cdot) \equiv \tilde{\nu}(x, \cdot)$ for all $x \in C^1$, and the number of cycles in $(X, \tilde{O}^1)$ is strictly smaller than the number of cycles in $(X, \tilde{O})$. Thus, the above arguments can be applied inductively, terminating in a finite number steps, when the resulting graph $(X, \tilde{O}^m)$ contains no cycles. The resulting $\tilde{\nu}^m$ satisfies (4.17).

**Step 3: solving both (4.17) and (4.19).** Finally, let us construct $(\hat{C}^b, \hat{C}^t, \hat{\nu})$ satisfying (4.17) and (4.19). As $(X, \tilde{O}^m)$ has no cycles, there must exist at least one node that has no out-coming edges (i.e., does not lead within the network): i.e., there exists $x \in X$, s.t. $\tilde{\nu}^m(x, \cdot) \equiv 0$. Denote the collection of all such nodes by $X^0$. We define
\[
\hat{C}^b(x) := 0, \quad \hat{C}^t(x) := \hat{C}^t \mathbf{1}_{\{ \tilde{R}(x) > \alpha(x) \}}(= 0), \quad \hat{\nu}(x, \cdot) := \tilde{\nu}^m(x, \cdot), \quad \forall x \in X^0.
\]
Since the nodes within $\mathcal{X}^0$ are not directly connected to each other, it is easy to notice that $(\bar{C}^{b,0}, \bar{C}^{l,0}, \bar{v})$ satisfy (4.17) and (4.19), with $\mathcal{X}$ replaced by $\mathcal{X}^0$. Next, we define $\mathcal{X}^1 := \left\{ x \in \mathcal{X} : \exists y \in \mathcal{X}^0, \text{ s.t. } \bar{v}^m(x, y) > 0 \right\}$, and, for every $x \in \mathcal{X}^1$, we set $\bar{v}(x, \cdot)$ to be proportional to $\bar{v}^m(x, \cdot)$ on $\mathcal{X}^0$, and to be zero on $\mathcal{X} \setminus \mathcal{X}^0$. We define $\bar{C}^{b,1}(x) = 0$, $\bar{C}^{l,1}(x) = \bar{C}$, $\forall x \in \mathcal{X}^1$, and update $\bar{C}^{b,1}(x) := \sum_{y \in \mathcal{X}^1} \bar{v}(y, x) \bar{C}^{l,1}(y) \mu(\{y\})$, $\bar{C}^{l,1} := (\bar{C} + \bar{C}^{b,1}(x)) 1_{\{R(x) > \alpha(x)\}}$, $\forall x \in \mathcal{X}^1$.

Since the nodes within $\mathcal{X}^j$ are not directly connected to each other, for $j = 0, 1$, it is easy to see that $\mathcal{X}^0 \cap \mathcal{X}^1 = \emptyset$ and that $(\bar{C}^{b,1}, \bar{C}^{l,1}, \bar{v})$ satisfy (4.17) and (4.19), with $\mathcal{X}$ replaced by $\mathcal{X}^0 \cup \mathcal{X}^1$. We repeat this procedure iteratively. Namely, at the $i$-th step, we define $\mathcal{X}^i := \left\{ x \in \mathcal{X} \setminus \bigcup_{j=0}^{i-1} \mathcal{X}^j : \exists y \in \bigcup_{j=0}^{i-1} \mathcal{X}^j, \text{ s.t. } \bar{v}^m(x, y) > 0 \right\}$, for every $x \in \mathcal{X}^i$, we set $\bar{v}(x, \cdot)$ to be proportional to $\bar{v}^m(x, \cdot)$ on $\bigcup_{j=0}^{i-1} \mathcal{X}^j$, and to be zero on $\mathcal{X} \setminus \bigcup_{j=0}^{i-1} \mathcal{X}^j$, define $\bar{C}^{b,i}(x) = 0$, $\bar{C}^{l,i}(x) = \bar{C}$, $\forall x \in \mathcal{X}^i$, and update, iterating over $k = i, i - 1, \ldots, 1$, $\bar{C}^{b,i}(x) := \sum_{y \in \mathcal{X}^k} \bar{v}(y, x) \bar{C}^{l,i}(y) \mu(\{y\})$, $\bar{C}^{l,i} := (\bar{C} + \bar{C}^{b,i}(x)) 1_{\{R(x) > \alpha(x)\}}$, $\forall x \in \mathcal{X}^k$.

It is easy to see, by induction, that, for every $i$: the nodes within $\mathcal{X}^i$ are not directly connected to each other, for $j = 0, \ldots, i$, that $(\bigcup_{j=0}^{i-1} \mathcal{X}^j) \cap \mathcal{X}^i = \emptyset$, and that $(\bar{C}^{b,i}, \bar{C}^{l,i}, \bar{v})$ satisfy (4.17) and (4.19), with $\mathcal{X}$ replaced by $\bigcup_{j=0}^{i-1} \mathcal{X}^j$. The construction stops after the $p$-th step, where $p$ is the smallest integer s.t. there are no nodes in $\mathcal{X} \setminus \bigcup_{j=0}^{p-1} \mathcal{X}^j$ directly connected to $\bigcup_{j=0}^{p-1} \mathcal{X}^j$, according to $\bar{v}^m$. As there are no cycles in $(\mathcal{X}, \bar{O}^m)$, we must have: $\mathcal{X} = \bigcup_{j=0}^{p} \mathcal{X}^j$. Setting $\bar{C}^* := \bar{C}^{b,p}$ and $\bar{C} := \bar{C}^{l,p}$, we complete the proof.

\begin{remark}
Theorem 3.17 in [BCOQ93] shows that the solution to (4.16) is unique if all entries of $\lambda$ are strictly negative, which is expected to hold at a.e. time $t$.

Therefore, typically, one may expect to have a unique interest rate $r$ in all “dominant lenders” (or “competitive borrowers”) equilibria, even though the sizes and allocations of credit exposures may not be unique.
\end{remark}

\begin{remark}
Note that it is important to find the smallest solution $r$ to the max-plus linear system (4.16), in order to avoid cycles and, in turn, to satisfy the market clearing condition. To see why, consider, for example, $\mathcal{X} = \{1, 2, 3\}$, $\alpha \equiv 0$, $\lambda \equiv 0$, $O(1, 2) = O(2, 3) = O(3, 1) = 1$, with all the other entries of $\bar{O}$ being zero. Note that $r(1) = r(2) = r(3) = z$, with any constant $z \geq 0$, solve (4.16). However, if $z > 0$, it is easy to see that the market clearing condition (4.19) cannot be satisfied. Indeed, this
\end{remark}

This fact can be proven easily by comparing $Y^r$ to a Brownian motion with drift and noticing that the drift $b$, given by (4.6), is absolutely bounded in equilibrium.
situation reminds a credit bubble, in which no particle can make positive returns on external investment, but the interest rate is strictly positive, hence, each particle aims to lend to the ones it is connected to. Since the network is a circle, the initially lent funds reach the creditor, thus encouraging it to lend more. It is clear that this problem can be easily fixed by choosing \(z = 0\), which is the smallest solution, and also the only economically meaningful solution, in this case.

Collecting the results of the last two subsections, we can describe an equilibrium in the proposed network game.

**Theorem 4.8.** Assume that, for every \(x \in \mathcal{X}\), \(Y_0^x \sim p_0(y;x)dy\), with \(p_0(\cdot;x) \in W^{2,1}_2(\mathbb{R}_+)\). Denote by \(S\) the mapping constructed in Proposition 4.7 with the associated \((\hat{r}, \hat{C}^b, \hat{C}^l, \hat{\nu})\), and by \(\hat{\lambda}\) the unique fixed point of the associated mapping \(D\). Then, \((r, \theta, C^b, C^l, \nu)\), with

\[
(r_t(x) := \hat{r}(\hat{\lambda}; x), \theta_t(x) := \hat{\theta}(\hat{\lambda}; x), \hat{C}^b_t(x) := \hat{C}^b(\hat{\lambda}; x), \\
\hat{C}^l_t(x) := \hat{C}^l(\hat{\lambda}; x), \hat{\nu}_t(x, x') := \hat{\nu}(\hat{\lambda}; x, x'),
\]

form an equilibrium.

**Proof.** The proof follows by collecting the established results. First, we consider \(S\) from Proposition 4.7 and use Lemma 4.6, equation (4.21), as well as the Lipschitz and boundedness properties of the associated \(R_t\), to conclude that \(S\) satisfies Assumption 1. Finally, we consider the fixed point \(\hat{\lambda}\) from Proposition 4.5 and use Corollary 2 to conclude that \((r, \theta, C^b, C^l, \nu)\) form an equilibrium. \(\square\)

It is important to note that the results of this part of the paper are not limited to proving the existence of equilibrium: the existence proof provided herein is, in fact, constructive, and it also produces a computational algorithm. Indeed, the proof of Theorem 4.8 shows that the equilibrium quantities \((r, \theta, C^b, C^l, \nu)\) are computed through the associated \(S\) and \(\hat{\lambda}\). The mapping \(S\) can be easily implemented (i.e. programmed) via the algorithm described in the proof of Proposition 4.7. Finally, the fixed point \(\hat{\lambda}\) can be obtained by the Picard iteration described in the proof of Proposition 4.5 which, in turn, can be computed numerically using \(S\) and an approximation scheme for the associated linear parabolic PDEs.

### 5. Appendix A

**Proof of Theorem 2.2.** Step 1. Notice that the theorem amounts to the existence of a family \(\{\Lambda^x\}_{x \in \mathcal{X}}\) of (“loss”) processes satisfying the fixed-point constraints

\[
\Lambda_t^x = C^x \int_{\mathcal{X}} \log \mathbb{P} \left( \min_{s \in [0,t]} (Z^x_s + \Lambda^x_s) > 0 \right) \kappa(x, dx'), \quad t \in [0,T], \quad x \in \mathcal{X},
\]

and converging to zero as \(t \downarrow 0\). We aim to employ Theorem 2.3 and, to this end, start with the construction of a domain \(\mathcal{D}\) as therein.

Notice that Assumption 2.1(a) and the continuity of \(Z^x\) yield the existence of a family \(\{(\delta_n^x, \eta_n^x)_{n \in \mathbb{N}}\}_{x \in \mathcal{X}}\) of positive sequences, satisfying \(\delta_n^x, \eta_n^x \downarrow 0\), as \(n \to \infty\), for all \(x \in \mathcal{X}\), and

\[
\int_{\mathcal{X}} \log \mathbb{P} \left( \min_{s \in [0,\delta_n^x]} Z^x_s > \eta_n^x \right) \kappa(x, dx') > - \frac{\eta_n^x}{C^x}, \quad x \in \mathcal{X}, \quad n \in \mathbb{N}
\]
Next, we factor the map associated with the right-hand side of (5.1) into a composition. To this end, we define the strong M1-metric in terms of graph parametrizations (see e.g. [Whi02, section 12.3]). 

Clearly, \( \mathcal{D} \subset D([0, T], \mathbb{R})^X \) is convex. In addition, since every convergent sequence in \( D([0, T], \mathbb{R}) \) converges pointwise on a dense subset of \([0, T]\) which includes 0 (see e.g. [Whi02 theorem 12.5.1(iv)]) and each function \( t \mapsto \sum_{n=1}^{\infty} -\eta_n \mathbf{1}_{[\delta_{n+1}, \delta_n]}(t) \) is right-continuous on \([0, \delta_1]\), the set \( \mathcal{D} \) is also closed. Moreover, \( S([0, T], \mathbb{R})^X \cap \mathcal{D} \) is dense in \( \mathcal{D} \) with respect to the componentwise uniform convergence, as can be inferred from the definition of the strong M1-metric in terms of graph parametrizations (see e.g. [Whi02 section 12.3]).

**Step 2.** Next, we factor the map associated with the right-hand side of (5.1) into a composition \( F_2 \circ F_1 \) of two auxiliary maps \( F_1 \) and \( F_2 \). The components of the map (5.4)

\[
F_1 : \mathcal{D} \rightarrow \left( \mathcal{P}(D([0, T], \mathbb{R})^2) \right)^X
\]

send each family \( \{\Lambda^x\}_{x \in \mathcal{X}} \in \mathcal{D} \) to the joint law of the pair \((Z^x, \tilde{\Lambda}^x)\), where

\[
\tilde{\Lambda}^x := \begin{cases} 
\Lambda^x & \text{if } t \in [0, \tau^x \wedge T), \\
-Z^x & \text{if } t \in (\tau^x \wedge T, T].
\end{cases}
\]

Here, \( \mathcal{P}(D([0, T], \mathbb{R})^2) \) is the space of Borel probability measures on \( D([0, T], \mathbb{R})^2 \) with the topology of weak convergence, and \( (\mathcal{P}(D([0, T], \mathbb{R})^2))^X \) is the corresponding product space with the product topology. The two key properties of \( F_1 \) are

\[
P \left( \min_{s \in [0, t]} \left( Z^x_s + \tilde{\Lambda}^x_s \right) > 0 \right) = P \left( \min_{s \in [0, t]} \left( Z^x_s + \Lambda^x_s \right) > 0 \right), \quad t \in [0, T], \quad x \in \mathcal{X},
\]

\[
Z^x_t + \tilde{\Lambda}^x_t = 0, \quad t \in (\tau^x \wedge T, T], \quad x \in \mathcal{X}.
\]

In particular, with

\[
F_2 : \text{ran } F_1 \rightarrow D([0, T], \mathbb{R})^X,
\]

\[
\{\mathcal{L}(Z^x, \tilde{\Lambda}^x)\}_{x \in \mathcal{X}} \mapsto \left\{ C^x \int_X \log P \left( \min_{s \in [0, t]} \left( Z^x_s + \tilde{\Lambda}^x_s \right) > 0 \right) \kappa(x, dx'), \quad t \in [0, T] \right\},
\]

the composition \( F_2 \circ F_1 \) is given by the right-hand side of (5.1).

**Step 3.** The goal of this step is to show that the range of \( F_1 \) in \( (\mathcal{P}(D([0, T], \mathbb{R})^2))^X \) is compact. Since each component \( F_{1x}^x \) of \( F_1 \) is defined in terms of \( \Lambda^x \) and \( Z^x \) only, the range of \( F_1 \) is of product form and, by Tychonoff’s theorem, it is enough to verify the compactness of the range of each \( F_{1x}^x \) in \( \mathcal{P}(D([0, T], \mathbb{R})^2) \). By Prokhorov’s theorem, the latter is precompact if, for any \( \epsilon \in (0, 1) \), one can find a compact subset of \( D([0, T], \mathbb{R})^2 \) which is assigned a probability of at least \( 1 - \epsilon \) by all the probability measures in the range of \( F_{1x}^x \). In view of the compactness criterion for subsets of \( D([0, T], \mathbb{R}) \) (cf. [Whi02 theorem 12.12.2]), such is possible thanks to the first marginal of \( F_{1x}^x(\{\Lambda^x\}_{x \in \mathcal{X}} \in \mathcal{D} \), the inequalities

\[
0 \geq \tilde{\Lambda}^x_t \geq \min_{s \in [0, T]} (-Z^x_s), \quad t \in [0, T]
\]

(see the definition of \( \tilde{\Lambda}^x \) in (5.5)), the estimates

\[
\tilde{\Lambda}^x = \Lambda^x \geq -\eta_n, \quad t \in [0, \delta_n], \quad \text{on } \{\tau^x \geq \delta_n\}
\]

(5.10)
for all \( n \in \mathbb{N} \) (cf. (5.3)), and the monotonicity of \( \tilde{\Lambda}^x \).

It remains to prove that the range of \( F^x_1 \) is closed in \( \mathcal{P}(D([0,T],\mathbb{R})^2) \) and, thus, compact. Consider a sequence \( L(Z^x,\tilde{\Lambda}^x,k) \), \( k \in \mathbb{N} \) in the range of \( F^x_1 \) converging weakly to some \( L(Z^x,\tilde{\Lambda}^{x,\infty}) \in \mathcal{P}(D([0,T],\mathbb{R})^2) \). Due to the Skorokhod representation theorem in the form of \(^{[Dud14]}\) theorem 3.5.1, there exist \( (Z^{x,k},\tilde{\Lambda}^{x,k}) \overset{d}{=} (Z^x,\tilde{\Lambda}^x,k) \), \( k \in \mathbb{N} \) and \( (\tilde{\Lambda}^{x,\infty},\tilde{\Lambda}^{x,\infty}) \overset{d}{=} (\tilde{\Lambda}^x,\tilde{\Lambda}^{x,\infty}) \) on some probability space \((\Omega,\mathcal{F},\mathbb{P})\) such that \( (Z^{x,k},\tilde{\Lambda}^{x,k}) \to (\tilde{\Lambda}^{x,\infty},\tilde{\Lambda}^{x,\infty}) \) as \( k \to \infty \) on \( \Omega \setminus \Omega_0 \) for some \( \mathbb{P} \)-null set \( \Omega_0 \). For any \( \omega \in \Omega \setminus \Omega_0 \), the sequence

\[
\tilde{\tau}^{x,k}(\omega) := \inf \{ t \in [0,T] : \tilde{\tau}^{x,k}(\omega) + \tilde{\Lambda}^{x,k}(\omega) = 0 \} \land T, \quad k \in \mathbb{N}
\]

has a subsequence converging to a limit \( \tilde{\tau}^{x,\infty}(\omega) \in [0,T] \). In the rest of this step, we fix an \( \omega \in \Omega \setminus \Omega_0 \) and take all limits \( k \to \infty \) along the subsequence associated with that \( \omega \).

One has \( \tilde{\Lambda}^{x,k}(\omega) \to \tilde{\Lambda}^{x,\infty}(\omega) \) on a dense subset of \([0,T]\) that includes 0, which together with the definition of \( \tilde{\Lambda}^x \) in (5.5) implies

\[
\tilde{\Lambda}^{x,\infty}(\omega) = \begin{cases} \Lambda^{x,\infty}_t, & t \in [0,\tilde{\tau}^{x,\infty}(\omega)), \\ -\lim_{k \to \infty} Z^{x,k}_t(\omega), & t \in (\tilde{\tau}^{x,\infty}(\omega), T], \end{cases}
\]

where \( \Lambda^{x,\infty} \) is the M1-limit of \( \Lambda^{x,k} \) on a suitable sequence of closed intervals increasing to \([0,\tilde{\tau}^{x,\infty}(\omega)]\), and \( \Lambda^{x,k} \) belongs to the preimage of \( L(Z^x,\tilde{\Lambda}^x,k) \) under \( F^x_1 \). The closedness assertion now follows from

\[
Z^{x,\infty}_t(\omega) + \tilde{\Lambda}^{x,\infty}_t(\omega) = \lim_{t \downarrow \tilde{\tau}^{x,\infty}(\omega)} \lim_{k \to \infty} \left( Z^{x,k}_t(\omega) + \tilde{\Lambda}^{x,k}_t(\omega) \right) = \lim_{t \downarrow \tilde{\tau}^{x,\infty}(\omega)} \lim_{k \to \infty} \left( Z^{x,k}_t(\omega) - Z^{x,k}_{\tilde{\tau}^{x,k}(\omega)}(\omega) \right) = 0,
\]

with the last equality being a result of the uniform convergence \( Z^{x,k}_t(\omega) \overset{k \to \infty}{\rightarrow} Z^{x,\infty}_t(\omega) \) (see e.g. \(^{[DIRT15b]}\) theorem 4.2). 

**Step 4.** Next, we argue that the map \( F_2 \) is continuous. Combined with the step 3 this yields the compactness of the range of \( F_2 \). To deduce the continuity of \( F_2 \) we pick a sequence \( \{L(Z^x,\tilde{\Lambda}^x,k)\}_{x \in \mathcal{X}} = F_1(\{\Lambda^{x,k}\}_{x \in \mathcal{X}}) \), \( k \in \mathbb{N} \) converging to some \( \{L(Z^x,\tilde{\Lambda}^{x,\infty})\}_{x \in \mathcal{X}} = F_1(\{\Lambda^{x,\infty}\}_{x \in \mathcal{X}}) \). For any \( x \in \mathcal{X} \) and any continuity point \( t \in [0,T] \) of \( \Lambda^{x,\infty} \), the process \( \Lambda^{x,\infty} \) is continuous at \( t \), so that one can infer \( \lim_{k \to \infty} L(Z^x,\tilde{\Lambda}^x,k) = L(Z^x,\tilde{\Lambda}^{x,\infty}) \) in \( \mathcal{P}(D([0,T],\mathbb{R})^2) \) from the Skorokhod representation theorem and the characterizations of M1-convergence for continuous and monotone functions (see e.g. \(^{[DIRT15b]}\) theorem 4.2). Hence, \( \lim_{k \to \infty} L(Z^x + \tilde{\Lambda}^x,k) = L(Z^x + \tilde{\Lambda}^{x,\infty}) \) in \( \mathcal{P}(D([0,T],\mathbb{R})) \) (see e.g. \(^{[Whi02]}\) theorem 12.7.3). A subsequent application of the Skorokhod representation theorem and the definition of the strong M1-metric in terms of graph parametrizations gives the convergences in distribution

\[
\lim_{k \to \infty} \min_{x \in [0,t]} (Z^x_s + \tilde{\Lambda}^x_s,k) = \min_{x \in [0,t]} (Z^x_s + \tilde{\Lambda}^{x,\infty}_s)
\]

for all continuity points \( t \in [0,T] \) of \( \Lambda^{x,\infty} \) and all \( x \in \mathcal{X} \). The latter argument shows (5.14) also for \( t = T \) and all \( x \in \mathcal{X} \).

Further, we claim that, for any \( x \in \mathcal{X} \),

\[
\lim_{k \to \infty} \log \mathbb{P}\left( \min_{s \in [0,t]} (Z^x_s + \tilde{\Lambda}^x_s,k) > 0 \right) = \log \mathbb{P}\left( \min_{s \in [0,t]} (Z^x_s + \tilde{\Lambda}^{x,\infty}) > 0 \right)
\]
at all continuity points \( t \in [0, T] \) of \( \Lambda^{x,\infty} \) and at \( t = T \). By (5.14) the convergence of the probabilities in (5.15) can be reduced to

\[
(5.16) \quad \mathbb{P} \left( \min_{s \in [0,t]} \left( Z_s^x + \Lambda_s^{x,\infty} \right) = 0 \right) = 0.
\]

The identity (5.16) holds, since the intersection of the events \( \min_{s \in [0,t]} (Z_s^{x} + \Lambda_s^{x,\infty}) = 0 \) and

\[
(5.17) \quad \inf \{ s \in [0,t] : Z_s^{x} + \Lambda_s^{x,\infty} \leq 0 \} = t
\]

is contained in the zero probability event \( \inf (Z_t^x = -\Lambda_t^{x,\infty}) \) (see Assumption 2.1(b)), and the intersection of the events \( \min_{s \in [0,t]} (Z_s^{x} + \Lambda_s^{x,\infty}) = 0 \) and

\[
(5.18) \quad \inf \{ s \in [0,t] : Z_s^{x} + \Lambda_s^{x,\infty} \leq 0 \} < t
\]

has a probability of zero by Assumption 2.1(c). At this point, (5.15) is readily obtained from the convergence of the probabilities involved and the positivity of the probability on the right-hand side of (5.15) (recall (5.6) and Assumption 2.1(d)).

**Step 5.** Lastly, for any sequence \( \{ \Lambda^{x,k} \}_{x \in \mathcal{X}} \), \( k \in \mathbb{N} \) in \( \mathcal{D} \) converging to some \( \{ \Lambda^{x,\infty} \}_{x \in \mathcal{X}} \in \mathcal{D} \), we proceed as in the step 4 to conclude that \( (Z^x + \Lambda^{x,k}) \xrightarrow{k \to \infty} Z^x + \Lambda^{x,\infty} \) in \( D([0,t], \mathbb{R}) \) almost surely for all \( x \in \mathcal{X} \) and all continuity points \( t \in [0,T] \) of \( \Lambda^{x,\infty} \) and \( t = T \). As in the step 4, this leads to

\[
(5.19) \quad \lim_{k \to \infty} \min_{s \in [0,t]} \left( Z_s^x + \Lambda_s^{x,k} \right) = \min_{s \in [0,t]} \left( Z_s^x + \Lambda_s^{x,\infty} \right) \text{ almost surely,}
\]

\[
(5.20) \quad \lim_{k \to \infty} \log \mathbb{P} \left( \min_{s \in [0,t]} \left( Z_s^x + \Lambda_s^{x,k} > 0 \right) \right) = \log \mathbb{P} \left( \min_{s \in [0,t]} \left( Z_s^x + \Lambda_s^{x,\infty} > 0 \right) \right),
\]

for all \( t \) and \( x \) as described. We can therefore infer from the M1-convergence criterion for monotone functions in [DIRT15b, theorem 4.2] that the map

\[
F_2 \circ F_1 : \mathcal{D} \longrightarrow D([0,T], \mathbb{R})^X,
\]

\[
(5.21) \quad \{ \Lambda^x \}_{x \in \mathcal{X}} \mapsto \left\{ C^x \int_{\mathcal{X}} \log \mathbb{P} \left( \min_{s \in [0,t]} \left( Z_s^x + \Lambda_s^x \right) > 0 \right) \kappa(x, dx') \mid t \in [0,T] \right\}_{x \in \mathcal{X}},
\]

is continuous.

In addition, for any \( \{ \Lambda^x \}_{x \in \mathcal{X}} \in \mathcal{D} \), it holds

\[
(5.22) \quad C^x \int_{\mathcal{X}} \log \mathbb{P} \left( \min_{s \in [0,t]} \left( Z_s^x + \Lambda_s^x \right) > 0 \right) \kappa(x, dx') \geq C^x \int_{\mathcal{X}} \log \mathbb{P} \left( \min_{s \in [0,t]} Z_s^x > \eta_n^x \right) \kappa(x, dx') > -\eta_n^x, \quad t \in [\delta_{n+1}^x, \delta_n^x], \quad n \in \mathbb{N}
\]

thanks to (5.2), so that the compact ran \( (F_2 \circ F_1) = \text{ran} F_2 \) (see the step 4) is a subset of \( \mathcal{D} \). In summary, \( F_2 \circ F_1 \) maps the convex closed set \( \mathcal{D} \), in which \( S([0,T], \mathbb{R})^X \cap \mathcal{D} \) forms a dense subset with respect to the componentwise uniform convergence, continuously onto a compact subset of \( \mathcal{D} \). Thus, \( F_2 \circ F_1 \) has a fixed-point by Theorem 2.3 \( \square \)

### 6. Appendix B

**Lemma 6.1.** Fix an arbitrary \( t \in [0,T] \). Suppose that Assumptions 2.1(b) and 3.2 hold. Assume, in addition, that a solution \( \{Y^x\}_{x \in \mathcal{X}} \) of (2.1) admits \( \{c^x, s^x\} \), satisfying
Assume the contrary: i.e., there exist least one \( x \) for some (6.1) and such that (6.5) \( \Lambda \) (6.4)

where \( \varepsilon \) \( x \) (0, \( z \)), for all \( x \in X \). Then, \( Y_t^x > Y_t^z \) for at least one \( x \in X \).

**Proof.** Assume the contrary: i.e., there exist \( t, \{ Y_t^x \} \subset X \), \( x_0 \), \((s_n)_{n \in \mathbb{N}}\) satisfying the assumptions of the lemma, and such that \( Y_t^x = Y_t^z \) for all \( x \in X \). Then, Assumption (2.1) gives

At this point, we let

\[
\Lambda^x_s := C \int_X \log \mathbb{P}(\tau^x > s) \kappa(x, dx'), \quad s \in [0, T], \quad x \in X,
\]

\[
\zeta^x(\cdot) := \frac{\mathbb{P}(\tau^x > t, Y_t^x \in \cdot)}{\mathbb{P}(\tau^x > t)}, \quad x \in X.
\]

We assume without loss of generality that

\[
\Lambda^x_t - \Lambda^{x_1}_t \leq \varepsilon^x, \quad x \in X,
\]

since (6.1) remains valid for any \((s_n)_{n \geq N}\) in place of \((s_n)_{n \in \mathbb{N}}\). Next, we employ the monotonicity of \( \{ \Lambda^x \} \subset X \), the inequality (3.6), the stochastic dominance in (6.2), the formula \( \int_0^\infty \mathbb{P}(R \geq z) \, dz = E[R] \) for non-negative random variables \( R \), the elementary bound \((z_1 + z_2) \land z_3 \geq z_1 \land z_3 + z_2 1_{\{z_1 + z_2 \leq z_3\}}\), \( z_1, z_2, z_3 \in \mathbb{R} \) in conjunction with (6.5), and \( \log(1 + z) \leq z, \, z \in \mathbb{R} \), consecutively, to obtain

\[
\Lambda^{x_0}_s - \Delta^x_0 = C^{x_0} \int_X \log \mathbb{P}\left( \min_{s \in [t, s_1]} \left( Y_t^{x_1} + Z_s - Z_t^{x_1} + \Lambda^{x_1}_s - \Lambda^x_t \right) > 0 \bigg| \tau^x > t \right) \kappa(x_0, dx_1)
\]

\[
\leq C^{x_0} \int_X \log \left( 1 - \int_0^{x_1} \mathbb{P}(\xi_{s_2, s_1}^x \geq z + \Lambda^{x_1}_{s_2} - \Lambda^x_t) \zeta^x_i(dz) \right) \kappa(x_0, dx_1)
\]

\[
\leq C^{x_0} \int_X \log \left( 1 - c^{x_1} \int_0^{x_1} \mathbb{P}(\xi_{s_2, s_1}^x \geq z + \Lambda^{x_1}_{s_2} - \Lambda^x_t) \right) \kappa(x_0, dx_1)
\]

\[
= C^{x_0} \int_X \log \left( 1 - c^{x_1} \mathbb{E}[\xi_{s_2, s_1}^x + \Lambda^{x_1}_{s_2} - \Lambda^x_t] \right) \kappa(x_0, dx_1)
\]

\[
\leq - C^{x_0} \int_X c^{x_1} \mathbb{E}[\xi_{s_2, s_1}^x] \kappa(x_0, dx_1)
\]

\[
+ C^{x_0} \int_X c^{x_1}(\Lambda^{x_1}_{s_2} - \Lambda^x_t) \mathbb{P}(\xi_{s_2, s_1}^x \leq z^{x_1} - \varepsilon^x) \kappa(x_0, dx_1).
\]
To conclude we repeat the estimates of (6.6) mutatis mutandis for the term $\Lambda_{s_2}^x - \Lambda_t^x$ inside the last expression in (6.6) to find

(6.7)

$\Lambda_{s_1}^x - \Lambda_t^x \leq -C_{a_0} \int_{\mathcal{X}} c_{a_1} \mathbb{E}[\xi_{s_2, s_1} \wedge z_{s_1}] \kappa(x_0, dx_1)$

$- C_{a_0} \int_{\mathcal{X}} c_{a_1} \mathbb{P}(\xi_{s_2, s_1} \leq z_{s_1} - \varepsilon_{s_1}) C_{a_1} \int_{\mathcal{X}} c_{a_2} \mathbb{E}[\xi_{s_3, s_2} \wedge z_{s_2}] \kappa(x_1, dx_2) \kappa(x_0, dx_1)$

$+ C_{a_0} \int_{\mathcal{X}} c_{a_1} \mathbb{P}(\xi_{s_2, s_1} \leq z_{s_1} - \varepsilon_{s_1}) C_{a_1} \int_{\mathcal{X}} c_{a_2} (\Lambda_{s_3}^x - \Lambda_t^x) \mathbb{P}(\xi_{s_3, s_2} \leq z_{s_2} - \varepsilon_{s_2}) \kappa(x_1, dx_2) \kappa(x_0, dx_1)$

Iterating $N \in \mathbb{N}$ times and dropping the term involving $\Lambda_{s_{N+1}}^x - \Lambda_t^x \leq 0$ we arrive at

(6.8)

$\Lambda_{s_1}^x - \Lambda_t^x \leq -\sum_{n=1}^{N} \int_{\mathcal{X}_n} \prod_{m=0}^{n-1} (C_{a_m} c_{a_{m+1}}) \prod_{m=1}^{n-1} \mathbb{P}(\xi_{s_{m+1}, s_m} \leq z_{s_m} - \varepsilon_{s_m}) \mathbb{E}[\xi_{s_{m+1}, s_m} \wedge z_{s_m}] \kappa(x_0, dx_1) \kappa(x_1, dx_2) \ldots \kappa(x_{n-1}, dx_n)$.

We now take $N \to \infty$ and end up with $\Lambda_{s_1}^x - \Lambda_t^x \leq -\infty$ (recall (6.1)), which is the desired contradiction.

**Proof of Theorem 3.1.** We note that Assumption 2.1(b) holds in the setting of (3.2) thanks to [McN85] inequality (3)] in conjunction with Girsanov’s theorem (see e.g. [KS91] chapter 3, theorem 5.1 and corollary 5.13]). Moreover, for any $\theta \in (0, T-t)$, $t \leq t_1 < t_2 \leq t + \theta$ and $x \in \mathcal{X}$ we write

(6.9)

$$
\min_{t_1 \leq s \leq t_2} (Z_s^x - Z_t^x) = Z_{t_1}^x - Z_t^x + \max_{t_1 \leq s \leq t_2} (Z_s^x - Z_{t_1}^x),
$$

infer from [Haj85] inequality (1.5)] that, conditionally on $Z_s^x$, $s \in [0, t]$,

(6.10)

$$
Z_{t_1}^x - Z_t^x \leq \bar{\sigma}(t_1 - t) + \frac{\beta_{t_1-t}^x + \tilde{\beta}_{t_1-t}^x}{2}
$$

with (possibly dependent) standard Brownian motions $\beta^x, \tilde{\beta}^x$, and dominate the conditional law of $\min_{t_1 \leq s \leq t_2} (Z_s^x - Z_{t_1}^x)$ given $Z_{t_1}^x - Z_t^x$ and $Z_{t_1}^x$, $s \in [0, t]$ stochastically by $\bar{\sigma}(t_2 - t_1) + \min_{0 \leq s \leq (t_2-t_1)} \hat{\beta}_s$ for a standard Brownian motion $\hat{\beta}$ by relying on the Dambis-Dubins-Schwarz theorem (see e.g. [KS91] chapter 3, theorem 4.6]). Thus, Assumption 3.2 holds with

(6.11)

$$
\xi_{t_1, t_2} := -\bar{\sigma}(t_2 - t) - \frac{\beta_{t_1-t}^x + \tilde{\beta}_{t_1-t}^x}{2} - \min_{0 \leq s \leq (t_2-t_1)} \hat{\beta}_s,
$$

where $\hat{\beta}$ is a standard Brownian motion independent of all $\beta^x, \tilde{\beta}^x$.

In view of the estimates

(6.12)

$$
\mathbb{P}(\xi_{t_1, t_2} \leq z^x - \varepsilon^x) = 1 - \mathbb{P}\left(-\frac{\beta_{t_1-t}^x + \tilde{\beta}_{t_1-t}^x}{2} + \left| \hat{\beta}_{t_2-t_1} \right| > z^x - \varepsilon^x + \bar{\sigma}(t_2-t_1) \right)
$$

$$
\geq 1 - \frac{4}{\bar{\sigma} \sqrt{2\pi(t_1 - t)}} \int_{-\infty}^{\infty} \Phi\left(\frac{z^x - \varepsilon^x + \bar{\sigma}(t_2-t_1) - z}{\sqrt{\bar{\sigma}^2(t_2-t_1)}}\right) e^{-\frac{z^2}{\bar{\sigma}^2(t_2-t_1)}} dz
$$

$$
= 1 - 4\Phi\left(\frac{z^x - \varepsilon^x + \bar{\sigma}(t_2-t_1)}{\sqrt{\bar{\sigma}^2(t_2-t_1) + \bar{\sigma}^2(t_1-t)}}\right)
$$
and

\begin{equation}
\mathbb{E}[\xi_{t_1,t_2} \land z^x] \geq \frac{1}{2} \mathbb{E} \left[ \left( -2\bar{\sigma}(t_1-t) - \sigma(\beta_{t_1-t} + \tilde{\beta}_{t_1-t}) \right) \land z^x \right]
\end{equation}

\begin{equation}
+ \frac{1}{2} \mathbb{E} \left[ \left( -2\bar{\sigma}(t_2-t_1) - 2 \min_{0 \leq s \leq \mathbb{E}(t_2-t_1)} \tilde{\beta}_s \right) \land z^x \right] = -\bar{\alpha}(t_2-t_1) + \sigma \sqrt{\frac{2(t_2-t_1)}{\pi}}
\end{equation}

\begin{equation}
+ \frac{1}{2} \mathbb{E} \left[ \left( z^x + 2\bar{\sigma}(t_1-t) + \sigma(\beta_{t_1-t} + \tilde{\beta}_{t_1-t}) \right) \mathbf{1}_{\{ -2\bar{\sigma}(t_1-t) - \sigma(\beta_{t_1-t} + \tilde{\beta}_{t_1-t}) > z^x \}} \right]
\end{equation}

\begin{equation}
+ \frac{1}{2} \mathbb{E} \left[ \left( z^x + 2\bar{\sigma}(t_2-t_1) + 2 \min_{0 \leq s \leq \mathbb{E}(t_2-t_1)} \tilde{\beta}_s \right) \mathbf{1}_{\{ -2\bar{\sigma}(t_2-t_1) - 2 \min_{0 \leq s \leq \mathbb{E}(t_2-t_1)} \tilde{\beta}_s > z^x \}} \right]
\end{equation}

= -\bar{\alpha}(t_1-t) + \sigma \sqrt{\frac{2(t_2-t_1)}{\pi}} + o(t_1-t) + o(\sqrt{t_2-t_1}) \text{ as } t_1,t_2 \downarrow 0,

the divergence of the series in (3.3) implies the same for the series in (6.1), with the selection of \(\xi_{t_1,t_2}\) in (6.11). The result now follows from Lemma 6.1.

\textbf{Proof of Theorem 3.4} In view of (3.7), (3.8) and (3.9), (6.14)

\begin{equation}
\liminf_{n \to \infty} \frac{1}{n} \log \left( \int_{\mathcal{A}^n} \prod_{m=0}^{n-1} (C^{x_m} e^{x_{m+1}}) \prod_{m=1}^{n-1} \mathbb{P} \left( \xi_{s_{m+1},s_m} \leq z^{x_m} - \varepsilon^{x_m} \right) \mathbb{E}[\xi_{s_{n+1},s_n} \land z^{x_n}] \kappa(x_0, dx_1) \ldots \kappa(x_{n-1}, dx_n) \right) \geq \liminf_{n \to \infty} \frac{1}{n} \log \left( \prod_{m=1}^{n-1} \left( \min_{x \in \mathcal{A}} \mathbb{P} \left( \xi_{s_{m+1},s_m} \leq z^x - \varepsilon^x \right) \right) \min_{x \in \mathcal{A}} \mathbb{E}[\xi_{s_{n+1},s_n} \land z^x] \right)
\end{equation}

\begin{equation}
\int_{\mathcal{A}^n} \prod_{m=0}^{n-1} (C^{x_m} e^{x_{m+1}}) \kappa(x_0, dx_1) \ldots \kappa(x_{n-1}, dx_n) \right) \geq \theta > 0.
\end{equation}

Consequently, the summands in the series (6.1) grow exponentially fast, so that the result readily follows from Lemma 6.1.

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[BH17] A. Babus and T.-W. Hu, 

[BJR07] B. Bollobás, S. Janson, and O. Riordan, 


Department of Mathematics, University of Michigan, Ann Arbor, MI, USA
*Email address:* sergeyn@umich.edu

ORFE Department, Princeton University, Princeton, NJ, USA
*Email address:* mshkolni@gmail.com