Endogenous Formation of Limit Order Books: the Effects of Trading Frequency.*

Roman Gayduk and Sergey Nadtochiy†

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Abstract

In this work, we present a modeling framework in which the shape and dynamics of a Limit Order Book (LOB) arise endogenously from an equilibrium between multiple market participants (agents). On the one hand, the new framework captures very closely the true, micro-level, mechanics of an auction-style exchange. On the other hand, it uses the standard abstractions of games with continuum of players (in particular, the mean field game theory) to obtain a tractable macro-level description of the LOB. We use the proposed modeling framework to analyze the effects of trading frequency on the liquidity of the market in a very general setting. In particular, we show that the higher trading frequency increases market efficiency if the agents choose to provide liquidity in equilibrium. However, the higher trading frequency also makes markets more fragile, in the following sense: in a high-frequency trading regime, the agents choose to provide liquidity in equilibrium “if and only if” they are market-neutral (i.e. their beliefs satisfy certain martingale property). The theoretical results are illustrated with numerical examples.

1 Introduction

The technological development presents new challenges to the mathematical modeling of social and economic phenomena. In particular, the rapid growth of electronic trading has changed significantly the existing approaches to modeling financial markets. The classical mathematical models used to focus on the macroscopic description of the financial processes, often, taking as input the price levels at which certain assets can be purchased or sold. However, in reality, the price arises as an outcome of the interaction between market participants, and understanding the mechanics of the price formation process has become an important problem on its own. A famous example of a problem that arises in this context is how to characterize the effects of high-frequency traders (HFTs), who account for more than a half of all US equity trading volume. These traders buy and sell the asset in relatively small quantities but at a very high rate, making significant profits due to the large overall volume. On the one hand, the HFTs compete with each other to provide liquidity to the market, allowing the other investors to buy or sell the asset in the quantity they need and increasing the market efficiency. On the other hand, the HFTs can manipulate the price by trading very aggressively, in order to exploit the long term investors trying to buy or sell large quantities of the asset. Such a manipulation creates a new type of risk, which reveals itself in the unusually high price deviations, that cannot be explained by the changes in the present, or projected, fundamental value of the asset. The most famous example of this phenomenon is the “flash crash” of 2010. This example motivates the need for a comprehensive study of the market microstructure

* Partial support from the NSF grant DMS-1411824 is acknowledged by both authors.
† Address the correspondence to: Mathematics Department, University of Michigan, 530 Church St, Ann Arbor, MI 48104; sergeyn@umich.edu.
that would focus on describing the behavior of a large number of agents participating in the price formation. In particular, such a study needs to investigate the tradeoff between the liquidity providing role of the strategic players and the liquidity risk they generate. In this paper we analyze the market microstructure in the context of an order-driven exchange (as most exchanges currently are), in which the participating agents can post limit or market orders. The collective liquidity effect of the agents (i.e., their liquidity providing vs. liquidity risk role) is captured by the shape and dynamics of the Limit Order Book (LOB), which contains all the limit buy and sell orders. The goal of the present paper is to develop a realistic framework for modeling the LOB, which would strike the right balance between the generality and tractability. Most importantly, we propose a modeling framework in which the shape of the LOB, and its dynamics, arise endogenously from the interactions between the agents. This is in contrast to many of the existing results on market microstructure, which assume that the shape and dynamics of the LOB are given exogenously. Among the many advantages of the equilibrium-based approach is the possibility of modeling phenomena that go beyond what is normally observed. For example, such a model would be beneficial for studying the extreme events in the market (e.g., flash crashes) or testing the effects of proposed regulation (e.g., limited trading frequency or transaction tax). Herein, we illustrate this idea by using the proposed modeling framework to analyze the effects of trading frequency on the efficiency and stability of the market.

In the recent years, there has been an explosion in the amount of literature devoted to the study of market microstructure. In addition to various empirical studies, a large part of the existing theoretical work focuses on the problem of optimal execution, in which an investor needs to liquidate her position in the asset within a given time horizon, by submitting smaller (limit or market) orders and aiming to minimize the losses. The relevant publications, include, among others, [26], [1], [29], [30], [15], [25], [11], [3], [2], [12], [28], [18], [10], [19], [32], and references therein. In these publications, the dynamics and shape of the LOB are modeled exogenously, or, equivalently, the arrival processes of the limit and market orders are specified exogenously. In particular, none of them attempt to explain the shape and dynamics of the LOB, arising directly from the interaction between the market participants. A different approach to the analysis of market microstructure has its roots in the economic literature. For example, in [27], [14], [17], [8], [22], introduce the models of endogenous formation of the LOB, and, to the best of our knowledge, are the closest existing publication to the analysis presented herein. However, the models proposed in these papers do not represent the mechanics of the order-driven markets closely enough, and, in particular, are not well suited to analyze the liquidity role of participating agents and its dependence on the specific rules of the exchange. Indeed, these questions do not appear to be addressed in the aforementioned works, but they are the main focus of the present paper. The liquidity role of the agents is analyzed, for example, in [4] and [6], but not in the context of a market microstructure. The results of this analysis demonstrate that, depending on the model parameters, the agents may either serve as liquidity providers for each other, or attempt to manipulate the price, in order to benefit at the expense of the distressed agents, and, effectively, reduce the overall liquidity. There also exists a fairly large amount of literature on a related topic: namely, the endogenous formation of the LOB in a market with a designated market maker (cf. [16], [21], [13], [9]). In these papers, the LOB does arise from an equilibrium, but it is controlled by a single agent – the market maker – which is not the case in many exchanges, and, in particular, is not assumed in the present paper. Finally, several recent papers have considered an equilibrium-based approach to the problem of optimal execution (cf. [31], [20]). Namely, they consider an equilibrium between several agents competing for liquidity, but the LOB (or, market), against which these agents trade, is still specified exogenously, rather than modeled as an output of the general equilibrium. In the present paper, we model the entire LOB as an output of the equilibrium between a large number of agents, each of whom is allowed to both consume and provide liquidity (in particular, we have no designated market maker). We formulate the problem as a game with a continuum of players – this abstraction allows us to obtain computationally tractable results. As the state processes of individual agents interact only through the empirical distribution of their controls, our approach falls within the framework of mean field games. The latter topic has received a lot of attention in the
recent years: we refer the interested reader to [24], [5], [7], [23], and references therein, for more information on the subject. It is worth mentioning, however, that, to the best of our knowledge, the type of mean field games presented herein are different from any of the models that have been considered in the literature. As a result, we cannot use much of the existing machinery of mean field games, and have to prove all the desired results by hand.

In addition to the introduction of a novel modeling framework, the main contribution of this paper is the description of liquidity effects of the market participants as the trading frequency grows. More specifically, we manage to show that, as the trading frequency grows to infinity, the market participants choose to provide liquidity in equilibrium (as opposed to be taking it or doing nothing) “if and only if” their beliefs satisfy a certain martingale property (we put the quotation marks here, as the sufficiency is proven under more restrictive assumptions). These results are formulated in Theorems 1 and 2. In addition, Corollaries 3 and 4 show that, when the agents do choose provide liquidity in equilibrium, the higher trading frequency leads to higher market efficiency. On the other hand, numerical examples in Section 6 show that, at a lower trading frequency, the market can tolerate a certain deviation of the agents’ beliefs from market-neutrality, in the sense that it is possible to find an equilibrium in which the agents provide liquidity. However, Theorem 1 shows that the latter which is impossible in a high-frequency regime, demonstrating that the market becomes more fragile as frequency increases. Thus, we describe precisely the dual role that the trading frequency plays.

The paper is organized as follows. Section 2 contains the main building blocks and the descriptions of underlying assumptions of the proposed discrete time models. Section 3 contains the definitions and preliminary results related to the notion of equilibrium in the proposed setting. It also introduces the notion of degeneracy of the market, which occurs when the agents choose not to provide liquidity, and, hence, is associated with a liquidity crises (which is due to the interaction between the agents, as opposed to any real shortage of the asset). In Section 4, we consider a setting in which the discrete time models (corresponding to a finite trading frequency) converge to a limiting continuous time model. In this setting, we show that, as the frequency becomes high enough, a non-degenerate equilibrium may only exist if the agents are market-neutral (i.e. their beliefs in a limiting model satisfy a certain martingale property). The main results of this section are Proposition 1, Corollaries 3 and 4, and Theorem 1. In addition, Lemma 6 may be interesting from a mathematical point of view. Section 5 is concerned with the existence of a non-degenerate equilibrium. Its main results are Theorems 2, 3, and Corollary 6. In Section 6, we illustrate numerically all the phenomena discussed in preceding sections, using a simple Gaussian model. Finally, we provide a summary of the results and discuss further extensions in Section 7.

2 Discrete time model: preliminaries

We assume that time is discrete: \( n = 0, 1, \ldots, N \). We also assume that the market participants are split into two groups: the external investors, who are “impatient”, in the sense that they only submit market orders and need to execute immediately, and the strategic players, who can submit both market and limit orders, and are willing to spend time doing so, in order to get a better execution price. In our model, we will focus on the strategic players, who we refer to as agents, and will model the behavior of the external investors exogenously, as the exogenous demand. The exogenous demand for the asset is described by the random field \( D = (D_n(p))_{p \in \mathbb{R}, n=1,\ldots,N} \) on a filtered probability space \( (\Omega, \mathcal{F} = (\mathcal{F}_n)_{n=0}^N, \mathbb{P}) \), such that \( \mathcal{F}_0 \) is complete with respect to \( \mathbb{P} \). The random variable \( D_n(p) \) denotes the incremental demand for the asset at price \( p \) and at all more favorable price levels, in the \( n \)th time period. In other words, \( D_n^+(p) \) denotes the amount of asset that the external investors and the agents submitting market orders are willing to purchase at the price \( p \) in the time period \( (n-1, n] \), and \( D_n^-(p) \) denotes the amount of asset that the external investors and the agents submitting market orders are willing to sell at the price \( p \) in the same time period. We assume that \( D_n(\cdot) \) is a.s. nonincreasing and measurable with
respect to $\mathcal{F}_n \otimes \mathcal{B}(\mathbb{R})$. For convenience, we also assume that $\mathcal{F}_0$ is trivial.

At any time $n$, every agent (i.e. strategic player) is allowed to submit a market order, post a limit buy or sell order, or wait (i.e. do nothing). We use the convention that all limit orders expire at the beginning of the next period, so that the limit orders have to be re-posted. Notice that this assumption cause no loss of generality, as we do not allow for any time-priority in the limit orders. Instead, we assume that the tick size is zero (the set of possible price levels is $\mathbb{R}$), and, hence an agent can achieve a priority by posting her order slightly above or below the other ones. The mechanics of order execution are explained in the subsequent paragraphs. We also assume that we are given a Borel space of beliefs, $\mathcal{A}$, and, for each $\alpha \in \mathcal{A}$, there exists a subjective probability measure $\mathbb{P}^\alpha$ on $(\Omega, \mathcal{F}_N)$, which is absolutely continuous with resect to $\mathbb{P}$. We assume that, for any $n = 0, \ldots, N$ and any $\alpha \in \mathcal{A}$, there exists a regular version of the conditional probability $\mathbb{P}^\alpha$ given $\mathcal{F}_n$, denoted $\mathbb{P}_n^\alpha$.\footnote{This assumption holds if, for example, the filtration is generated by a sequence of random elements with values in a standard Borel space.} We denote the associated conditional expectations by $\mathbb{E}_n^\alpha$. However, we also need to assume that, for any $\alpha \in \mathcal{A}$, there exists a modification of the family $\{\mathbb{P}_n^\alpha\}_{n=0}^N$, which satisfy the tower property with respect to $\mathbb{P}$, in the following sense: for any $n \leq m$ and any r.v. $\xi$, such that $\mathbb{E}_n^\alpha \xi < \infty$, we have

$$\mathbb{E}_n^\alpha \mathbb{E}_m^\alpha \xi = \mathbb{E}_n^\alpha \xi, \quad \mathbb{P}\text{-a.s.}$$

Clearly, such a modification exists if $\mathbb{P}^\alpha \sim \mathbb{P}$. In any market model, for every $\alpha$, we fix such a modification of conditional probabilities (up to a set of $\mathbb{P}$-measure zero) and assume that all conditional expectations $\{\mathbb{E}_n^\alpha\}$ are taken under this family of measures.

We denote $\mathbb{I} = \mathbb{R}$ and define the state space of an agent as $S = \mathbb{I} \times \mathcal{A}$, where the first component denotes the inventory of an agent (i.e. how much asset she currently holds), and the second component denotes her beliefs. Every agent in state $(s, \alpha)$ models the future outcomes using the subjective probability measure $\mathbb{P}^\alpha$. There are infinitely many agents, and their distribution over the state space is given by the empirical distribution process $\mu = (\mu_n)_{n=0}^{N-1}$, such that every $\mu$ is a finite sigma-additive random measure on $S$ (defined with respect to $\mathcal{F}_n \otimes \mathcal{B}(S)$). We allow for the possibility that the agents can arrive and leave the market, hence, for the most of the paper, we treat $\mu$ as being given exogenously, but arbitrary. Notice that this causes no loss of generality for the results of Section 4, in which we analyze the necessary properties of the equilibria. The case of endogenous $\mu$ is analyzed in Subsection 5.1. It is worth mentioning how we interpret the inventory levels in the case of infinitely many agents. In fact, one should not think of the inventory level $s$ as the actual number of shares, but rather, as a number of shares measured relative to the overall mass of of agents, $\mu_n(S)$, held by the agents at state $(s, \alpha)$ (assuming that, within the state, all agents hold equal amounts). Then, the total number of shares held by all agents in states $(s, \alpha) \in S \subset S$ (which is now measured in shares) is given by $\int_S s \mu_n(ds, d\alpha)$. Of course, a proper justification of this setting should come from taking a limit of a finite-agent game – we leave this for further research.

The Limit Order Book (LOB) is given by a pair of adapted process $\nu = (\nu_n^+, \nu_n^-)_{n=0}^N$, such that every $\nu_n^+$ and $\nu_n^-$ is a finite sigma-additive random measure on $\mathbb{R}$ (defined with respect to $\mathcal{F}_n \otimes \mathcal{B}(\mathbb{R})$). Herein, $\nu_n^+$ corresponds to the cumulative limit sell orders, and $\nu_n^-$ corresponds to the cumulative limit buy orders, posted at time $n$.\footnote{For convenience, we sometimes refer to $\nu_n$ as a “measure”, rather than a “pair of measures”.} The bid and ask prices at any time $n = 0, \ldots, N$ are given by the random variables

$$p_n^b = \sup \text{supp}(\nu_n^-), \quad p_n^a = \inf \text{supp}(\nu_n^+),$$

respectively. Notice that these extended random variables are always well defined but may take infinite values.

Let us now discus the dynamics of the state process, and, in particular, the order execution rules. First, we assume that $\alpha$, representing the beliefs of an agent, does not change over time.\footnote{This restriction causes no loss of generality, as we allow for very general measures $\mathbb{P}^\alpha$.} Therefore, the state process of
an agent, denoted \((S_n)\), represents only her inventory and it is an adapted process with values in \(\mathbb{I}\). The control of every agent is given by a triplet of adapted processes \((p, q, r) = (p_n, q_n, r_n)_{n=0}^{N-1}\) on \((\Omega, \mathcal{F})\), with values in \(\mathbb{R}^2 \times \{0, 1\}\). The first coordinate, \(p_n\), indicates the location of a limit order placed at time \(n\), and \(q_n\) indicates the size of the order (with negative values corresponding to buy orders). The last coordinate \(r_n\) shows whether the agent submits a market order (if \(r_n = 1\)) or a limit order (if \(r_n = 0\)).\(^4\) Assume that an agent posted a limit sell order at a price level \(p_n\). If the incremental demand to buy the asset at the price level \(p_n\), \(D^+_{n+1}(p)\), exceeds the amount of all limit sell orders posted below \(p\) at time \(n\), then (and only then) the limit sell order of the agent is executed. Market orders are always executed at the bid or ask price available at the time when the order is submitted. Summing up the above, we obtain the following dynamics for the state process of an agent, starting with initial inventory \(s \in \mathbb{I}\) at time \(m = 0, \ldots, N - 1\):

\[
S^m_{n+1}(s, p,q,r) = S^m_n(s, p,q,r) - q_n 1_{\{r_n=1\}} - 1_{\{r_n=0\}} \left( q_n^+ 1_{\{D^+_{n+1}(p_n) > \nu^+((-\infty,p_n])\}} - q_n^- 1_{\{D^-_{n+1}(p_n) > \nu^-((p_n,\infty))\}} \right), \quad n = m, \ldots, N - 1.
\]  

The above dynamics make use of the assumption that each agent is infinitesimally small, and, hence, her order is necessarily executed once the demand reaches it. They also use the following two implicit assumptions: each agent believes that her limit order will be executed first among all orders at the same price level, and her market order will be executed at the best price available. It is easy to see, from the two properties mentioned above, that the evolution of the state process, given by (1), may not be consistent with the market clearance condition: e.g. the total executed demand from the external investors, over any given period, may not coincide with the total change in the cumulative inventory of all agents present in the market in that period. For example, if \(\nu^+\) is proportional to a delta function located at \(p^0\), and if the incremental demand becomes positive at \(p^0\), then, according to (1), all agents who posted their limit sell orders at \(p^0\) will have their orders executed in full. However, if the size of the demand is strictly less than \(\nu^+\{\{p^0\}\}\), then, we obtain an example of the aforementioned inconsistency with market clearance. Similar phenomenon is observed when the agents submit market orders: such orders get executed without having any effect on the existing limit orders or on the external demand. This issue is resolved if \((\nu, p, q, r)\) satisfy, almost surely for all \(n = 0, \ldots, N - 1\): \(\nu_n\) is continuous, and \(r_n = 0\). This case is discussed in more detail in Subsection 5.1.

The objective function of an agent, starting at the initial state \((s, \alpha) \in S\) at any time \(m = 0, \ldots, N\) and using the control \((p, q, r)\), is given by the \(\mathcal{F}_m\)-measurable random variable:

\[
J^\alpha_m(p,q,r)(s, \alpha) = \mathbb{E}_m^\alpha \left[ \left( S^{m,s,(p,q,r)}_N \right)^+ P^b_N - \left( S^{m,s,(p,q,r)}_N \right)^- P^a_N \right] - \sum_{n=m}^{N-1} \left( p_n^1 1_{\{r_n=0\}} + p_n^d 1_{\{r_n=1, q_n<0\}} + p_n^b 1_{\{r_n=1, q_n>0\}} \right) \Delta S^{m,s,(p,q,r)}_{n+1} \right] \Delta S^{m,s,(p,q,r)}_{n+1}
\]

where we assume that \(0 \cdot \infty = 0\). In the above expression, we assume that, at the final time \(n = N\), each agent will be forced to liquidate her position at the bid or ask price available at that time. Alternatively, one can think of it as “marking to market” of the residual inventory, right after the last external market order is executed.

3 Equilibrium: definitions, assumptions and representation results

All constructions, notation and assumptions made in the previous section hold throughout.

\(^4\)Note that we allow each agent to place a limit order at one price level only. However, this results in no loss of generality, because, as follows from the subsequent paragraphs, the agents objective is a linear function of the submitted orders, provided they have the same sign. Posting limit orders of different signs simultaneously is not optimal because only one type of orders is exercised in a given period.
Definition 1. A given LOB $\nu$ is admissible if, for any $m = 0, \ldots, N - 1$ and any $\alpha \in A$, we have, $P$-a.s.:
\[
\mathbb{E}_m^\alpha |p_m^b| \vee |p_m^b| < \infty
\]

Definition 2. For a given LOB $\nu$, integer $m = 0, \ldots, N - 1$, and state $(s, \alpha) \in S$, the control $(p, q, r)$ is admissible if the positive part of the expression inside the expectation in (2) has a finite expectation under $\mathbb{P}^\alpha$.

Definition 3. For a given LOB $\nu$, we call the triplet of progressively measurable random fields $(\hat{p}, \hat{q}, \hat{r})$ an optimal control if, for any $m = 0, \ldots, N$ and $(s, \alpha) \in S$, we have:

- $(\hat{p}, \hat{q}, \hat{r})$ is admissible,
- $J_m^\nu((\hat{p}, \hat{q}, \hat{r}))(s, \alpha) \geq J_m^\nu(p, q, r)(s, \alpha)$, $P$-a.s., for any admissible control $(p, q, r)$, where, for a fixed $(m, s, \alpha)$, we identify the stochastic processes with random fields as follows: $\hat{p}_n = \hat{p}_n \left( s_m, s, \alpha \right)$, $\hat{q}_n = q_n \left( s_m, s, \alpha \right)$, $\hat{r}_n = r_n \left( s_m, s, \alpha \right)$.

In the above, we make the standard assumption of games with continuum of players (also used in the mean field game theory): each agent is too small to affect the distribution of cumulative controls (described by $\nu$) when she changes her control. Note also that our definition of the optimal control implies that it is time consistent. re-evaluation of the optimality at any future step, using the same terminal criteria, must lead to the same optimal strategy.

Let us now consider the (stochastic) value function of an agent for a fixed $(m, s, \alpha, \nu)$:
\[
V_m^\nu(s, \alpha) = \mathbb{E}_m^\alpha \left[ J_m^\nu(p, q, r)(s, \alpha) \right],
\]
where the essential supremum is taken under $\mathbb{P}$, over all admissible controls $(p, q, r)$, and $J_m^\nu(p, q, r)$ is given by (2). It is clear that, if an optimal control exists, then the value function is well defined. In addition, as follows from (1) and (2), if $\nu$ is admissible, then, for any $(\alpha, m, p, q, r)$, we have, $P$-a.s.:
\[
\left| J_m^\nu(p, q, r)(s, \alpha) - J_m^\nu(p, q, r)(s', \alpha) \right| \leq |s - s'| \mathbb{E}_m^\alpha |p_N^b| \vee |p_N^b|, \quad \forall s, s' \in \mathbb{I}
\]
This implies that every $J_m^\nu(p, q, r)(s, \alpha)$ and $V_m^\nu(s, \alpha)$ has a continuous modification under $\mathbb{P}$. Thus, whenever $\nu$ is admissible, we define the value function of an agent as the aforementioned continuous modification of the left hand side of (3).

Lemma 1. Assume that an optimal control exists for an admissible LOB $\nu$. Assume also that, for any $\alpha \in A$, the associated value function $V_m^\nu(s, \alpha)$, defined in (3), is measurable with respect to $\mathcal{F}_n \otimes \mathcal{B}(\mathbb{R})$. Then, it satisfies the following Dynamic Programming Principle.

- For $n = N$ and all $(s, \alpha) \in S$, we have, $P$-a.s.:
\[
V_N^\nu(s, \alpha) = s^+ p_N^b - s^- p_N^b
\]

- For all $n = N - 1, \ldots, 0$ and all $(s, \alpha) \in S$, we have:
\[
V_n^\nu(s, \alpha) = \mathbb{E}_n^\alpha \left[ \left( q_n p_n + V_{n+1}^\nu(s, \alpha) - V_n^\nu(s, \alpha) \right) \cdot \left( 1_{\{q_n < 0, D_{n+1}(p_n) > v_n^*((\infty, p_n)) \}} + 1_{\{q_n > 0, D_{n+1}(p_n) < v_n^*((\infty, p_n)) \}} \right) \right]
\]
\[
+ \mathbb{E}_n^\alpha \left[ \left( q_n p_n - q_n^* p_n + E_n^\alpha V_{n+1}^\nu(s, \alpha) \right) \cdot \left( 1_{\{q_n > 0, D_{n+1}(p_n) > v_n^*((\infty, p_n)) \}} + 1_{\{q_n < 0, D_{n+1}(p_n) < v_n^*((\infty, p_n)) \}} \right) \right],
\]
where the essential supremum is taken under $\mathbb{P}$, over all admissible controls $(p, q, r)$. 


The proof is completed easily by plugging the dynamics of the state process, (1), into (6).

where

\[ g_n^\nu(p_n, q_n, r_n, D_{n+1}) = (p_n1_{r_n=0} + p_n^q1_{r_n=1, q_n<0} + q_n^p1_{r_n=1, q_n>0}) \Delta S_{n+1}^{m,s,p,q,r} \]

does not depend on \( s \). Assume that \( J_n^\nu(p,q,r)(\cdot, \alpha) \) is a continuous modification of the objective function. Notice that, for all \( m \leq k \leq n \), we have, \( \mathbb{P} \)-a.s.:

\[
\mathbb{E}_k^\alpha J_n^\nu(p,q,r) (S_m^{m,s,p,q,r}, \alpha) = J_k^\nu(p,q,r) (S_k^{m,s,p,q,r}, \alpha) + \mathbb{E}_k \sum_{j=k}^{n-1} g_j^\nu(p, q, r, D_{j+1})
\]

Notice also that, for any \( (p, q, r) \) we have, \( \mathbb{P} \)-a.s.:

\[ J_n^\nu(p,q,r)(s, \alpha) \leq V_n^\nu(s, \alpha), \quad \forall s \in S \]

Let us show that the left hand side of (6) is less than its right hand side:

\[
V_n^\nu(s, \alpha) = \text{esssup}_{p,q,r} J_n^\nu(p,q,r) (S_m^{m,s,p,q,r}, \alpha)
\]

\[ = \text{esssup}_{p,q,r} \mathbb{E}_m^\alpha \left( J_{m+1}^\nu(p,q,r) (S_{m+1}^{m,s,p,q,r}, \alpha) - g_m^\nu(p_m, q_m, r_m, D_{m+1}) \right)
\]

\[ \leq \text{esssup}_{p,q,r} \mathbb{E}_m^\alpha \left( V_{m+1}^\nu (S_{m+1}^{m,s,p,q,r}, \alpha) - g_m^\nu(p_m, q_m, r_m, D_{m+1}) \right)
\]

Next, we show that the right hand side of (6) is less than its left hand side. For any \( (p, q, r) \), we have, \( \mathbb{P} \)-a.s.:

\[
\mathbb{E}_m^\alpha \left( V_{m+1}^\nu (S_{m+1}^{m,s,p,q,r}, \alpha) - g_m^\nu(p_m, q_m, r_m, D_{m+1}) \right)
\]

\[ = \mathbb{E}_m^\alpha \left( J_{m+1}^\nu(p, q, r) (S_{m+1}^{m,s,p,q,r}, \alpha) - g_m^\nu(p_m, q_m, r_m, D_{m+1}) \right)
\]

\[ = J_{m+1}^\nu(p, q, r)(s, \alpha) \leq V_m^\nu(s, \alpha), \]

where \((\tilde{p}_n, \tilde{q}_n, \tilde{r}_n)\) coincide with \((\hat{p}_n, \hat{q}_n, \hat{r}_n)\), for \( n \geq m + 1 \), while they are equal to \((p_m, q_m, r_m)\), for \( n = m \). The proof is completed easily by plugging the dynamics of the state process, (1), into (6).

The following corollary provides a more explicit recursive formula for the value function and optimal control. In particular, it states that the value function of an agent at any time \( n \) remains linear in \( s \) in both positive and negative half lines (with possibly different slopes).

**Corollary 1.** Assume that an admissible LOB \( \nu \) has an optimal control \((\hat{p}, \hat{q}, \hat{r})\). Then, for any \((s, \alpha) \in S\), the following holds \( \mathbb{P} \)-a.s., for all \( n = 0, \ldots, N - 1 \):

1. \( V_n^\nu(s, \alpha) = s^+ \lambda_n^a(\alpha) - s^- \lambda_n^b(\alpha) \), with some adapted processes \( \lambda_n^a(\alpha) \) and \( \lambda_n^b(\alpha) \), such that \( \lambda_n^a(\alpha) = p_n^b \) and \( \lambda_n^b(\alpha) = p_n^a \);

2. \( p_n^a \geq \mathbb{E}_n^\alpha (\lambda_{n+1}^a(\alpha)) \) and \( p_n^b \leq \mathbb{E}_n^\alpha (\lambda_{n+1}^b(\alpha)) \);
3. if for some \( p \in \mathbb{R}, \ P_n^a\left(D_{n+1}^+(p) > \nu^+_{n}(\langle -\infty, p \rangle)\right) > 0 \), then
\[
p \leq \mathbb{E}_n^a\left(\lambda^b_{n+1}(\alpha) | D_{n+1}^+(p) > \nu^+_{n}(\langle -\infty, p \rangle)\right);\]

4. if for some \( p \in \mathbb{R}, \ P_n^a\left(D_{n+1}^-(p) > \nu^-_{n}(\langle p, \infty \rangle)\right) > 0 \), then
\[
p \geq \mathbb{E}_n^a\left(\lambda^a_{n+1}(\alpha) | D_{n+1}^-(p) > \nu^-_{n}(\langle p, \infty \rangle)\right);\]

5. for all \( s > 0 \),

- \( \lambda^a_n(\alpha) = \max\left\{ \min\{s, \alpha\}, \mathbb{E}_n^a\lambda^a_{n+1}(\alpha) + \sup_{p \in \mathbb{R}} \mathbb{E}_n^a\left(\left(p - \lambda^a_{n+1}(\alpha)\right)1\{D_{n+1}^+(p) > \nu^+_{n}(\langle -\infty, p \rangle)\}\right)\right\}\),
- if \( \bar{q}_n(s, \alpha) \neq 0 \) and \( \bar{r}_n(s, \alpha) = 0 \), then
\[
\lambda^a_n(\alpha) = \mathbb{E}_n^a\lambda^a_{n+1}(\alpha) + \sup_{p \in \mathbb{R}} \mathbb{E}_n^a\left(\left(p - \lambda^a_{n+1}(\alpha)\right)1\{D_{n+1}^+(p) > \nu^+_{n}(\langle -\infty, p \rangle)\}\right),
\]

and \( p = \bar{p}_n(s, \alpha) \) attains the above supremum,
- if \( \bar{q}_n(s, \alpha) = 0 \) and \( \bar{r}_n(s, \alpha) = 0 \), then \( \lambda^a_n(\alpha) = \mathbb{E}_n^a\lambda^a_{n+1}(\alpha)\),
- if \( \bar{r}_n(s, \alpha) = 1 \), then \( \lambda^a_n(\alpha) = p^n_n;\)

6. for all \( s < 0 \),

- \( \lambda^b_n(\alpha) = \min\left\{ \min\{s, \alpha\}, \mathbb{E}_n^a\lambda^b_{n+1}(\alpha) - \sup_{p \in \mathbb{R}} \mathbb{E}_n^a\left(\left(\lambda^b_{n+1}(\alpha) - p\right)1\{D_{n}^-(p) > \nu^-_{n-1}(\langle p, \infty \rangle)\}\right)\right\}\),
- if \( \bar{q}_n(s, \alpha) \neq 0 \) and \( \bar{r}_n(s, \alpha) = 0 \), then
\[
\lambda^b_n(\alpha) = \mathbb{E}_n^a\lambda^b_{n+1}(\alpha) - \sup_{p \in \mathbb{R}} \mathbb{E}_n^a\left(\left(\lambda^b_{n+1}(\alpha) - p\right)1\{D_{n}^-(p) > \nu^-_{n-1}(\langle p, \infty \rangle)\}\right),
\]

and \( p = \bar{p}_n(s, \alpha) \) attains the above supremum,
- if \( \bar{q}_n(s, \alpha) = 0 \) and \( \bar{r}_n(s, \alpha) = 0 \), then \( \lambda^b_n(\alpha) = \mathbb{E}_n^a\lambda^b_{n+1}(\alpha)\),
- if \( \bar{r}_n(s, \alpha) = 1 \), then \( \lambda^b_n(\alpha) = p^n_n;\)

Proof:

Let us plug the piecewise-linear form of the value function into (5):
\[
V^\nu_n(s, \alpha) = \text{esssup}_{p,q,r} \left\{ 1_{\{r_n = 0\}} \left[ s^+ \mathbb{E}_n^a\lambda^a_{n+1}(\alpha) - s^- \mathbb{E}_n^a\lambda^b_{n+1}(\alpha) + \mathbb{E}_n^a \left[ (g_n p_n + (s - q_n)^+ - \lambda^a_{n+1}(\alpha) - (s - q_n)^- - \lambda^b_{n+1}(\alpha) + s^+ \lambda^a_{n+1}(\alpha) + s^- \lambda^b_{n+1}(\alpha) - \mathcal{D}_{n+1}^+(p_n) > \nu^+_{n}(\langle -\infty, p_n \rangle)\right)\right] + 1_{\{r_n = 1\}} \left[ g_n^+ p_n^+ - g_n^- p_n^- + (s - q_n)^+ \mathbb{E}_n^a\lambda^a_{n+1}(\alpha) - (s - q_n)^- \mathbb{E}_n^a\lambda^b_{n+1}(\alpha)\right] \right\}
\]

First, notice that it suffices to consider the essential supremum over all random variables \((p_n, q_n, r_n)\). It is easy to see that, for any fixed \((p_n, r_n, s)\), the above function is piece-wise linear in \( q_n \), with the slope changing\(^5\).
at \( q_n = 0 \) and \( q_n = s \). Hence, for a finite maximum to exists, the slope of this function must be nonnegative, at \( q_n \to -\infty \), and non-positive, at \( q_n \to \infty \). This must hold for any \((p_n, r_n, s)\), to ensure that the value function of an agent is finite: otherwise, an agent can scale up her position to increase the value function arbitrarily. Considering \( r_n = 1 \), we obtain condition 2 of the corollary. The case \( r_n = 0 \) yields conditions 3 and 4. Notice also that the maximum of the aforementioned function is always attained at \( q_n = 0 \) or \( q_n = s \). Considering all possible cases – \( r_n = 0, 1, q_n = 0, s, s = 0, s > 0 \) and \( s < 0 \) – we obtain the recursive formulas for \( \lambda_n^a \) and \( \lambda_n^b \) (i.e. conditions 5 and 6 of the corollary). In addition, as the optimal \( q_n \) takes values 0 and \( s \), it is easy to see that the piece-wise linear structure of the value function in \( s \) is propagated backwards, and, hence, condition 1 of the corollary holds.

It is also useful to have a converse statement.

**Corollary 2.** Consider an admissible LOB \( \nu \) and admissible control \((\tilde{p}, \tilde{q}, \tilde{r})\), such that \( \hat{q}_n(s, \alpha) \in \{0, s\}\). Assume that, for any \( \alpha \in \mathbb{A} \) and any \( n = 0, \ldots, N \), there exists a \( \mathcal{F}_n \otimes \mathcal{B}(\mathbb{I}) \)-measurable random function \( V_n^\nu(\cdot, \alpha) \), such that, for any \( s \in \mathbb{I} \), \( \mathbb{P}\text{-a.s.} \), \((\hat{p}, \hat{q}, \hat{r}, V^\nu)\) satisfy the conditions 1–6 of Corollary 1. Then, \((\hat{p}, \hat{q}, \hat{r})\) is an optimal control for the LOB \( \nu \).

**Proof:**

It suffices to revert the arguments in the proof of Corollary 1, and recall that \( \hat{q} \) can always be chosen to be equal to 0 or \( s \), without compromising the optimality.

The values of \( \lambda^a(\alpha) \) and \( \lambda^b(\alpha) \) can be interpreted as the expected execution prices of agents with beliefs \( \alpha \) who are long and short the asset, respectively. Next, we discuss the notion of equilibrium in the proposed game.

**Definition 4.** Consider a given empirical distribution process \( \mu = (\mu_n)_{n=0}^N \) and a market model, as described in Section 2. Then, we say that a given LOB process \( \nu \) and a control \((p, q, r)\) form an equilibrium, if there exists a Borel set \( \mathbb{A} \subset \mathbb{A} \), called the support of the equilibrium, such that:

1. \( \mu_n \left( \mathbb{I} \times \left( \mathbb{A} \setminus \mathbb{A} \right) \right) = 0, \mathbb{P}\text{-a.s.}, \text{for all } n, \)
2. \( \nu \) is admissible, and \((p, q, r)\) is an optimal control for \( \nu \), on the state space \( \hat{S} = \mathbb{I} \times \mathbb{A} \),
3. and, for any \( n = 0, \ldots, N - 1 \), we have, \( \mathbb{P}\text{-a.s.}:
   \[
   \nu_n^+(\langle -\infty, x \rangle) = \int_{\hat{S}} 1_{\{p_n(s, \alpha) \leq x, r_n(s, \alpha) = 0\}} q_n^+(s, \alpha) \mu_n(ds, d\alpha), \quad \forall x \in \mathbb{R}, \tag{7}
   \]
   \[
   \nu_n^-(\langle -\infty, x \rangle) = \int_{\hat{S}} 1_{\{p_n(s, \alpha) \leq x, r_n(s, \alpha) = 0\}} q_n^-(s, \alpha) \mu_n(ds, d\alpha), \quad \forall x \in \mathbb{R}. \tag{8}
   \]

**Remark 1.** Notice that, because the optimal controls are chosen as random fields and are required to be time consistent under \( \mathbb{P} \), the above definition, in fact, defines a sub-game perfect equilibrium.

Notice that the above definition is very similar to the notion of an equilibrium in a mean field game with common noise. Indeed, the state processes corresponding to different initial states depend on the common noise process (the demand) and on the distribution of the controls of other players, given by the LOB \( \nu \) (notice that the right hand sides of (7) and (8) can be represented as the integrals with respect to the empirical distribution of control values at time \( n \)). However, as mentioned in the introduction, the present setting, does not fall within any class of mean field games that have been studied before. One of the main reasons for it is the complicated
nonlinear dependence between the increments of the common noise process and the increments of the state process, which makes it impossible to formulate this game directly as a differential game model.

Another difference consists in the fact that Definition 4 only provides a partial equilibrium, with the measure-valued process $\mu$ given exogenously. This feature, in principle, makes our analysis different from the existing mean field game approaches, even though it can be eliminated under some additional assumptions. This issue is important on its own and deserves a brief discussion. Recall that a more traditional, stronger, version of equilibrium (which is typically used in the mean field game framework) would require $\mu$ to be the empirical distribution of the states of all agents. This amounts to the following additional constraint:

$$\mu_n = \mu_0 \circ \left( (s, \alpha) \mapsto \left( S^{0,s,(p,q,r)}_n, \alpha \right) \right)^{-1}, \quad (9)$$

which must hold $\mathbb{P}$-a.s., for all $n = 0, \ldots, N$, in addition to the other conditions in the definition of an equilibrium. In the above, $S^{0,s,(p,q,r)}_n$ is defined for all $s \in \mathbb{I}$ and all random outcomes via (1). However, for the most part of this paper, we choose not to enforce the condition (9), for the following two reasons. First, it seems unrealistic to assume that no new agents will arrive to the market during the time span of the game. Hence, one needs to model their arrivals, which, in effect, amounts to modeling $\mu$ exogenously. Of course, one can require that $\mu$ is a given function of the empirical distribution of the states of all players and some additional stochastic factors. This will lead to a slightly stronger version of the equilibrium, with an additional constraint on $\mu$, which does not necessarily coincide with (9). We leave this, more realistic, approach for further investigation. With the exception of Subsection 5.1, herein, we assume that $\mu$ is a given measure-valued stochastic process. Second, the issue of potential inconsistency with the market clearance, discussed after equation (1), often makes it impossible to construct an equilibrium satisfying (9), even in the most basic cases. To see this, recall that, in order for an equilibrium to exist, we must have a non-zero total mass of agents having positive and negative inventories, at the terminal time $n = N$. Assume that, at time $n = N - 1$, all agents post limit orders of the sizes equal to their inventories (this is natural in view of Corollary 2). If, in addition, the agents have the same beliefs, then, in view of Corollary 1, it is natural to assume that the price level at which an agent posts her order depends only on the sign of her inventory (in fact, we construct such an equilibrium in Sections 5 and 6). Then, whenever one agent’s order gets executed, all of the limit orders of the same type (buy or sell) get executed, according to (9). Thus, if it is possible for any order to be executed, it is also possible that the LOB becomes degenerate at time $n = N$, which means that the equilibrium does not exist (as no strategy is admissible). Notice that this occurs even if the total size of external demand is smaller than the total positive and negative inventories held by the agents at time $n = N - 1$. This phenomenon is, clearly, artificial, and one must not make any conclusions about the market microstructure based on such a non-existence result. This example illustrates that the fixed-point constraint (9) can only be enforced if the market clearance condition holds as well. As discussed in the paragraph after equation (1), the market clearance condition is satisfied if the LOB is continuous and the agents never post market orders. In Subsection 5.1, we take a closer look at the condition (9) and construct an equilibrium that satisfies these properties, and such that (9) holds. However, for the most part of the paper, we assume that $\mu$ is given exogenously. Notice that, with the exception of Subsection 5 (where we show existence of an equilibrium), this assumption causes no loss of generality, as we make statements about the necessary properties of all equilibria, which, in particular, remain valid for the equilibria satisfying the additional constraint (9).

Using the Dynamic Programming Principle and the available representation of a value function, we can characterize all equilibria by a sequence of “local” fixed-point problems, given by conditions 1–6 of Corollary 1, with the additional constraint given by condition 3 of Definition 4. We do not investigate the question of existence of an equilibrium in the generality presented above. The reason being that, under certain natural assumptions, this question becomes trivial. Namely, assume that the maximum total demand for the asset in any given time period (either to buy or to sell) is less than the total inventory of the agents meeting that demand:
i.e. \( \sup_{\nu} D_n^+(p) \leq \mu_n^1(\mathbb{R}) \) and \( \sup_{\nu} D_n^-(p) \leq \mu_n^2(\mathbb{R}) \), for all \( n \), where

\[
\mu_n^1(dx) = \int_{(0,\infty)} s\mu_n(dx, ds), \quad \mu_n^2(dx) = \int_{(-\infty,0)} |s|\mu_n(dx, ds)
\]  

(10)

Notice that this assumption is natural if we believe that the trading frequency is very high (i.e. the length of every time interval is very small): it is hard to expect a large external demand arriving within a small time interval. Then, it is easy to see that an equilibrium can be constructed by fixing any constant \( c \in \mathbb{R} \) and setting \( \nu_n^+ = \mu_n^1(\mathbb{R})\delta_c, \nu_n^- = \mu_n^2(\mathbb{R})\delta_c \), for all \( n = 0, \ldots, N \). Indeed, the expected execution price for any agent in this LOB is equal to \( c \), regardless of the external demand, hence, it is always optimal for every one of them to post a limit order at \( c \). It is clear that such an equilibrium is not realistic, as it ignores the external demand (representing the “true” value of the asset) completely. The latter is partially due to the flexibility in determining the execution price at the terminal time \( n = N \): notice that at time \( n = N \), there is no fixed-point problem that the LOB has to satisfy, so, without any further constraints, one can define \( \nu_N \) arbitrarily (as long as the subsequent fixed-point problems have solutions). The above construction also shows that there is no hope for uniqueness of the equilibrium: different choices of \( c \) lead to different equilibria. Thus, to make the setting more realistic, we have to restrict the set of possible LOBs, in particular, by adding a constraint on \( \nu_N \). We choose to model \( \nu_N \) as a shifted \( \nu_{N-1} \), where the shift is due to a change in the so-called fundamental price.

**Assumption 1.** \( \mathbb{P} \)-a.s., for any \( n = 1, \ldots, N \) there exists a unique \( p_n^0 \) satisfying \( D_n(p_n^0) = 0 \).

**Definition 5.** Let Assumption 1 hold. Then, the adapted process \( (p_n^0)_{n=1}^N \) is called the fundamental price process.

The intuition behind \( p_n^0 \) is clear: it is a price level at which the external demand is balanced. However, it is important to stress that we do not assume that the asset can be traded at the fundamental price level. Rather, \( p_n^0 \) denotes some fundamental forecast about external demand, whereas all actual trading happens on the exchange, against the current LOB. This makes our setting different from many other approaches existing in the literature.

**Definition 6.** Let Assumption 1 hold, and denote \( \xi_N = p_N^0 - p_{N-1}^0 \). Then, an equilibrium with LOB \( \nu \) is linear at terminal crossing (LTC) if

\[
\nu_N = \nu_{N-1} \circ (x \mapsto x + \xi_N)^{-1}, \quad \mathbb{P} \text{-a.s.}
\]

Notice that this definition connects the LOB at the terminal time with the demand process, ruling out the trivial “constant” equilibrium discussed above. In particular, the question of existence of an equilibrium becomes non-trivial. However, existence is not the only important question that needs to be addressed. As it is demonstrated in the examples section, in some cases the agents may reach an equilibrium in which one side of the LOB is empty. We call such LOB, and the associated equilibrium, degenerate. In what follows, we analyze the existence along with of degeneracy of LTC equilibria, as the trading frequency increases. A rigorous definition of degeneracy is provided below.

**Definition 7.** We say that an equilibrium with LOB \( \nu \) is non-degenerate if \( \nu_n^{\Delta t,+}(\mathbb{R}) > 0 \) and \( \nu_n^{\Delta t,-}(\mathbb{R}) > 0 \), for all \( n = 0, \ldots, N - 1 \), \( \mathbb{P} \)-a.s.

Intuitively, it refers to a situation when one side of the LOB disappears from the market: i.e. \( \nu_n^{\Delta t, +}(\mathbb{R}) \) or \( \nu_n^{\Delta t, -}(\mathbb{R}) \) becomes zero (or sufficiently small). Clearly, this happens when the agents who are supposed to provide liquidity choose to post market orders (i.e. consume liquidity) or wait (neither provide nor consume liquidity). Such a degeneracy can be interpreted as a liquidity crisis – more specifically, a liquidity crisis that arises purely from the interaction between the agents, not due to any fundamental economic reasons. Taking an optimistic point of view, we assume that the agents choose a non-degenerate equilibrium, whenever one is available. Hence, a market becomes degenerate when there is no non-degenerate equilibrium available. The main contribution of the subsequent part of this paper is the characterization of the existence of non-degenerate equilibria via the agents’ beliefs and the trading frequency.
4 Limit Order Book in a High Frequency Regime

In this section, we analyze the question of existence of a non-degenerate equilibrium in a more specific, although still quite general, setting. We begin our discussion with the continuous time model, which can be viewed as a limit of discrete time models, as the trading frequency increases. Consider a fixed \( T > 0 \) and a stochastic basis \( (\Omega, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P}) \), where the filtration supports a Brownian motion \( W \), and every \( \mathcal{F}_t \) is generated by a random variable taking values in a standard Borel space (the latter is needed to be consistent with the setting of Section 2). We define the adapted process \( \tilde{p}^0 \) via

\[
\tilde{p}_t^0 = p_0^0 + \int_0^t \sigma_s dW_s, \quad p_0^0 \in \mathbb{R},
\]

where \( \sigma \) is a progressively measurable locally square integrable process. We make the following assumption on \( \sigma \).

**Assumption 2.** There exists a constant \( C > 1 \), such that, \( 1/C \leq \sigma \leq C \), \( \mathbb{P}\)-a.s..

The set of agents’ beliefs \( \mathcal{A} \) and the associated measures \( \{\mathbb{P}^\alpha\}_{\alpha \in \mathcal{A}} \) satisfy the assumptions of Section 2. In particular, every \( \mathbb{P}^\alpha \) is absolutely continuous with respect to \( \mathbb{P} \). As a consequence, for any \( \alpha \in \mathcal{A} \), we have

\[
\tilde{p}_t^0 = p_0^\alpha + A_t^\alpha + \int_0^t \sigma_s dW_s^\alpha, \quad p_0^\alpha \in \mathbb{R},
\]

where \( W^\alpha \) is a Brownian motion under \( \mathbb{P}^\alpha \), and \( A^\alpha \) is a process of finite variation. We make the following regularity assumption on \( A^\alpha \).

**Assumption 3.** There exists a constant \( C > 0 \), such that, \( \mathbb{P}\)-a.s., for all \( \alpha \in \mathcal{A} \) and all \( t \in [0,T] \), we have:

\[
A_t^\alpha = \int_0^t \mu_s^\alpha ds,
\]

with some progressively measurable and right-continuous \( \mu^\alpha \), absolutely bounded by \( C \).

Under the above assumption, we can rewrite the dynamics of \( \tilde{p}^0 \), under each \( \mathbb{P}^\alpha \), as follows:

\[
\tilde{p}_t^0 = p_0^\alpha + \int_0^t \mu_s^\alpha ds + \int_0^t \sigma_s dW_s^\alpha, \quad p_0^\alpha \in \mathbb{R}
\]

(12)

As usual, \( \mathbb{E}_t^\alpha \) denotes the conditional expectation given \( \mathcal{F}_t \), under \( \mathbb{P}^\alpha \). In many cases, we will need to analyze the future dynamics of \( \tilde{p}^0 \) under various measures \( \mathbb{P}^\alpha \), conditional on \( \mathcal{F}_t \). This is why we need the following joint regularity assumption.

**Assumption 4.** There exists a modification of regular conditional probabilities 

\[
\{\mathbb{P}_t^\alpha = \mathbb{P}^\alpha (\cdot | \mathcal{F}_t)\}_{t \in [0,T], \alpha \in \mathcal{A}},
\]

such that it satisfies the tower property with respect to \( \mathbb{P} \) (as described in Section 2), and, \( \mathbb{P}\)-a.s., for all \( \alpha \in \mathcal{A} \) and all \( t \in [0,T] \), the future price process \( \tilde{p}_{s}^0 \) satisfies (12) under \( \mathbb{P}_t^\alpha \). All conditional expectations are taken under this modification of regular conditional probabilities, in all statements where this assumption is invoked.
For any uniform partition of $[0, T]$, with diameter $\Delta t = T/N$, we consider an associated discrete time market model described in Section 2, defined on a stochastic basis $(\Omega, (\mathcal{F}_{n}\alpha_{t})_{t=0}^{\infty}, \mathbb{P}^{\alpha})$. We denote the resulting set of beliefs by $\mathcal{A}^{\Delta t}$ and the associated measures by $\{\mathbb{P}^{\alpha, \Delta t}\}$. The random field $D^{\Delta t} = (D_{n}^{\Delta t})_{n=1}^{N}$ represents the demand in any such model, and the measure-valued process $\mu^{\Delta t} = (\mu_{n}^{\Delta t})_{n=1}^{N}$ represents the empirical distribution of the agents’ states. Let Assumption 1 hold for $D^{\Delta t}$ and denote the fundamental price at time $n$ by $p_{n}^{0, \Delta t}$ (cf. Definition 5). We assume that the elements of any such discrete time model are obtained by restricting the corresponding elements of the above continuous time model to the discrete set of times: $t = 0, \Delta t, \ldots, N \Delta t$.

**Assumption 5.** $\mathcal{F}_{n}^{\Delta t} = \mathcal{F}_{n}\Delta t$, for all $n = 0, \ldots, N$, $\mathcal{A}^{\Delta t} = \mathcal{A}$, $\mathbb{P}^{\alpha} = \mathbb{P}$, and $\mathbb{P}^{\alpha, \Delta t} = \mathbb{P}^{\alpha}$ for all $\alpha \in \mathcal{A}$.

**Assumption 6.** $p_{n}^{\alpha, \Delta t} = p_{n}^{\alpha}$.

The main results of this Section require several additional assumptions. The first one can be viewed as a stronger version of $\mathbb{L}^{2}$-continuity of $\sigma$.

**Assumption 7.** There exists a function $\varepsilon(\cdot) \geq 0$, such that $\varepsilon(\Delta t) \to 0$, as $\Delta t \to 0$, and, $\mathbb{P}$-a.s.,

$$\mathbb{P}^{\alpha}_{t} \left( \mathbb{E}^{\alpha} \left( (\sigma_{s\wedge \tau} - \sigma_{\tau})^{2} \mid \mathcal{F}_{\tau} \right) \right) \leq \varepsilon(\Delta t) = 1$$

holds for all $t \in [0, T - \Delta t]$, all $s \in [t, t + \Delta t]$, all stopping times $t \leq \tau \leq s$, and all $\alpha \in \mathcal{A}$, where the family of conditional probabilities $\{\mathbb{P}^{\alpha}_{t}\}$ is the modification of regular conditional probabilities appearing in Assumption 4.

The above assumption holds, for example, if $\sigma$ is an Itô process with bounded drift and diffusion coefficient. The following technical lemmas are useful in the proofs of the subsequent results. In words, they show that, in the present setting, the conditional distributions of the normalized increments of the fundamental price are asymptotically Gaussian. For any market model on the time interval $[0, T]$, associated with a uniform partition with diameter $\Delta t > 0$ and having a fundamental price process $p^{0}$ (given that Assumption 1 holds), we define

$$\xi_{n} = p_{n}^{0, \Delta t} - p_{n-1}^{0, \Delta t}, \quad \tilde{\xi}_{n} = 0, \quad t_{n} = n \Delta t, \quad n = 1, \ldots, N = T/\Delta t$$

(13)

The next lemma shows that the conditional distribution of a fundamental price increment becomes close to normal, as the trading frequency increases. In order to formulate it, we construct a standard normal random variable $\eta_{0}$, on a possible extended probability space, such that it is independent of $\mathcal{F}_{\tau}$ under every $\mathbb{P}^{\alpha}$.

**Lemma 2.** Let Assumptions 2, 3, 4 and 7 hold. There exists a function $\varepsilon(\cdot) \geq 0$, such that $\varepsilon(\Delta t) \to 0$, as $\Delta t \to 0$, and the following holds $\mathbb{P}$-a.s., for all $p \in \mathbb{R}$, all $\alpha \in \mathcal{A}$ and all $n = 1, \ldots, T/\Delta t$:

(i) $\left| \mathbb{E}^{\alpha}_{t_{n-1}} \left[ 1 \{\xi_{n}/\sqrt{\Delta t} > p\} \right] - \mathbb{E}^{\alpha}_{t_{n-1}} \left[ 1 \{\sigma_{t_{n-1}} \eta_{01} > p\} \right] \right| \leq \varepsilon(\Delta t)$

(ii) $\left| \mathbb{E}^{\alpha}_{t_{n-1}} \left[ \frac{\tilde{\xi}_{n}}{\sqrt{\Delta t}} 1 \{\xi_{n}/\sqrt{\Delta t} > p\} \right] - \mathbb{E}^{\alpha}_{t_{n-1}} \left[ \sigma_{t_{n-1}} \eta_{0} 1 \{\sigma_{t_{n-1}} \eta_{01} > p\} \right] \right| \leq \varepsilon(\Delta t)$

(iii) $\left| \mathbb{E}^{\alpha}_{t_{n-1}} \left[ p1 \{\xi_{n}/\sqrt{\Delta t} > p\} \right] - \mathbb{E}^{\alpha}_{t_{n-1}} \left[ p1 \{\sigma_{t_{n-1}} \eta_{01} > p\} \right] \right| \leq \varepsilon(\Delta t)$

In addition, the above estimates hold if we replace $(\tilde{\xi}_{n}, \eta_{0}, p)$ by $(-\tilde{\xi}_{n}, -\eta_{0}, -p)$.

**Proof:**

The proof is given in Appendix A. ■
The next assumption ensures that the demand decreases with the price level at a rate bounded from below.

**Assumption 8.** For a given \( \Delta t > 0 \) and any \( n = 1, \ldots, T/\Delta t \), there exists a \( \mathcal{F}_{n \Delta t} \otimes \mathcal{B}(\mathbb{R}) \)-measurable random function \( \kappa_{n \Delta t} > 0 \), such that \( \kappa_{n \Delta t}^{-1}(0) = 0, \ k_{n \Delta t}^{-1}(\cdot) \) strictly decreasing, and the following holds \( \mathbb{P} \)-a.s.:

\[
|D_{n \Delta t} (p + p_{n \Delta t}^0) | \geq |\kappa_{n \Delta t}^{-1}(p)|, \quad \forall p \in \mathbb{R}
\]

The next assumption states that every empirical distribution of the agents’ states, \( \mu_n^{\Delta t} \), is dominated by a deterministic measure.

**Assumption 9.** For any \( \Delta t > 0 \) and any \( n = 0, \ldots, T/\Delta t - \Delta t \), there exists a finite sigma-additive measure \( \mu_n^{\Delta t} \) on \( (\mathbb{S}, \mathcal{B}(\mathbb{S})) \) and \( \mathcal{F}_{n \Delta t} \otimes \mathcal{B}(\mathbb{S}) \)-measurable \( p_n \geq 0 \), such that, \( \mathbb{P} \)-a.s., for all \( B \in \mathcal{B}(\mathbb{S}) \), we have

\[
\mu_n^{\Delta t}(B) = \int_B p_n(s, \alpha) \mu_n^{\Delta t}(ds, d\alpha)
\]

The following proposition shows that, if every market model in a given sequence admits a non-degenerate equilibrium, then, the terminal bid-ask spread converges to zero as the trading frequency goes to infinity.

**Proposition 1.** Consider a family of uniform partitions of a given time interval \([0, T]\), with diameters \( \{\Delta t > 0\} \), and with the associated family of market models satisfying Assumptions 1, 2–7 and 8–9. Assume that every such model admits a non-degenerate LTC equilibrium, with a LOB \( \nu^{\Delta t} \). Then, there exists a deterministic function \( \varepsilon(\cdot) \), such that \( \varepsilon(\Delta t) \to 0 \), as \( \Delta t \to 0 \), and, for all small enough \( \Delta t > 0 \), the following holds \( \mathbb{P} \)-a.s.:

\[
\left| p_{T/\Delta t}^{\Delta t} - p_{T/\Delta t}^{0,\Delta t} \right| + \left| p_{T/\Delta t}^{\Delta t} - p_{T/\Delta t}^{0,\Delta t} \right| \leq \varepsilon(\Delta t)
\]

**Proof:**

Within the scope of this proof, we use the following notational convention.

**Notational Convention 1.** The LOB, the bid and ask prices, the expected execution prices, and the demand are all centered around \( p^0 \). Namely, we use the notation \( \nu \) to denote \( \nu \circ (x \mapsto +p^0)^{-1} \), \( p^\alpha \) to denote \( p^x - p^0 \), \( p^b \) to denote \( p^0 - p^0 \), \( \lambda^a \) to denote \( \lambda^x - p^0 \), \( \lambda^b \) to denote \( \lambda^0 - p^0 \), and \( D_n(p) \) to denote \( D_n(p_0 + p) \).

Herein, we are only concerned with what happens at the last trading period – at time \( \{N = T/\Delta t \} \). Hence, we omit the subscript \( N - 1 \) when it’s clear from the context, in particular we write \( p^\alpha \) and \( p^b \) for \( p_N^\alpha \) and \( p_N^b \), \( \nu \) for \( \nu_N \), and \( \xi \) for \( \xi_N \). Note also that, in an LTC equilibrium, we have: \( p^\alpha = p_N^\alpha = p_{N-1}^\alpha \), with similar equalities for \( p^b \) and \( \nu \). For convenience, we also drop the superscript \( \Delta t \) in the LOB and the associated bid and ask prices. Finally, we denote by \( \hat{A} \) the support of a given equilibrium, and by \( \mu \) the measure of states. As the roles of \( p^\alpha \) and \( p^b \) in our model are symmetric, we will only prove the statement of the proposition for \( p^b \).

We are going to show that there exists a constant \( C_0 > 0 \), depending only on the constant \( C \), from Assumptions 2 and 3, such that, for all small enough \( \Delta t \), we have, \( \mathbb{P} \)-a.s.:

\[
-C_0 \leq p^b / \sqrt{\Delta t} < 0
\]

First, we need to introduce the simplified (random) objective functions \( \hat{A}^\alpha(p ; x) \):

\[
\hat{A}^\alpha(p ; x) = \mathbb{E}_{N-1}^\alpha \left[ (p - x - \xi) \mathbb{1}_{\{ x > p \}} \right]
\]

The simplified objective \( \hat{A}^\alpha(p ; x) \) is somewhat similar to the true objective function:

\[
A^\alpha(p ; x) = \mathbb{E}_{N-1}^\alpha \left[ (p - x - \xi) \mathbb{1}_{\{ D^+(p) > \nu^+((p^0, -\infty)) \}} \right]
\]
In particular, \( \hat{A}^\alpha(p; x) = A^\alpha(p; x) \) for \( p \leq p^\alpha \). Recall Corollary 1, which states that, in equilibrium, \( \mathbb{P}\text{-a.s.} \), if the agents in the state \((s, \alpha)\) post limit sell orders, then they post them at a price level \( p \) that maximizes \( A^\alpha(p; p^\alpha) \). The following lemma shows that the value of the modified objective becomes close to the value of the true objective, for the agents posting limit sell orders close to the ask price.

**Lemma 3.** \( \mathbb{P}\text{-a.s.}, either } \nu^+(\{p^\alpha\}) > 0 \text{ or we have:} \]

\[
\left| A^\alpha(p; p^\alpha) - \hat{A}^\alpha(p^\alpha; p^\alpha) \right| \to 0,
\]

as \( p \downarrow p^\alpha \), uniformly over all \( \alpha \in \hat{A} \).

**Proof:**

If \( \nu^+(\{p^\alpha\}) = 0 \), then \( \nu^+ \) is continuous at \( p^\alpha \), and \( \nu^+((-\infty, p]) \to 0 \), as \( p \downarrow p^\alpha \). Then, we have

\[
\left| A^\alpha(p; p^\alpha) - \hat{A}^\alpha(p^\alpha; p^\alpha) \right| = |\mathbb{E}^\alpha_{N-1}(p - p^\beta - \xi_N)1_{D^+(p - \xi_N) > \nu^+((-\infty, p))} - \mathbb{E}^\alpha_{N-1}(p^\alpha - p^\beta - \xi_N)1_{\xi_N > p^\alpha}| \\
\leq |p - p^\alpha| + \|p^\alpha - p^\beta - \xi_N\|_{L^2(\mathbb{P}^\alpha_{N-1})} \mathbb{P}^\alpha_{N-1}\left[\xi_N > p^\alpha, D^+(p - \xi_N) \leq \nu^+((-\infty, p))\right]
\]

Thus, it suffices to show that: (i) \( \|p^\alpha - p^\beta - \xi_N\|_{L^2(\mathbb{P}^\alpha_{N-1})} \) is bounded by a constant independent of \( \alpha \), and (ii) \( \mathbb{P}^\alpha_{N-1}\left[\xi_N > p^\alpha, D^+(p - \xi_N) \leq \nu^+((-\infty, p))\right] \to 0 \) as \( p \downarrow p^\alpha \), uniformly over \( \alpha \). For (i), we have:

\[
\|p^\alpha - p^\beta - \xi\|_{L^2(\mathbb{P}^\alpha_{N-1})} \leq |p^\alpha - p^\beta| + \|\xi\|_{L^2(\mathbb{P}^\alpha_{N-1})} \leq 2C\sqrt{\Delta t},
\]

where the constant \( C \) appears in Assumptions 2 and 3. For (ii), we note that

\[
\{\xi_N > p^\alpha, D^+(p - \xi_N) \leq \nu^+((-\infty, p))\} = \{\xi_N > p^\alpha, \xi_N \leq p - D^{-1}(\nu^+((-\infty, p)))\},
\]

as \( D(\cdot) \) is strictly decreasing, with \( D(0) = 0 \). Assumption 8 implies that

\[
\kappa^{-1}(\nu^+((-\infty, p))) \leq D^{-1}(\nu^+((-\infty, p))) < 0,
\]

where \( \kappa \) is known at time \( N - 1 \). Therefore,

\[
\mathbb{P}^\alpha_{N-1}\left[\xi_N > p^\alpha, D^+(p - \xi_N) \leq \nu^+((-\infty, p))\right] \leq \mathbb{P}^\alpha_{N-1}\left(\xi_N \in (p^\alpha, p - \kappa^{-1}(\nu^+((-\infty, p))))\right)
\]

It remains to show that, \( \mathbb{P}\text{-a.s.}, \) the right hand side of the above converges to zero, uniformly over all \( \alpha \). Assume that it does not hold. Then, with positive probability \( \mathbb{P} \), there exists \( \varepsilon > 0 \) and a sequence of \((p_k, \alpha_k)\), such that \( p_k \downarrow p^\alpha \) and

\[
\mathbb{P}^\alpha_{N-1}\left(\xi_N \in (p^\alpha, p_k - \kappa^{-1}(\nu^+((-\infty, p_k))))\right) \geq \varepsilon
\]

Notice that, \( \mathbb{P}\text{-a.s.}, \) the family of measures \( \{\hat{\mu}_k = \mathbb{P}^\alpha_{N-1} \circ \xi_N^{-1}\}_k \) is tight. The latter follows, for example, from the fact that, \( \mathbb{P}\text{-a.s.}, \) the conditional second moments of \( \xi_N \) are bounded uniformly over all \( \alpha \) (which, in turn, is a standard exercise in stochastic calculus). Prokhorov’s theorem, then, implies that there is a subsequence of these measures that converges weakly to some measure \( \hat{\mu} \) on \( \mathbb{R} \). Next, notice that, for any fixed \( k \) in the chosen subsequence, there exists a large enough \( k' \), such that

\[
|\mu((p^\alpha, p_k - \kappa^{-1}(\nu^+((-\infty, p_k))))\}) - \mu_{k'}((p^\alpha, p_k - \kappa^{-1}(\nu^+((-\infty, p_k))))\})| \leq \varepsilon/2
\]
Thus, for any $k$ in the subsequence, we have
\[ \mu \left( (p^\alpha, p_k - \kappa^{-1}(\nu^+((-\infty, p_k)))) \right) \geq \varepsilon/2 \]

The above is a contradiction, as the intersection of the corresponding intervals, $(p^\alpha, p_k - \kappa^{-1}(\nu^+((-\infty, p_k))))$, over all $k$ is empty. ■

Now we are ready to prove the upper bound in (14).

**Lemma 4.** In any non-degenerate LTC equilibrium, $p^b < 0 < p^a$, $\mathbb{P}$-a.s.

**Proof:**
We only prove the $p^b < 0$ part, the $p^a > 0$ one being symmetric. Assume that “$p^b < 0$, $\mathbb{P}$-a.s.” does not hold. Then $p^b \geq 0$ on some positive $\mathbb{P}$-probability set $\Omega' \in \mathcal{F}_{N-1}$. We are going to show that this results in a contradiction. First, Corollary 1 implies that, $\mathbb{P}$-a.s., if the agents in state $(s, \alpha)$ post a limit sell order, then we must have:
\[ \sup_{p \in \mathbb{R}} A^\alpha(p; p^b) \geq 0 \]

In addition, on $\Omega'$, we have: $\hat{A}^\alpha(p^a; p^b) < 0$ for all $\alpha \in \hat{\alpha}$, as $\xi_N$ has full support in $\mathbb{R}$ under every $\mathbb{P}_{N-1}$ (which, in turn, follows from the fact that $\sigma$ is bounded uniformly away from zero). Then, Lemma 3 implies that there exists a $\mathcal{F}_{N-1}$-measurable $\hat{p} \geq p^a$, such that, on $\Omega'$ (possibly, without a set of $\mathbb{P}$-measure zero), the following holds: if $\nu^+(\{p^a\}) = 0$ then $\hat{p} > p^a$, and, in all cases,
\[ A^\alpha(p; p^b) < 0, \quad \forall p \in [p^a, \hat{p}], \quad \forall \alpha \in \hat{\alpha} \]  \hspace{1cm} (17)

Clearly, it is suboptimal for an agent to post a limit sell order below $\hat{p}$. However, an agent’s strategy only needs to be optimal up to a set of $\mathbb{P}$-measure zero, and these sets can be different for different $(s, \alpha)$. Therefore, a little more work is required to obtain the desired contradiction. Consider the set $B \subset \Omega' \times \mathbb{I} \times \hat{\alpha}$:
\[ B = \{ (\omega, s, \alpha) \mid \hat{q}(s, \alpha) > 0, \ \hat{p}(s, \alpha) \leq \hat{p} \} \]

This set is measurable with respect to $\mathcal{F}_{N-1} \otimes \mathcal{B} \left( \mathbb{I} \times \hat{\alpha} \right)$, due to the measurability properties of $\hat{q}$ and $\hat{p}$. Notice that, due to the above discussion and the optimality of agents’ actions (cf. Corollary 1), for any $(s, \alpha) \in \mathbb{I} \times \hat{\alpha}$, we have:
\[ \mathbb{P}(\{\omega \mid (\omega, s, \alpha) \in B\}) = 0, \]
and hence
\[ \mathbb{E}_{N-1} \int_{\mathbb{I} \times \hat{\alpha}} 1_B(\omega, s, \alpha) \mu_{N-1}(ds, d\alpha) = \int_{\mathbb{I} \times \hat{\alpha}} \mathbb{E}_{N-1}(1_B(\omega, s, \alpha)\rho_{N-1}(\omega, s, \alpha)) \mu_{N-1}(ds, d\alpha) = 0, \]

which implies that, $\mathbb{P}_{N-1}$-a.s., we have: $1_B(\omega, s, \alpha)\rho_{N-1}(\omega, s, \alpha) = 0$, for $\mu_{N-1}$-a.e. $(s, \alpha)$. Notice also that, for all $(\omega, s, \alpha) \in \Omega' \times \mathbb{I} \times \hat{\alpha}$,
\[ 1_{\{\hat{p}(s, \alpha) \leq \hat{p}\}} \hat{q}^+(s, \alpha)1_{B^c} = 0 \]

From the above observations and the condition (7) in the definition of equilibrium (cf. Definition 4), we conclude that, on $\Omega'$, without, possibly, a set of $\mathbb{P}$-measure zero, we have:
\[ \nu^+_{N-1}(\{p^\alpha, \hat{p}\}) = 0, \]
where $\hat{p} \geq p^a$, and, if $\nu^+(\{p^a\}) = 0$, then $\hat{p} > p^a$. Thus contradicts the definition of $p^a$ (recall that $p^a$ is $\mathbb{P}$-a.s. finite, due to non-degeneracy of the LOB). ■
It only remains to prove the lower bound on $p^b$ in (14). Assume that it does not hold. That is, assume that there exists a family of equilibria, with arbitrary small $\Delta t$, and positive $\mathbb{P}$-probability $\mathcal{F}_{N-1}$-measurable sets $\Omega^{\Delta t}$, such that $p^b < -C_0\sqrt{\Delta t}$ on $\Omega^{\Delta t}$. We are going to show that this leads to a contradiction with $p^a > 0$. To this end, assume that the agents maximize the simplified objective function, $A^a$, instead of the true one, $\hat{A}^a$. Then, it turns out that, if $p^b$ is negative enough, the optimal price levels become negative for all $\alpha$. The precise formulation of this is given by the following lemma.

**Lemma 5.** There exists a constant $C_0 > 0$, such that for any small enough $\Delta t$, there exist constants $\epsilon, \delta > 0$, such that, $\mathbb{P}$-a.s., we have:

$$\hat{A}^a(-\delta; x) \geq \epsilon + \sup_{y \geq 0} \hat{A}^a(y; x),$$

for all $\alpha \in \hat{A}$ and all $x \leq -C_0\sqrt{\Delta t}$.

**Proof:**

Denote $\xi = \xi_N/\sqrt{\Delta t}$ and consider the random function

$$\hat{A}^a(p; x) = \mathbb{E}_{N-1}^\alpha \left[ (p - x - \tilde{\xi})1_{(\xi > p)} \right]$$

Notice that

$$\hat{A}^a(p; x) = \sqrt{\Delta t} \hat{A}^a \left( \frac{p}{\sqrt{\Delta t}}; \frac{x}{\sqrt{\Delta t}} \right),$$

and, hence, we can reformulate the statement of the lemma as follows: There exists a constant $C_0 > 0$, such that for any small enough $\Delta t$, there exist constants $\epsilon, \delta > 0$, such that, $\mathbb{P}$-a.s., we have:

$$\hat{A}^a(-\delta; x) \geq \epsilon + \sup_{y \geq 0} \hat{A}^a(y; x),$$

for all $\alpha \in \hat{A}$ and all $x \leq -C_0$. Notice that

$$\hat{A}^a(-\delta; x) - \hat{A}^a(y; x) = -x\mathbb{E}_{N-1}^\alpha \left[ 1_{(\xi < \xi^y)} \right] - \mathbb{E}_{N-1}^\alpha \left[ \tilde{\xi}1_{(-\delta < \xi \leq y)} \right] - \delta \mathbb{E}_{N-1}^\alpha \left[ 1_{(\xi > -\delta)} \right] - y\mathbb{E}_{N-1}^\alpha \left[ 1_{(\xi > y)} \right]$$

is non-increasing in $x$, and, hence, such is $\hat{A}^a(-\delta; x) - \sup_{y \geq 0} \hat{A}^a(y; x)$. Hence, it suffices to prove the above statement for $x = -C_0$.

Next, consider the deterministic function $A_\sigma(p; x)$ is defined via

$$A_\sigma(p; x) = \mathbb{E} \left[ (p - x - \sigma \eta_0)1_{(\sigma \eta_0 > p)} \right],$$

where $\eta_0$ is a standard normal random variable on some auxiliary probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$. It follows from Lemma 2 that there exists a function $\varepsilon_2(\cdot) \geq 0$, such that $\varepsilon_2(\Delta t) \to 0$, as $\Delta t \to 0$, and, $\mathbb{P}$-a.s., we have:

$$\left| \hat{A}^a(p; -C_0) - A_{\sigma_{N-1}}(p; -C_0) \right| \leq \varepsilon_2(\Delta t)$$

for all $\alpha \in \hat{A}$ and all $p \in \mathbb{R}$. Then, as we can always choose $\Delta t$ small enough so that $\varepsilon_2(\Delta t) < \epsilon$, the statements of the lemma would follow if we can show that there exist constants $\epsilon, \delta, C_0 > 0$ such that, $\mathbb{P}$-a.s.,

$$A_{\sigma_{N-1}}(-\delta; -C_0) \geq 3\epsilon + \sup_{y \geq 0} A_{\sigma_{N-1}}(y; -C_0)$$
As $\sigma_{t,N-1}(\omega) \in \left[ \frac{1}{C}, C \right]$, P-a.s., it is sufficient to find $\epsilon, \delta, C_0 > 0$ such that

$$A_\sigma(-\delta; -C_0) \geq 3\epsilon + \sup_{y \geq 0} A_\sigma(y; -C_0), \quad \forall \sigma \in \left[ \frac{1}{C}, C \right]$$

Note that this statement does not involve $\omega$ or $\xi$ anymore and is simply a property of a deterministic function.

Notice that $A_\sigma(p; x) = \sigma A_1 \left( \frac{x}{\sigma}; \frac{p}{\sigma} \right)$, where

$$A_1(p; x) = \mathbb{E} \left[ (p - x - \eta) \mathbf{1}_{\{\eta > p\}} \right],$$

with $\eta$ being a standard normal. Then, if we denote by $F(x)$ and $f(x)$, respectively, the cdf and pdf of a standard normal, we obtain:

$$A_1(p; x) = (p - x)(1 - F(p)) - \int_p^\infty tf(t)dt$$

And a straightforward calculation gives us the following useful properties of $A_1$ and $A_\sigma$:

(i) For any $\sigma > 0$ and any $x < 0$, the function $p \mapsto A_\sigma(p; x)$ has a unique maximizer $p_\sigma(x)$, this function is increasing in $p \leq p_\sigma(x)$ and decreasing in $p \geq p_\sigma(x)$;

(ii) The function

$$x \mapsto p_\sigma(x) = \sigma p_1(x/\sigma) = \sigma \left( \frac{1 - F}{f^{-1}(-x/\sigma)} \right)$$

is increasing in $x < 0$ and converges to $-\infty$, as $x \to -\infty$.

Then, choosing $C_0$ large enough, so that $p_1(-C_0/C) < 0$, ensures $p_\sigma(-C_0) < 0$, for all $\sigma \in \left[ \frac{1}{C}, C \right]$. Setting

$$\delta = -p_1(-C_0/C)/C$$

guarantees that $p_\sigma(-C_0) \leq -\delta$, for all $\sigma \in [1/C, C]$. Then, by property (i) above, we have, for all $\sigma \in \left[ \frac{1}{C}, C \right]$:

$$A_\sigma(-\delta; -C_0) > A_\sigma(0; -C_0) = \sup_{y \geq 0} A_\sigma(y; -C_0)$$

Finally, as $A_\sigma(-\delta; -C_0) - A_\sigma(0; -C_0)$ is a continuous function of $\sigma$, changing in a compact $[1/C, C]$, we can find $\epsilon$, such that

$$A_\sigma(-\delta; -C_0) \geq 3\epsilon + \sup_{y \geq 0} A_\sigma(y; -C_0), \quad \forall \sigma \in [1/C, C]$$

Recall that our assumption is that $p^b < -C_0\sqrt{\Delta t}$ holds on a set of positive $\mathbb{P}$-measure, $\Omega^\Delta t$. Recall also that $p^a > 0$, $\mathbb{P}$-a.s., due to Lemma 4. Then, Lemmas 3 and 5 imply that there exists $\mathcal{F}_{N-1}$-measurable $\bar{p} \geq p^a$, such that, for any random outcome in $\Omega^\Delta t$ (possibly, without a set of measure zero), we have: if $\nu^+(\{p^a\}) = 0$ then $\bar{p} > p^a$, and, in all cases,

$$A^\alpha(p; p^b) < \sup_{p' \in \mathbb{R}} A^\alpha(p'; p^b), \quad \forall p \in [p^a, p], \quad \forall \alpha \in \tilde{\alpha}$$

It is intuitively clear that posting limit sell orders at the above price levels $p$ must be suboptimal for the agents. However, the above inequality, on its own, does not yield a contradiction, as the agents’ strategies are only optimal up to a set of $\mathbb{P}$-probability zero, and these sets may be different for different states $(s, \alpha)$. To obtain a contradiction with the definition of $p^a$, we simply repeat the last part of the proof of Lemma 4 (following equation (17)). This ensures that (14) holds and completes the proof of the proposition.
The above proposition has a useful corollary, which can be interpreted as follows: if the market does not degenerate as the frequency increases, then such increase improves market efficiency, in the sense that the expected execution price of every agent converges to the fundamental price, at any time.

**Corollary 3.** Under the assumptions of Proposition 1, denote the support of every equilibrium by \( \tilde{\Lambda}^\Delta t \) and the associated expected execution prices by \( \lambda^{a,\Delta t} \) and \( \lambda^{b,\Delta t} \). Then, there exists a deterministic function \( \varepsilon(\cdot) \), such that \( \varepsilon(\Delta t) \to 0 \), as \( \Delta t \to 0 \), and, \( \mathbb{P} \)-a.s.,

\[
\sup_{n=0,\ldots,T/\Delta t,\alpha\in\tilde{\Lambda}^\Delta t} \left( |\lambda^{a,\Delta t}_n(\alpha) - \hat{p}_n^0| + |\lambda^{b,\Delta t}_n(\alpha) - \bar{p}_n^0| \right) \leq \varepsilon(\Delta t),
\]

for all small enough \( \Delta t > 0 \).

**Proof:**

It follows from Corollary 1 and the definition of a LTC equilibrium that \( \lambda^{a,\Delta t}_n(\alpha) \) and \( \lambda^{b,\Delta t}_n(\alpha) \) belong to the interval \( \left[ p^{a,\Delta t}_{N-1}, p^{b,\Delta t}_{N-1} \right] \). It also follows from Corollary 1 (or, more generally, from the definition of a value function) that \( \lambda^{a}(\alpha) \) is a supermartingale, and \( \lambda^{b}(\alpha) \) is a martingale, under \( \mathbb{P}^\alpha \). Hence, for any \( n = 0, \ldots, N-1 \), both \( \lambda^{a,\Delta t}_n(\alpha) \) and \( \lambda^{b,\Delta t}_n(\alpha) \) belong to the interval

\[
\left[ \mathbb{E}^{\alpha,\Delta t}_N p^{a,\Delta t}_{N-1}, \mathbb{E}^{\alpha,\Delta t}_N p^{b,\Delta t}_{N-1} \right],
\]

which, in turn, converges to zero, as \( \Delta t \to 0 \), due to the deterministic bounds obtained in the proof of Proposition 1.

To formalize the economic meaning of the above corollary, we formulate another statement, whose proof follows trivially from Corollary 3.

**Corollary 4.** Let the assumptions and notation of Proposition 1 hold. Then, there exists a deterministic function \( \varepsilon(\cdot) \), such that \( \varepsilon(\Delta t) \to 0 \), as \( \Delta t \to 0 \), and, for all \( \Delta t > 0 \), the strategy given by \( \hat{r}_n(s,\alpha) = 0 \), \( \hat{q}_n(s,\alpha) = s \), \( \hat{p}_n(s,\alpha) = p^{0,\Delta t}_n \) is \( \varepsilon(\Delta t) \)-optimal, in the sense that, \( \mathbb{P} \)-a.s.,

\[
J_n^{\nu^{\Delta t},(\hat{p},\hat{\alpha},\varepsilon)}(s,\alpha) \geq V_n^{\nu^{\Delta t}}(s,\alpha) - s\varepsilon(\Delta t), \quad \forall (s,\alpha) \in I \times \tilde{\Lambda}^\Delta t,
\]

where \( \nu^{\Delta t},(\hat{p},\hat{\alpha},\varepsilon) \) and \( V_n^{\nu^{\Delta t}} \) are defined in (2) and (3) respectively.

In other words, trading at the fundamental price produces an \( \varepsilon \)-equilibrium, with \( \varepsilon \) vanishing as the trading frequency increases. We interpret this fact as follows: for a family of equilibria in which the agents choose to provide liquidity (i.e. for a family of non-degenerate equilibria) the markets become asymptotically efficient as the trading frequency increases. This is justified by the fact that, in a high frequency regime, both the strategic players (agents) and the external investors have the same expected execution prices, which “levels the playground” for the two types of market participants.

### 4.1 Market-neutrality as a necessary condition

The following theorem shows that, under the above assumptions, a non-degenerate equilibrium exists for an arbitrarily high trading frequency, only if the agents’ beliefs are market-neutral: i.e. \( \bar{p}^0 \) is a martingale under every \( \mathbb{P}^\alpha \). To prove this, we need an additional assumption, which can be interpreted as a uniform right-continuity in probability of the martingale

\[
\mathbb{E}^\alpha \int_t^T \mu^\alpha_s ds
\]
Assumption 10. For any \( \alpha \in \mathbb{R} \) and any \( t \in [0, T] \), there exists a deterministic function \( \varepsilon(\cdot) \geq 0 \), such that \( \varepsilon(\Delta t) \to 0 \), as \( \Delta t \to 0 \), and, for any \( t \leq t' \leq t'' \leq t + \Delta t \), the following holds \( \mathbb{P}^\alpha \)-a.s.:

\[
\mathbb{P}^\alpha_t \left( \mathbb{E}^\alpha_t \int_t^{t''} \mu_s^\alpha ds - \mathbb{E}^\alpha_t \int_t^{t'} \mu_s^\alpha ds \geq \varepsilon(\Delta t) \right) \leq \varepsilon(\Delta t),
\]

where all conditional expectations are taken under the modification of regular conditional probabilities appearing in Assumption 4.

The above assumption holds, for example, if \( \mathbb{E}^\alpha \mu_s^\alpha \) has an integral representation with respect to some Brownian motion \( B \),

\[
\mathbb{E}^\alpha_t \mu_s^\alpha = \mu_s^\alpha + \int_s^t \sigma^\alpha u dB_u,
\]

and \( \sigma^\alpha \) satisfies the \( L^2 \)-continuity property, stated for \( \sigma \) in Assumption 7, uniformly over \( s \in (0, T] \). In a diffusion-based model (i.e. assuming there exists a multivariate process \( Y \), such that \( (\tilde{p}^0, Y) \) form a diffusion process), this condition amounts to imposing bounds on derivatives of the coefficients of this diffusion.

Before we present the main result of this subsection, we state the following lemma, which, we believe, is valuable in its own right. In short, this lemma shows that, if the quadratic variation of a continuous semimartingale does not oscillate too much, and if its drift is small enough, then, the marginal distributions of this process and its running maximum behave similar to those of a Brownian motion. This conclusion sounds very natural, to the point that it may even seem trivial. However, the precise formulation of the desired similarity makes the result far from obvious. Namely, the first statement of the lemma is meant to provide uniform estimates of the tails of the running maximum of a process \( X \), conditional on the event that this maximum exceeds a given threshold. The second statement connects the tails of a marginal distribution of this process with the tails of its running maximum – it can be viewed as an asymptotic version of the Bachelier theorem for a continuous semimartingale. This connection, then, allows us to estimate the tails of the conditional marginal distribution of \( X \), in the same way it is done for the running maximum of \( X \) (cf. Lemma 8). Note that, even in the case of a diffusion process \( X \), the classical Gaussian-type bounds for the tails of the marginal distributions of \( X \) are not sufficient to establish the desired estimates, as the bounds from above and from below have different orders of decay, for the large values of the argument. The main difficulty in establishing the desired estimates is their uniformity over the values of the argument.

Lemma 6. Consider the following continuous semimartingale on a stochastic basis \((\tilde{\Omega}, (\tilde{\mathcal{F}}_t)_{t \in [0, 1]}, \tilde{\mathbb{P}})\):

\[
X_t = \int_0^t \tilde{\mu}_u du + \int_0^t \tilde{\sigma}_u dB_u, \quad t \in [0, 1],
\]

where \( B \) is a Brownian motion (with respect to the given stochastic basis), \( \tilde{\mu} \) and \( \tilde{\sigma} \) are progressively measurable processes, such that the above integrals are well defined. Assume that there exists a constant \( C > 1 \), such that, for any stopping time \( \tau \), with values in \([0, 1]\), we have, \( \tilde{\mathbb{P}} \)-a.s.: \(|\tilde{\sigma}_\tau| \leq C \) and \(|\tilde{\mu}_\tau| \leq C \). Then the following holds.

1. For any constant \( c > 0 \), there exists a constant \( C_1 > 0 \) (depending only on \( C \) and \( c \)), such that

\[
\tilde{\mathbb{P}} \left( \sup_{t \in [0, 1]} X_t > x + z \right) \leq C_1 e^{-cz} \tilde{\mathbb{P}} \left( \sup_{t \in [0, 1]} X_t > x \right), \quad \forall x, z \geq 0.
\]

2. For any constant \( c > 0 \), there exist constants \( \varepsilon, C_2 > 0 \) (depending only on \( C \) and \( c \)), such that,

\[
\tilde{\mathbb{P}} \left( \sup_{t \in [0, 1]} X_t > x \right) \leq C_2 \tilde{\mathbb{P}}(X_1 > x), \quad \forall x \geq 0,
\]
provided the following additional conditions hold for any stopping time $\tau$, with values in $[0, 1]$, and any $s \in [0, 1]$, $\tilde{P}$-a.s.: $|\tilde{\sigma}_\tau| \geq c$, $\tilde{\mu}_\tau^2 \leq \epsilon$ and

$$\tilde{E} \left( (\tilde{\sigma}_{s \wedge \tau} - \tilde{\sigma}_\tau)^2 | \tilde{F}_\tau \right) \leq \epsilon$$

Proof:

In the course of this proof, we will use the shorthand notation, $\tilde{E}$ and $\tilde{P}$, to denote the conditional expectation and conditional probability with respect to $\tilde{F}_\tau$. We also denote

$$A_t = \int_0^t \tilde{\mu}_u \, du, \quad C_t = \int_0^t \tilde{\sigma}_u \, dB_u$$

For any $x \geq 0$, let us introduce

$$\tau_x = 1 \wedge \inf \{ t \in [0, 1] : X_t = x \}$$

Then

$$\tilde{P}(X_1 > x + z) \leq \tilde{P}( \sup_{t \in [0, 1]} X_t > x + z) = \tilde{E} \left( 1_{\{\tau_x < 1\}} \mathbf{1}_{\{\sup_{s \in [\tau_x, 1]} (X_s - x) > z\}} \right)$$

$$= \tilde{E} \left( 1_{\{\tau_x < 1\}} \tilde{P}_{\tau_x} \left( \sup_{s \in [\tau_x, 1]} (X_s - x) > z \right) \right)$$

Notice that, on $\{\tau_x \leq s\}$, we have:

$$X_s - x = A_{s \wedge \tau_x} - A_{\tau_x} + C_{s \wedge \tau_x} - C_{\tau_x}$$

In addition, the process $(Y)_{s \in [0, 1]}$, with

$$Y_s = A_{s \wedge \tau_x} - A_{\tau_x}, \quad (18)$$

is adapted to the filtration $\left( \tilde{F}_{\tau_x \vee s} \right)_{s \in [0, 1]}$, while the process $Z$,

$$Z_s = C_{s \wedge \tau_x} - C_{\tau_x}, \quad s \in [0, 1], \quad (19)$$

is a martingale with respect to it. Next, we obtain:

$$\tilde{P}_{\tau_x} \left( \sup_{s \in [\tau_x, 1]} (X_s - x) > z \right) = \tilde{P}_{\tau_x} \left( \sup_{s \in [0, 1]} (Y_s + Z_s) > z \right)$$

$$\leq \tilde{P}_{\tau_x} \left( \sup_{s \in [0, 1]} \exp \left( cZ_s - \frac{1}{2} c^2 \langle Z \rangle_s \right) > \exp \left( cz - cC - \frac{1}{2} c^2 C^2 \right) \right),$$

where we used the fact that $\langle Z \rangle_s \leq \langle X \rangle_s \leq C^2$, for all $s \in [0, 1]$. Using the Novikov’s condition, it is easy to check that

$$M_s = \exp \left( cZ_s - \frac{1}{2} c^2 \langle Z \rangle_s \right), \quad s \in [0, 1],$$

is a true martingale, and, hence, we can apply the Doob’s martingale inequality:

$$\tilde{P}_{\tau_x} \left( \sup_{s \in [0, 1]} \exp \left( cZ_s - \frac{1}{2} c^2 \langle Z \rangle_s \right) > \exp \left( cz - cC - \frac{1}{2} c^2 C^2 \right) \right) \leq \exp \left( -cz + cC + \frac{1}{2} c^2 C^2 \right)$$
Collecting the above inequalities, we obtain
\[
\tilde{\mathbb{P}}(X_1 > x + z) \leq \tilde{\mathbb{P}}(\sup_{t \in [0,1]} X_t > x + z) \leq C_1 e^{-cz} \tilde{\mathbb{P}}(\tau_x < 1) = C_1 e^{-cz} \tilde{\mathbb{P}}(\sup_{t \in [0,1]} X_t > x),
\]
which yields the first statement of the lemma.

The next step is to estimate the distribution tails of a running maximum via the tails of the distribution of \(X_1\). To do this, we proceed as before:
\[
\tilde{\mathbb{P}}(X_1 > x) = \tilde{E} \left( \mathbf{1}_{\{\tau_x < 1\}} \tilde{\mathbb{P}}_{\tau_x} (Y_1 + Z_1 > 0) \right),
\]
where \(Y\) and \(Z\) is defined in (18) and (19). Notice that
\[
\tilde{\mathbb{P}}_{\tau_x} (Y_1 + Z_1 > 0) = \tilde{\mathbb{P}}_{\tau_x} \left( \bar{\sigma}_{\tau_x} \frac{B_1 - B_{\tau_x}}{\sqrt{1 - \tau_x}} + \frac{1}{\sqrt{1 - \tau_x}} \int_{\tau_x}^1 \bar{\mu}_u du + \frac{1}{\sqrt{1 - \tau_x}} \int_0^1 (\bar{\sigma}_{u \lor \tau_x} - \bar{\sigma}_{\tau_x}) dB_u^x > 0 \right),
\]
where \(B_s^x = B_{s \lor \tau_x}\) is a continuous square-integrable martingale with respect to \((\mathcal{F}_{s \lor \tau_x})_{s \in [0,1]}\). Denote
\[
R_s = \int_0^s (\bar{\sigma}_{u \lor \tau_x} - \bar{\sigma}_{\tau_x}) dB_u^x,
\]
and notice that it is a square-integrable martingale with respect to \((\mathcal{F}_{s \lor \tau_x})_{s \in [0,1]}\). Then, on \(\{\tau_x < 1\}\) (possibly, without a set of \(\tilde{\mathbb{P}}\)-measure zero), we have:
\[
\tilde{\mathbb{E}}_{\tau_x} \left( \frac{1}{\sqrt{1 - \tau_x}} R_1 \right)^2 = \frac{1}{1 - \tau_x} \tilde{\mathbb{E}}_{\tau_x} R_1^2 = \frac{1}{1 - \tau_x} \tilde{\mathbb{E}}_{\tau_x} (R_1 - R_{\tau_x})^2 \leq \frac{1}{1 - \tau_x} \int_{\tau_x}^1 \tilde{\mathbb{E}}_{\tau_x} (\bar{\sigma}_{u \lor \tau_x} - \bar{\sigma}_{\tau_x})^2 du \leq \varepsilon
\]
In addition,
\[
\tilde{\mathbb{E}}_{\tau_x} \left( \frac{1}{\sqrt{1 - \tau_x}} \int_{\tau_x}^1 \bar{\mu}_u du \right)^2 \leq \varepsilon
\]
Collecting the above and using Chebyshev’s inequality, we obtain
\[
\left| \tilde{\mathbb{P}}_{\tau_x} (Z_1 > 0) - \tilde{\mathbb{P}}_{\tau_x} \left( \bar{\sigma}_{\tau_x} \frac{B_1 - B_{\tau_x}}{\sqrt{1 - \tau_x}} \leq -\varepsilon^{1/3} \right) \right|
\leq \tilde{\mathbb{P}}_{\tau_x} \left( \frac{1}{\sqrt{1 - \tau_x}} R_1 + \frac{1}{\sqrt{1 - \tau_x}} \int_{\tau_x}^1 \bar{\mu}_u du \right) \geq \varepsilon^{1/3} \leq 4\varepsilon^{1/3}
\]
On the other hand, due to the strong Markov property of a Brownian motion, on \(\{\tau_x < 1\}\) (possibly, without a set of \(\tilde{\mathbb{P}}\)-measure zero), we have:
\[
\tilde{\mathbb{P}}_{\tau_x} \left( \bar{\sigma}_{\tau_x} \frac{B_1 - B_{\tau_x}}{\sqrt{1 - \tau_x}} \leq -\varepsilon^{1/3} \right) = \tilde{\mathbb{P}} \left( \xi \leq \frac{-\varepsilon^{1/3}}{\sigma} \right)\bigg|_{\sigma=\bar{\sigma}_{\tau_x}},
\]
where \(\xi\) is a standard normal random variable under \(\tilde{\mathbb{P}}\). As \(\bar{\sigma}_{\tau_x} \in [\varepsilon, C]\), we conclude that the right hand side of the above converges to 1/2, as \(\varepsilon \to 0\), uniformly over all random outcomes in \(\{\tau_x < 1\}\) (possibly, without a set of \(\tilde{\mathbb{P}}\)-measure zero). In particular, for all small enough \(\varepsilon\), we have:
\[
\mathbf{1}_{\{\tau_x < 1\}} \left| \tilde{\mathbb{P}}_{\tau_x} (Z_1 > 0) - \frac{1}{2} \right| \leq \mathbf{1}_{\{\tau_x < 1\}} \frac{1}{4}
\]
Similarly, we can show that
\[ 1_{\{\tau x < 1\}} \left| \tilde{P}_{\tau x}(Z_1 \leq 0) - \frac{1}{2} \right| \leq 1_{\{\tau x < 1\}} \frac{1}{4}, \]
which yields
\[ 1_{\{\tau x < 1\}} \left| \tilde{P}_{\tau x}(Z_1 \leq 0) - \tilde{P}_{\tau x}(Z_1 > 0) \right| \leq 1_{\{\tau x < 1\}} \frac{1}{2}, \]
and, in view of (21),
\[ \tilde{P}(X_1 > x) \geq \tilde{E} \left( 1_{\{\tau x < 1\}} \tilde{p}_{\tau x}(Z_1 \leq 0) \right) - \frac{1}{2} \tilde{P}(\tau x < 1) \]
Summing up the above inequality and (21), we obtain
\[ 2\tilde{P}(X_1 > x) \geq \frac{1}{2} \tilde{P}(\tau x < 1) = \frac{1}{2} \tilde{P}(\sup_{t \in [0,1]} X_t > x), \]
which yields the second statement of the lemma.

Finally, we are ready to formulate the main result of this section.

**Theorem 1.** Consider a family of uniform partitions of a given time interval \([0, T]\), with diameters \(\{\Delta t > 0\}\) containing arbitrarily small \(\Delta t\), and an associated family of market models satisfying Assumptions 1, 2–7, 8–9, and 10. Assume that every such model admits a non-degenerate LTC equilibrium, with the same support \(\tilde{A}\). Then, for all \(\alpha \in \tilde{A}\), we have: \(\tilde{p}^0\) is a martingale under \(P^\alpha\).

**Proof:**
Within the scope of this proof, we use the Notational Convention 1 (i.e. we shift the LOB, the expected execution prices, and the demand, by \(p^0\), without changing the notation). For convenience, we also drop the superscript \(\Delta t\) in many variables which do, in fact, depend on \(\Delta t\). We hope it does not cause any confusion, and we emphasize the dependence on \(\Delta t\) whenever it is important. We also use the notation introduced in (13).

Assume the contrary: i.e. there exists \(\alpha_0 \in \tilde{A}\), such that \(\tilde{p}^0\) is not a martingale under \(P^{\alpha_0}\). Then, there exists \(s \in [0, T)\), such that, with positive probability \(P^{\alpha_0}\), we have:
\[ E^{\alpha_0}\left[ p^0_s \right] \neq p^0_s \]
Without loss of generality, we assume that there exists a constant \(\delta > 0\) and a set \(\Omega' \in \mathcal{F}_s\), having positive probability \(P^{\alpha_0}\) (and hence \(P\)), such that, for all random outcomes in \(\Omega'\), we have:
\[ E^{\alpha_0}\left[ p^0_T - p^0_s \right] \geq \delta \] (22)
(the case of negative values is completely symmetric). Next, we fix an arbitrary \(\Delta t\) from a given family and consider the associated non-degenerate LTC equilibrium.

**Lemma 7.** There exists a deterministic function \(\varepsilon(\cdot) \geq 0\), such that \(\varepsilon(\Delta t) \to 0\), as \(\Delta t \to 0\), and, for any small enough \(\Delta t > 0\), there exists \(n = 0, \ldots, N - 3\) and \(\Omega'' \in \mathcal{F}_n\), such that \(P^{\alpha_0}(\Omega'') > 0\), and the following holds on \(\Omega''\):
\[ P^{\alpha_0}_{n+2} \left( E^{\alpha_0}_{n+3} \left( p^0_N - p^0_{n+3} \right) \leq \delta / 2 \right) \leq \varepsilon(\Delta t) \]

**Proof:**
Finally, due to Assumption 3 and the smallness of $\Delta_t$ equations, we obtain

$$\Omega \quad \text{on}\quad \Omega$$

imply

$$P_{\alpha_0}^\alpha \left( \left| \mathbb{E}_{t_n+2}^{\alpha_0} \int_s^T \mu_u^\alpha du - \mathbb{E}_{s}^{\alpha_0} \int_s^T \mu_u^\alpha du \right| \geq \varepsilon(\Delta t) \right) \leq \varepsilon(\Delta t)$$

on $\Omega'$ (possibly, without a set of measure zero). Assuming that $\varepsilon(\Delta t)$ is small enough and recalling (22), we obtain

$$P_{\alpha_0}^\alpha \left( \left| \mathbb{E}_{t_n+2}^{\alpha_0} \int_s^T \mu_u^\alpha du \leq 3\delta/4 \right) \leq \varepsilon(\Delta t)$$

on $\Omega'$. Therefore, there exists a set $\Omega'' \subset F_s \subset F_n$, such that $P_{t_n}^{\alpha_0}(\Omega'') > 0$ and

$$\mathbb{E}_{t_n+2}^{\alpha_0} \int_s^T \mu_u^\alpha du \geq 3\delta/4$$

on $\Omega''$. Next, we choose $t = s$, $t' = t_{n+2}$ and $t'' = t_{n+3}$ and use Assumption 10 to obtain:

$$P_{t_{n+2}}^{\alpha_0} \left( \left| \mathbb{E}_{t_{n+3}}^{\alpha_0} \int_s^T \mu_u^\alpha du - \mathbb{E}_{t_{n+2}}^{\alpha_0} \int_s^T \mu_u^\alpha du \right| \geq \varepsilon(\Delta t) \right) \leq \varepsilon(\Delta t)$$

on $\Omega''$ (possibly, without a set of measure zero). Assuming that $\varepsilon(\Delta t)$ is small enough and using the last two equations, we obtain

$$P_{t_{n+2}}^{\alpha_0} \left( \left| \mathbb{E}_{t_{n+3}}^{\alpha_0} \int_s^T \mu_u^\alpha du \leq \delta/2 \right) \leq \varepsilon(\Delta t)$$

Finally, due to Assumption 3 and the smallness of $\Delta t$, we can replace $\int_s^T \mu_u^\alpha du$ by $\int_{t_{n+3}}^{T} \mu_u^\alpha du$, and $\delta/2$ by $\delta/4$, in the above equation. This completes the proof of the lemma.

Using the strategy at which the agent in state $(1, \alpha_0)$ waits until the last moment $n = N$, we conclude that the process $(\lambda_n^{\alpha}(\alpha_0) + p_0^\alpha)$ must be a supermartingale under $P^{\alpha_0}$. In particular, $P$-a.s.,

$$\lambda_n^{\alpha}(\alpha_0) \geq E_{n+2}^{\alpha_0} \lambda_n^{\alpha}(\alpha_0) + E_{n+2}^{\alpha_0} \left( E_{n+3}^{\alpha_0} \left( p_0^N - p_{n+3}^0 \right) + \xi_{n+3} \right)$$

Recall that $\lambda_N^{\alpha}(\alpha_0) = p_0^N$ and, due to Proposition 1, there exists a constant $C_0 > 0$, such that for all small enough $\Delta t > 0$, the following holds $P$-a.s.:

$$-C_0 \sqrt{\Delta t} \leq p_N^b < p_N^a \leq C_0 \sqrt{\Delta t}$$

Thus, we have, $P$-a.s.,

$$\lambda_{n+2}^{\alpha}(\alpha_0) \geq -C_0 \sqrt{\Delta t} + E_{n+2}^{\alpha_0} \left( E_{n+3}^{\alpha_0} \left( p_0^N - p_{n+3}^0 \right) \right) + E_{n+2}^{\alpha_0} \xi_{n+3}$$

(23)

Due to Assumption 3, we have, $P$-a.s.

$$E_{n+2}^{\alpha_0} \xi_{n+3} \leq C \Delta t, \quad \left| E_{n+3}^{\alpha_0} \left( p_0^N - p_{n+3}^0 \right) \right| \leq C T$$

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and, hence,
\[ \lambda_{n+2}(\alpha_0) \geq -C_0 \sqrt{\Delta t} + CT + C\Delta t \]

In addition, making use of Lemma 7, we conclude that, for any small enough \( \Delta t \), there exists \( n = 0, \ldots, N-2 \) and \( \Omega^\prime \in \mathcal{F}_n \), such that \( \mathbb{P}^{\alpha_0}(\Omega^\prime) > 0 \) and
\[ \mathbb{P}^{\alpha_0}_{n+2} \left( \mathbb{E}^{\alpha_0}_{n+3} \left( \bar{p}^{0}_{N} - \bar{p}^{0}_{n+3} \right) \leq \delta/2 \right) \leq \varepsilon(\Delta t) \quad \text{on } \Omega^\prime. \]

Using (23) and assuming that \( \Delta t \) is small enough, we obtain:
\[ \lambda_{n+2}(\alpha_0) \geq \delta/4 \quad \text{on } \Omega^\prime. \]

Next, Corollary 1 implies that, \( \mathbb{P} \)-a.s.,
\[ p^b_{n+1} \geq \mathbb{E}^{\alpha_0}_{n+1} \left[ \lambda_{n+2}(\alpha_0) + \xi_{n+2} | \xi_{n+2} < p^b_{n+1} \right] \]
Thus, on \( \Omega^\prime \), we obtain:
\[ p^b_{n+1} - \mathbb{E}^{\alpha_0}_{n+1} \left[ \xi_{n+2} | \xi_{n+2} < p^b_{n+1} \right] \geq \delta/4 \quad (24) \]

The following lemma shows that any price increment \( \xi \) behaves similarly to a small variance gaussian in a sense that expected shortfall above certain threshold, \( \mathbb{E}[\xi] > p \), is close to the value of the threshold, \( p \).

**Lemma 8.** There exists a constant \( C_3 > 0 \), such that for all small enough \( \Delta t > 0 \) and for any \( t \in [0, T - \Delta t] \), the following holds \( \mathbb{P} \)-a.s.:
\[ \sup_{p \leq 0} \left| p - \mathbb{E}^{\alpha_0}_t \left[ \bar{p}^0_{t+\Delta t} - \bar{p}^0_t | \bar{p}^0_{t+\Delta t} - \bar{p}^0_t < p \right] \right| \leq C_3 \sqrt{\Delta t} \]

**Proof:**

Fix \( t \) and \( \Delta t > 0 \) and consider the evolution of \( \bar{p}^0_s \), for \( s \in [t, t + \Delta t] \), under \( \mathbb{P}^{\alpha_0}_t \):
\[ \bar{p}^0_s - \bar{p}^0_t = \int_t^s \mu^\alpha_u du + \int_t^s \sigma_u dW^\alpha_u, \]
where \( W^\alpha \) is a Brownian motion under \( \mathbb{P}^{\alpha_0} \). Rescaling by \( \sqrt{\Delta t} \), we obtain
\[ \frac{1}{\sqrt{\Delta t}} (\bar{p}^0_s - \bar{p}^0_t) = X_{(s-t)/\Delta t}, \]
where
\[ X_s = \int_0^s \hat{\mu}_u du + \int_0^s \hat{\sigma}_u d\bar{W}_u, \quad s \in [0, 1], \]
with
\[ \hat{\mu}_s = \sqrt{\Delta t} \mu^\alpha_{t+s\Delta t}, \quad \hat{\sigma}_s = \sigma_{t+s\Delta t}, \quad \bar{W}_s = \frac{1}{\sqrt{\Delta t}} (W^\alpha_{t+s\Delta t} - W^\alpha_t), \quad s \in [0, 1] \]
Notice that the above processes are adapted to the new filtration \( \bar{\mathcal{F}} \), with \( \bar{\mathcal{F}}_s = \mathcal{F}_{t+s\Delta t} \), and, \( \mathbb{P} \)-a.s., under \( \mathbb{P}^{\alpha_0}_t \), \( \bar{W} \) is a Brownian motion with respect to \( \bar{\mathbb{P}} \) (which follows form the strong Markov property and the scale invariance of Brownian motion). In particular, the above stochastic integral is well defined.

Next, due to Assumptions 2 and 7, for any small enough \( \Delta t > 0 \), \( \mathbb{P} \)-a.s., the dynamics of \( -X_s \), under \( \mathbb{P}^{\alpha_0}_t \), satisfy all the assumptions of Lemma 6. As a result, we obtain:
\[ \mathbb{P}^{\alpha_0}_t (X_1 < -x - z) \leq \mathbb{P}^{\alpha_0}_t \left( \sup_{t \in [0, 1]} (-X_t) > x + z \right) \leq C_1 e^{-\frac{z}{2}} \mathbb{P}^{\alpha_0}_t \left( \sup_{t \in [0, 1]} (-X_t) > x \right) \]
Finally, we notice that
\[ \sup_{p \leq 0} \left| p - \mathbb{E}_t^\alpha \left[ \tilde{p}_{t+\Delta t}^0 - \tilde{p}_t^0 | \tilde{p}_{t+\Delta t}^0 - \tilde{p}_t^0 < p \right] \right| = \sqrt{\Delta t} \sup_{p \leq 0} \left| p - \mathbb{E}_t^\alpha \left[ X_1 | X_1 < p \right] \right| \]
\[ = \sqrt{\Delta t} \sup_{p \leq 0} \left| p - \int_{-\infty}^\infty d\mathbb{P}_t^\alpha(X_1 < x) \right| \]
\[ = \sqrt{\Delta t} \sup_{p \leq 0} \left| p - \frac{\mathbb{E}_t^\alpha(X_1 < p) - \int_{-\infty}^\infty \mathbb{P}_t^\alpha(X_1 < -x) dx}{\mathbb{P}_t^\alpha(X_1 < p)} \right| \]
\[ = \sqrt{\Delta t} \sup_{p \leq 0} \left| \int_0^\infty \mathbb{P}_t^\alpha(X_1 < p - z) dz \right| \leq C_1 C_2 \sqrt{\Delta t} \]
This completes the proof of the lemma. 

Using (24) and Lemma 8, we conclude that, for all small enough \( \Delta t \), we have: \( p^b_{n+1} > 0 \) on \( \Omega'' \) (possibly, without a set of \( \mathbb{P} \)-measure zero). In addition, Corollary 1 implies that, for any \( \alpha \in \mathcal{A} \), the following holds \( \mathbb{P} \)-a.s.:
\[ \lambda^a_{n+1}(\alpha) \geq p^b_{n+1} \]
Next, with a slight abuse of notation (a similar, but more restrictive, notation was introduced in the proof of Proposition 1), we consider the simplified objective of an agent who posts a limit sell order at the ask price \( p^a_n \):
\[ \hat{A}^\alpha(p^a_n; \lambda^a_{n+1}) = \mathbb{E}_n^\alpha \left( p^a_n - \lambda^a_{n+1} - \xi_{n+1} | \xi_{n+1} > p^a_n \right) \]
The above estimates imply that, on \( \Omega'' \) (without, possibly, a set of \( \mathbb{P} \)-measure zero), we have:
\[ \hat{A}^\alpha(p^a_n; \lambda^a_{n+1}) \leq \mathbb{E}_n^\alpha \left( p^a_n - \xi_{n+1} | \xi_{n+1} > p^a_n \right) - \mathbb{E}_n^\alpha \left( p^b_{n+1} 1_{\Omega''} | \xi_{n+1} > p^a_n \right) < 0, \quad \forall \alpha \in \mathcal{A}, \quad (25) \]
as \( \Omega'' \in \mathcal{F}_n \), and we have, \( \mathbb{P} \)-a.s.:
\[ 1_{\Omega''} \mathbb{P}_n(\Omega \setminus \Omega'') = 0 \quad \text{and} \quad \mathbb{P}_n^\alpha(\xi_{n+1} < p^a_n) > 0, \quad \forall \alpha \in \mathcal{A}. \]
Next, repeating, essentially, the proof of Lemma 3 (and using the fact that \( \lambda^a_{n+1} \) is absolutely bounded, as shown in Corollary 3), we conclude that, \( \mathbb{P} \)-a.s., either \( \nu^+_n(\{p^a_n\}) > 0 \), or we have:
\[ \left| A^\alpha(p; \lambda^a_{n+1}) - \hat{A}^\alpha(p^a_n; \lambda^a_{n+1}) \right| \rightarrow 0, \]
as \( p \downarrow p^a \), uniformly over all \( \alpha \in \mathcal{A} \), where we introduce the true objective:
\[ A^\alpha(p; \lambda^a_{n+1}) = \mathbb{E}_n^\alpha \left[ (p - \lambda^a_{n+1} - \xi_{n+1}) 1_{\{D^+_{n+1} > p - \xi_{n+1} > -\infty \}} \right] \]
This convergence, along with (25), yields that there exists a \( \mathcal{F}_n \)-measurable \( \tilde{p} \geq p^a_n \), such that, on \( \Omega'' \) (possibly, without a set of \( \mathbb{P} \)-measure zero), the following holds: if \( \nu^+_n(\{p^a_n\}) = 0 \) then \( \tilde{p} > p^a_n \), and, in all cases,
\[ A^\alpha(p; \lambda^a_{n+1}) < 0, \quad \forall p \in [p^a_n; \tilde{p}], \quad \forall \alpha \in \mathcal{A} \]
Finally, we repeat the last part of the proof of Lemma 4 (following equation (17)), to obtain a contradiction with the definition of \( p^a_n \).

The proof of the above theorem yields the following corollary.
Corollary 5. Under the assumptions of Theorem 1, for any \(n = 0, \ldots, N\) and any \(\alpha \in \tilde{\mathbb{A}}\), the following holds \(\mathbb{P}^\alpha\text{-a.s.}:\)

\[
p_n^b < p_0^0 < p_n^a
\]

Notice that in many existing studies of market microstructure, the fundamental price process is defined to be a mid-point (or just a point) between the bid and ask prices. Herein, we obtain this conclusion as an output of the equilibrium.

5 Existence of a non-degenerate equilibrium for homogeneous beliefs

In this section, we show that, if the agents’ beliefs are market-neutral (i.e. the fundamental price is a martingale under the respective measures), then there exists a non-degenerate equilibrium provided certain assumptions are satisfied. Some of these assumptions are related to the fundamental price process. We show that these assumptions are, typically, satisfied when the trading frequency is high enough. The other assumptions concern the size of the demand process and, more importantly, the set of beliefs \(\mathbb{A}\). The results presented in this section may be interpreted without any connection to the continuous time model for \(\tilde{p}_0\) considered in Section 4 (except that, of course, it is useful to know that the associated assumptions are satisfied if we adopt the setting of Section 4). Therefore, herein, we work with arbitrary discrete time models described in Section 2.

Assumption 11. \(\mathbb{A}\) is a singleton.

The above assumption is the most serious limitation of the results of this section. We do believe that it can be relaxed, but our current method of proof does not allow us to do this. We, therefore, leave this for future research. As the agents’ beliefs are homogeneous, we also assume that \(\mathbb{P}^\alpha = \mathbb{P}\) and drop the superscript \(\alpha\) everywhere.

Assumption 12. The following holds for any \(n = 0, \ldots, N - 1:\)

1. \(\mathbb{P}\text{-a.s.}, the distribution of } \xi_{n+1}\text{ under } \mathbb{P}_n\text{ is continuous and has a finite first moment;}
2. \(\mathbb{P}\text{-a.s., for any } \varepsilon > 0, there exists } p \in \mathbb{R}, \text{ such that } \mathbb{E}_n(\xi_{n+1} - p \mid \xi_{n+1} > p) \leq \varepsilon;\)
3. \(\text{there exists a constant } \delta > 0, \text{ such that } \mathbb{P}_n(\xi > 0) \geq \delta, \mathbb{P}\text{-a.s., where } \xi \text{ can be } \xi_{n+1} \text{ and } -\xi_{n+1}.\)

Assumption 12 may seem complicated and potentially restrictive. However, the following lemma shows that this assumption is satisfied in the context of the continuous time model proposed at the beginning of Section 4.

Lemma 9. Let Assumptions 1, 2–7 and 11 hold. Then Assumption 12 holds for all small enough \(\Delta t > 0.\)

Proof:

The finiteness of the first moment of \(\xi_{n+1}\) is obvious. The continuity of the conditional distribution of \(\xi_{n+1}\) outside of zero follows directly from Lemma 6 and the rescaling used in the proof of Lemma 8. To show the continuity at zero, simply add a positive constant to the drift \(\mu\) and apply Lemma 6 and the rescaling to the
resulting price increment. The last part of Assumption 12 follows from Lemma 2. To show that the second part of Assumption 12 holds, we repeat the derivations in the proof of Lemma 8, to obtain:

$$
\mathbb{E}_n(\xi_{n+1} - p \mid \xi_{n+1} > p) = \frac{\int_p^\infty \mathbb{P}_n(\xi_{n+1} > x) dx}{\mathbb{P}_n(\xi_{n+1} > p)}
$$

Assume that the above expression is bounded from below by some $\varepsilon > 0$, for all $p > 0$. Then, we have

$$
-\partial_x \log \left( \int_x^\infty \mathbb{P}_n(\xi_{n+1} > x) dx \right) \leq \frac{1}{\varepsilon}, \quad \forall x > 0,
$$

$$
\int_x^\infty \mathbb{P}_n(\xi_{n+1} > x) dx \geq \exp \left( C - \frac{1}{\varepsilon} x \right), \quad \forall x > 0,
$$

with some $C \in \mathbb{R}$. The above contradicts the fact that the tails of the distribution of $\xi_{n+1}$ decay faster than any exponential, which, in turn, follows from Lemma 6.

The next assumption introduces further restrictions on the dynamics of the fundamental price process. This assumption holds for a given pair of constants $\bar{c}, \bar{\varepsilon} > 0$.

**Assumption 13.** For a given pair $\bar{c}, \bar{\varepsilon} > 0$, there exist martingales $(\bar{p}_{n+1}^\alpha, \bar{p}_{n+1}^b)$, such that, $\mathbb{P}$-a.s., the following holds:

- $-\bar{c} \leq \bar{p}_{n+1}^b \leq -\hat{\varepsilon} < \bar{\varepsilon} \leq \bar{p}_{n+1}^a \leq \bar{c}$, for all $n = 0, \ldots, N - 1$;

- $\mathbb{E}_{N-1} \left[ (\bar{p}_{N-1}^a - \bar{p}_{N-1}^b - \xi_N) \mathbf{1}_{\{\xi_N > \bar{p}_{N-1}^a\}} \right] \geq \bar{\varepsilon}$;

- $\mathbb{E}_{N-1} \left[ (\bar{p}_{N-1}^a - \bar{p}_{N-1}^b + \xi_N) \mathbf{1}_{\{\xi_N < \bar{p}_{N-1}^b\}} \right] \geq \bar{\varepsilon}$;

- for all $n = 0, \ldots, N - 2$, if $p^a \in \mathbb{R}$ satisfies

$$
p^a \in \arg\max_{p \in \mathbb{R}} \mathbb{E}_n \left[ (p - \bar{p}_{n+1}^b - \xi_{n+1}) \mathbf{1}_{\{\xi_{n+1} > p\}} \right],
$$

then $p^a \geq \bar{p}_{n+1}^a$;

- for all $n = 0, \ldots, N - 2$, if $p^b \in \mathbb{R}$ satisfies

$$
p^b \in \arg\max_{p \in \mathbb{R}} \mathbb{E}_n \left[ (\bar{p}_{n+1}^a - p + \xi_{n+1}) \mathbf{1}_{\{\xi_{n+1} < p\}} \right],
$$

then $p^b \leq \bar{p}_{n+1}^b$.

Assumption 13 holds if we accept the continuous time model proposed at the beginning of Section 4, and under the additional assumption that $\sigma_t$ does not deviate too far from its smallest possible value, in a neighborhood of $T$.

**Lemma 10.** Let Assumptions 1, 2–7 and 11 hold. Then, there exists a constant $c > 1$, such that, if $\sigma_t \geq \bar{\sigma}$, for all $t \in [0, T]$, and $\sigma_t \leq c\bar{\sigma}$, for all $t \in [T - \bar{\delta}, T]$, $\mathbb{P}$-a.s., with some constants $\bar{\sigma}, \bar{\delta} > 0$, then, for all small enough $\Delta t$, there exist $\bar{c}, \bar{\varepsilon} > 0$, such that Assumption 13 holds, with constant $\bar{p}^a$ and $\bar{p}^b$. Moreover, we can choose $\bar{c}$ and $\bar{\varepsilon}$ so that $\bar{c}$ is bounded and $\bar{\varepsilon} \geq \bar{\varepsilon}\sqrt{\Delta t}$, with the same constant $\bar{\varepsilon} > 0$, for all small enough $\Delta t$. 

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\textbf{Proof:}

The proof is given in Appendix B. ■

The following group of assumptions ensure that the size of the incremental demand is sufficiently small, relative to the total inventory held by the agents on each side (long and short). The latter are given by the measure-valued processes \( \mu^1 \) and \( \mu^2 \), introduced in (10). These assumptions make sense in a high frequency regime, and can be interpreted as follows: the external investors buy or sell the asset due to economic considerations, hence, they are expected to buy or sell at a finite rate per unit of time, even if the potential trading opportunities (i.e., trading frequency) increase to infinity. First, we impose this assumption at the last time period. This assumption holds for a given empirical distribution process \( \mu \) and a given pair of constants \( \bar{c}, \bar{\varepsilon} > 0 \).

\textbf{Assumption 14.} For a given \( \mu \) and \( \bar{c}, \bar{\varepsilon} > 0 \), the following holds \( \mathbb{P} \)-a.s.:

\[
\sup_{p > 0} \mathbb{P}_{N-1} \left( \varphi^+_N(p_{N-1}^0 + p) > \mu^1_{N-1}(\bar{A}) \right) < \bar{\varepsilon}/\bar{c}, \quad \sup_{p < 0} \mathbb{P}_{N-1} \left( \varphi^-_N(p_{N-1}^0 + p) > \mu^2_{N-1}(\bar{A}) \right) < \bar{\varepsilon}/\bar{c},
\]

where \( \mu^1 \) and \( \mu^2 \) are defined in (10).

The following two assumptions impose constraints on the demand size in the intermediate time steps. Either one of the two assumptions (but not necessarily both) can be used to prove the existence result that follows. The first one puts a very strong (almost sure) constraint on the size of the incremental demand. This assumption holds for a given empirical distribution process \( \mu \).

\textbf{Assumption 15.} For a given \( \mu \), the following holds \( \mathbb{P} \)-a.s., for all \( n = 1, \ldots, N - 1 \):

\[
\sup_{p \in \mathbb{R}} \varphi^+_n(p) < \mu^1_n(\bar{A}), \quad \sup_{p \in \mathbb{R}} \varphi^-_n(p) < \mu^2_n(\bar{A}).
\]

The next assumption is somewhat milder: essentially, it states that the size of the incremental demand at a price level that is fixed relative to the total inventory held by the agents on each side (long and short), is independent of the fundamental price itself.

\textbf{Assumption 16.} There exist filtrations \( \mathbb{G} \) and \( \mathbb{H} \), such that: \( \mathbb{F} = \mathbb{G} \lor \mathbb{H} \), \( p^0 \) is adapted to \( \mathbb{G} \), the random field \( \varphi_n(p^0_n + \cdot) \) is progressively measurable with respect to \( \mathbb{H} \otimes \mathcal{B}([0, \infty)) \), and \( \mathbb{G} \) is independent of \( \mathbb{H} \) under \( \mathbb{P} \).

Let us now clarify, in which cases Assumptions 14–16 hold, and, in addition, when Assumptions 13–14 hold with the same constants \( \bar{c}, \bar{\varepsilon} > 0 \). In the following model, Assumptions 14–15 are satisfied for any \( \bar{c}, \bar{\varepsilon} > 0 \), provided \( \mu^1(\bar{A}) \) and \( \mu^2(\bar{A}) \) are bounded away from zero by a constant and \( \Delta t > 0 \) is small enough. Namely, assume that

\[
\varphi_n(p^0_n + p) = \Delta t \kappa_n(p), \quad n = 1, \ldots, N,
\]

with an adapted random field \( \kappa \), which is strictly decreasing in \( p \), takes value zero at \( p = 0 \), and is absolutely bounded by a constant. In fact, in the above model for the demand size, for any fixed \( \epsilon > 0 \), satisfying \( \mu^1(\bar{A}), \mu^2(\bar{A}) \geq \epsilon \), whenever \( \Delta t \) is small enough, Assumption 14 holds with arbitrary \( \bar{c}, \bar{\varepsilon} > 0 \), and, in particular, we can choose \( \bar{\varepsilon} \) to decay faster than any \( O(\sqrt{\Delta t}) \), as \( \Delta t \to 0 \). If, in addition, the model for \( p^0 \) is obtained by discretizing the continuous time process \( p^0 \), and if the Assumptions 2–7 hold, then, Lemma 10 implies that Assumptions 13–14 hold with the same \( \bar{c}, \bar{\varepsilon} > 0 \), for all small enough \( \Delta t > 0 \).

However, Assumptions 14 and 16 allow for more general models of the demand size. In particular, the size of the incremental demand does not have to be bounded by a deterministic number. Rather, it suffices to require that the size of the demand convergence to zero in probability, as the trading frequency increases. For example, in the following model, Assumptions 14 and 16 are also satisfied for any \( \bar{c}, \bar{\varepsilon} > 0 \), provided \( \mu^1(\bar{A}) \) and \( \mu^2(\bar{A}) \) are bounded away from zero by a constant and \( \Delta t > 0 \) is small enough:

\[
\varphi_n(p) = \left( N_n \Delta t - N_{(n-1)\Delta t} \right) \kappa_n(p^0_n - p), \quad n = 1, \ldots, N,
\]

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where \((N_t)\) is any strictly increasing adapted process bounded intensity (e.g. a Poisson process with drift), independent of \(p^0\), and \(\kappa\) is as above. In this case, for any fixed \(\epsilon\), such that \(\mu^1(\hat{\mathcal{A}}), \mu^2(\hat{\mathcal{A}}) \geq \epsilon\), and any \(\bar{c} > 0\), we can choose \(\bar{\varepsilon} > 0\) so that it is of the order \(O(\Delta t)\) and Assumption 14 holds. Then, in the context of a continuous time model (i.e. if Assumptions 2–7 hold), Assumptions 13 and 14 hold with the same constants \(\bar{c}, \bar{\varepsilon} > 0\), for all small enough \(\Delta t > 0\).

The following theorem is the main result of this subsection. It shows that, under the above assumptions, there exists a non-degenerate LTC equilibrium, provided the agents’ beliefs are market-neutral and the empirical distribution does not degenerate.

**Theorem 2.** Consider a market model and an empirical distribution process \(\mu\), as described in Section 2. Let Assumptions 1, 11 and 12 hold for this model. Assume, in addition, that there exist constants \(\bar{c}, \bar{\varepsilon} > 0\), such that the following holds:

- Assumptions 13 and 14 hold with the same constants \(\bar{c}, \bar{\varepsilon} > 0\);
- either Assumption 15 or Assumption 16 holds;
- \(\mu^1_n(\hat{\mathcal{A}}), \mu^2_n(\hat{\mathcal{A}}) > 0\), \(\mathbb{P}\)-a.s., for all \(n = 0, \ldots, N - 1\);
- \(p^0\) is a martingale under \(\mathbb{P}\).

Then, the associated market model and \(\mu\) admit a non-degenerate LTC equilibrium. Moreover, this equilibrium can be constructed so that, \(\mathbb{P}\)-a.s., at every step \(n = 0, \ldots, N - 1\), all agents who are long the asset post limit sell orders at the ask price, while all agents who are short the asset post limit buy orders at the bid price, and the size of every order is equal to the agent’s inventory.

**Proof:**

Within the scope of this proof, we use the Notational Convention 1, introduced in the proof of Proposition 1 (i.e. we shift the LOB, the expected execution prices, and the demand, by \(p^0\), without changing the notation). As the set \(\mathcal{A}\) is a singleton, we drop the superscript (or argument) \(\alpha\) in some variables. Let us construct a non-degenerate LTC equilibrium, with the LOB given by a combination of two Dirac measures:

\[
\nu_n^a = h^a_n \delta_{p^0}, \quad \nu_n^b = h^b_n \delta_{p^0},
\]

where \((h^a_n)\) and \((h^b_n)\) are adapted processes, such that \(h^a_n = \mu^2_n(\hat{\mathcal{A}}) > 0\) and \(h^b_n = \mu^1_n(\hat{\mathcal{A}}) > 0\).

**Step 1.** Let us show that the LOB \(\nu_{N-1}\), given by (26), admits an equilibrium in the single-period subgame, with \(n = N - 1, N\). For convenience, in this step, we drop the subscript “\(N - 1\)” in some variables. We would like to find \(\mathcal{F}_{N-1}\)-measurable random variables \(-\bar{c} \leq p^b < 0 < p^a \leq \bar{c}\), such that:

\[
\hat{\nu}(-1) = p^b \leq \lambda^a < 0 < \lambda^b \leq p^a = \hat{\nu}(1)
\]

Due to Assumption 14, the incremental demand to buy at a price level \(p > 0\) exceeds \(\mu^1_{N-1}(\hat{\mathcal{A}})\) with a probability less than \(\bar{\varepsilon}/\bar{c}\). The latter, and the assumption that \(-\bar{c} \leq p^b < 0\), imply that, if \(\nu\) satisfies (26), then, for any \(p > p^b\), we have, \(\mathbb{P}\)-a.s.:

\[
\mathbb{E}_{N-1} \left[ (p - p^b - \xi N) \mathbf{1}_{\{D_{N-1}^+ (p - \xi N) > \nu_{N-1}^-((-\infty, p))\}} \right] < \bar{\varepsilon}
\]

Similarly, for all \(p < p^b\), we have, \(\mathbb{P}\)-a.s.:

\[
\mathbb{E}_{N-1} \left[ (p^a - p + \xi N) \mathbf{1}_{\{D_{N-1}^- (p - \xi N) > \nu_{N-1}^+((p, \infty))\}} \right] < \bar{\varepsilon}
\]
Then, due to Corollary 2, it suffices to find \( F_{N-1} \)-measurable \( p^b, p^a, \lambda^b \) and \( \lambda^a \), such that, \( \mathbb{P} \)-a.s., we have:

\[
\begin{align*}
-\bar{c} + \bar{\varepsilon} &\leq p^b + \bar{\varepsilon} \leq \lambda^a < 0 < \lambda^b \leq p^a - \bar{\varepsilon} \leq \bar{c} - \varepsilon, \\
p^a &\in \arg \max_{p \leq p^a} \mathbb{E}_{N-1} \left[ (p - p^b - \xi_N)1_{\{\xi_N > p^a\}} \right], \\
\lambda^a &\leq p^b + \mathbb{E}_{N-1} \left[ (p^a - p^b - \xi_N)1_{\{\xi_N > p^a\}} \right], \\
p^b &\in \arg \max_{p \geq p^b} \mathbb{E}_{N-1} \left[ (p^a + \xi_N - p)1_{\{\xi_N < p\}} \right], \\
\lambda^b &\leq p^a - \mathbb{E}_{N-1} \left[ (p^a - p^b + \xi_N)1_{\{\xi_N < p^b\}} \right]. 
\end{align*}
\] (27)

Notice that

\[
\lambda^a = p^b + \mathbb{E}_{N-1} \left[ (p^a - p^b - \xi_N)1_{\{\xi_N > p^a\}} \right] = \mathbb{E}_{N-1} \left[ p^b 1_{\{\xi_N \leq p^a\}} \right] + \mathbb{E}_{N-1} \left[ (p^a - \xi_N)1_{\{\xi_N > p^a\}} \right] < 0
\]

holds as long as \( p^b < 0 \). Similarly, \( \lambda^b > 0 \) as long as \( p^a > 0 \). Thus, setting \( p^a = \bar{p}^a_{N-1} \) and \( p^b = \bar{p}^b_{N-1} \), where

\[
-\bar{c} \leq \bar{p}^b_{N-1} \leq -\bar{\varepsilon} < 0 < \bar{\varepsilon} \leq \bar{p}^b_{N-1} \leq \bar{c} \]

appear in Assumption 13, we obtain the desired bid and ask prices.

Due to Corollary 2 and the properties of \( \lambda^a, \lambda^b, p^a \) and \( p^b \), whenever the LOB is in the form (26), it is optimal for all agents who are long the asset to post limit sell orders at the price level \( p^a \), and for all agents who are short the asset to post limit buy orders at the price level \( p^b \), with the sizes of the orders being equal to their inventories. Then, the total size of the limit orders at each of \( p^a \) and \( p^b \) is at least \( \epsilon \). From the above constructions, it is also easy to see that such actions are consistent with the LOB being in the form (26) and, hence, produce an equilibrium. Finally, let us emphasize a useful property of the equilibrium constructed above: \( \mathbb{P} \)-a.s.,

\[
\lambda^a = p^b + \mathbb{E}_{N-1} \left[ (p^a - p^b - \xi_N)1_{\{\xi_N > p^a\}} \right] \geq p^b = \bar{p}^b_{N-1}, \\
\lambda^a = \mathbb{E}_{N-1} \left[ p^b 1_{\{\xi_N \leq p^a\}} \right] + \mathbb{E}_{N-1} \left[ (p^a - \xi_N)1_{\{\xi_N > p^a\}} \right] \leq \delta \bar{p}^b_{N-1} \leq -\delta \bar{\varepsilon} < 0,
\]
due to Assumption 12. Analogous estimates hold for \( \lambda^b \).

**Step 2.** Here, we use the induction argument. Assume that we have constructed an equilibrium with the LOB \( \nu \), for the sub-games with times \( m, \ldots, N \), for all \( m = n + 1, \ldots, N - 1 \), such that the value function and the LOB satisfy, \( \mathbb{P} \)-a.s.: \( \bar{p}^b_m \leq \lambda^a_m \leq -\bar{\varepsilon}_m \) and \( \bar{\varepsilon}_m \leq \lambda^b_m \leq \bar{p}^a_m \), with some constants \( \varepsilon_m > 0 \). In addition, if Assumption 16 holds, then, we assume that \( \lambda^a_m \) and \( \lambda^b_m \) are adapted to \( \mathcal{G}_m \), for \( m = n + 1, \ldots, N - 1 \). Step 1 shows that these induction assumptions are satisfied for \( n = N - 2 \).

Assume that the LOB \( \nu_n \) is given by (26). Let us show that it admits an equilibrium and that the induction assumptions are satisfied for \( n \). First, notice that, if the ask price is chosen as

\[
p^a_n \in \arg \max_{p \in \mathbb{R}} \mathbb{E}_n \left[ (p - \lambda^a_{n+1} - \xi_{n+1})1_{\{\xi_{n+1} > p\}} \right],
\]

and if Assumption 15 holds, then an agent never better off posting a limit sell order at \( p \neq p^a_n \). This is obvious form the fact that the incremental demand never exceeds \( \mu^a_n (\hat{\lambda}_n) \). It turns out that the same conclusion remains valid under Assumption 16. To see this, notice that, for any \( p > p^a_n \), we have

\[
\mathbb{E}_n \left[ (p - \lambda^a_{n+1} - \xi_{n+1})1_{\{D^+_{n+1} (p - \xi_{n+1}) > \nu^a_n (\mathbb{R}) \}} \right] \\
= \mathbb{E}_n \mathbb{E} \left[ (p - \lambda^a_{n+1} - \xi_{n+1})1_{\{D^+_{n+1} (p - \xi_{n+1}) > h^a_n \}} | \mathcal{H}_{n+1} \cup \mathcal{G}_n \right] \\
\leq \mathbb{E}_n \mathbb{E} \left[ (p - \lambda^a_{n+1} - \xi_{n+1})1_{\{D^+_{n+1} (p - F^{-1} (h^a_n)) \}} | \mathcal{H}_{n+1} \cup \mathcal{G}_n \right] \\
\leq \mathbb{E}_n \mathbb{E} \left[ (p - F^{-1} (h^a_n) - \lambda^a_{n+1} - \xi_{n+1})1_{\{D^+_{n+1} (p - F^{-1} (h^a_n)) \}} | \mathcal{H}_{n+1} \cup \mathcal{G}_n \right]
\]
Similarly, for all \( n \in \mathbb{N} \), where we denote by \( F \) an arbitrary non-increasing Borel-measurable function, taking value zero at zero (with \( F^{-1} \) defined, for example, as the right inverse). It is easy to see that the same inequality holds for \( p < p_n^a \). Similarly, for all \( p \in \mathbb{R} \), we have:

\[
\mathbb{E}_n \left[ (\lambda_{n+1}^b + \xi_{n+1} - p) \mathbf{1}_{\{D_n^+(p-\xi_{n+1}) > \nu_n((r, \infty))\}} \right] \leq \mathbb{E}_n \left[ (\lambda_{n+1}^a + \xi_{n+1} - p_n^b) \mathbf{1}_{\{\xi_{n+1} < p_n^b\}} \right],
\]

provided

\[
p_n^b \in \arg \max_{p \in \mathbb{R}} \mathbb{E}_n \left[ (\lambda_{n+1}^b + \xi_{n+1} - p) \mathbf{1}_{\{\xi_{n+1} < p\}} \right].
\]

Thus, we need to find \( p_n^a, p_n^b, \lambda_n^a \) and \( \lambda_n^b \), such that, \( \mathbb{P} \)-a.s.,

\[
\begin{cases}
\lambda_n^a = \mathbb{E}_n \lambda_{n+1}^a + \mathbb{E}_n \left[ (p_n^a - \lambda_{n+1}^a - \xi_{n+1}) \mathbf{1}_{\{\xi_{n+1} > p_n^a\}} \right], \\
\lambda_n^b = \mathbb{E}_n \lambda_{n+1}^b + \mathbb{E}_n \left[ (p_n^b - \lambda_{n+1}^b - \xi_{n+1}) \mathbf{1}_{\{\xi_{n+1} > p_n^b\}} \right], \\
p_n^a \leq \mathbb{E}_n \lambda_{n+1}^a, \\
p_n^b \leq \mathbb{E}_n \lambda_{n+1}^b.
\end{cases}
\tag{28}
\]

As before,

\[
\lambda_{n+1}^a = \mathbb{E}_n \lambda_{n+1}^a + \mathbb{E}_n \left[ (p_n^a - \lambda_{n+1}^a - \xi_{n+1}) \mathbf{1}_{\{\xi_{n+1} > p_n^a\}} \right]
\]

\[
= \mathbb{E}_n \left[ \lambda_{n+1}^a \mathbf{1}_{\{\xi_{n+1} \leq p_n^a\}} \right] + \mathbb{E}_n \left[ (p_n^a - \xi_{n+1}) \mathbf{1}_{\{\xi_{n+1} > p_n^a\}} \right] \leq -\delta \varepsilon_{n+1} < 0
\]

is always satisfied, as long as \( p_n^a \geq 0 \). The only question is how to choose \( p_n^a \leq 0 \leq p_n^a \) so that they satisfy (28). Notice that, due to \( \lambda_{n+1}^a \leq -\varepsilon_{n+1} < 0 \) and Assumption 12, there exists \( p_n^a \) that satisfies

\[
p_n^a = \arg \max_{p \in \mathbb{R}} \mathbb{E}_n \left[ (p - \lambda_{n+1}^a - \xi_{n+1}) \mathbf{1}_{\{\xi_{n+1} > p\}} \right],
\]

and the value of the objective function at \( p = p_n^a \) is nonnegative. With this choice of \( p_n^a \), we have

\[
\lambda_{n+1}^a \geq \mathbb{E}_n \lambda_{n+1}^a \geq \mathbb{E}_n \bar{p}_{n+1}^a = \bar{p}_n^a
\]

In addition, since \( \lambda_{n+1}^b \leq \bar{p}_{n+1}^a \), we conclude that \( p_n^b \) is not larger than the largest maximum point of

\[
p \mapsto \mathbb{E}_n \left[ (\bar{p}_{n+1}^a - p + \xi_{n+1}) \mathbf{1}_{\{\xi_{n+1} < p\}} \right],
\]

which, in turn, is smaller than \( \bar{p}_n^b < 0 \), due to Assumption 13. Therefore, \( p_n^b \leq \bar{p}_n^b \leq \lambda_n^a \). Similarly, we construct the values of \( p_n^a \) and \( \lambda_n^b \) and verify their desired properties. Thus, we obtain a solution to (28). Most of the induction assumptions are verified by construction. To check the additional measurability properties of \( \lambda_n^a \) and \( \lambda_n^b \), under Assumption 16, we simply notice that these random variables also satisfy a version of (28) in which the conditioning on \( \mathcal{F}_n \) is replaced by conditioning on \( \mathcal{G}_n \). This, in turn, follows from the structure of the filtration and the induction assumption, and yields the desired measurability of \( \lambda_n^a \) and \( \lambda_n^b \) with respect to \( \mathcal{G}_n \).

\[\blacksquare\]
5.1 Equilibrium with endogenous $\mu$

Notice that the equilibrium constructed in Theorem 2 suffers from the inconsistency with the market clearance condition, discussed in Section 3 (after Remark 1). As follows from this discussion, such inconsistency makes it difficult to construct an equilibrium with endogenous $\mu$ – i.e. satisfying the additional condition (9). In this subsection, we take a closer look at this problem and show how it can be resolved, at least, in a more specific modeling framework. Namely, we construct an equilibrium in which the LOB is continuous and the agents do not post market orders, and, as a result, it becomes possible to construct the equilibrium so that (9) holds. Herein, we assume that we are given a discrete time market model, along with an empirical distribution process. Before we present the main existence result, we state the relevant assumptions and explain when they hold.

**Assumption 17.** There exists a constant $\tilde{\varepsilon} > 0$ and the martingales $(\bar{p}_n^b, \bar{p}_n^a)_{n=0}^{N-1}$, such that, \(\mathbb{P}\text{-a.s.},\) the following holds:

- $\bar{p}_n^b \leq -\tilde{\varepsilon} < 0 < \tilde{\varepsilon} \leq \bar{p}_n^a$, for all $n = 0, \ldots, N - 1$;
- $\bar{p}_{N-1}^a = \inf \{p \in \mathbb{R} | \mathbb{E}_{N-1} [(p - \bar{p}_{N-1}^a - \xi_N) 1_{\{\xi_N > p\}}] = 0\}$,
- $\bar{p}_{N-1}^b = \sup \{p \in \mathbb{R} | \mathbb{E}_{N-1} [(\bar{p}_{N-1}^a - p + \xi_N) 1_{\{\xi_N < p\}}] = 0\}$;
- for all $n = 0, \ldots, N - 2$, if $p^a \in \mathbb{R}$ satisfies $p^a = \inf \{p \in \mathbb{R} | \mathbb{E}_n [(p - \bar{p}_{n+1}^b - \xi_{n+1}) 1_{\{\xi_{n+1} > p\}}] = 0\}$,
  then $p^a \geq \bar{p}_n^a$;
- for all $n = 0, \ldots, N - 2$, if $p^b \in \mathbb{R}$ satisfies $p^b = \sup \{p \in \mathbb{R} | \mathbb{E}_n [(\bar{p}_{n+1}^a - p + \xi_N) 1_{\{\xi_N < p\}}] = 0\}$,
  then $p^b \leq \bar{p}_n^b$.

Assumption 17 holds, for example, if we accept the continuous time model proposed at the beginning of Section 4, and under the additional assumption that $\sigma_t$ becomes deterministic and decreasing in a neighborhood of $T$.

**Lemma 11.** Let Assumptions 1, 2–7 and 11 hold. Assume, in addition, that one of the following two properties holds:

- $\sigma_t$ is deterministic and non-increasing in $t \in [0, T]$, \(\mathbb{P}\text{-a.s.},\) and $\bar{p}^0$ is a martingale;
- there exist constants $\tilde{c}, \tilde{\delta} > 0$, such that, $\bar{p}^0$ is a martingale on $[T - \tilde{\delta}, T]$ and, \(\mathbb{P}\text{-a.s.},\) we have: $\sigma_t \geq \tilde{c} > 0$, for all $t \in [0, T - \tilde{\delta}]$, $\sigma_t$ is deterministic and non-increasing in $t \in [T - \tilde{\delta}, T)$, with $\sigma_{T-} < \tilde{c}$.

Then, for all small enough $\Delta t > 0$, Assumption 17 holds, with constant $\bar{p}^a$ and $\bar{p}^b$.

**Proof:**

The proof is given in Appendix C. ■

**Assumption 18.** The following holds \(\mathbb{P}\text{-a.s.},\) for all $n = 0, \ldots, N - 1$ and all $p, \Delta p > 0$:

$$
\mathbb{E}_n (\xi_{n+1} | \xi_{n+1} > p + \Delta p) \leq \mathbb{E}_n (\xi_{n+1} | \xi_{n+1} > p) + \Delta p
$$
Assumption 18 is satisfied if, for example, we accept the continuous time model proposed at the beginning of Section 4 and assume that $\mu^a$ and $\sigma$ are deterministic. The next assumption controls the form of the incremental demand, as well as its size relative to the agents’ inventory. It is formulated for a given empirical distribution process $\mu$. In order to formulate the assumption, we need to introduce $\mu^c = (\mu^c_n)_{n=1}^\infty$, which is an adapted family of random measures constructed as follows: $\mu^c_n$ is the continuous part of $\mu_n$. Then, we define $\mu^{1,c}$ and $\mu^{2,c}$ via (10), with $\mu$ in place of $\mu^c$.

**Assumption 19.** For a given $\mu$, there exists a family of strictly decreasing continuous (deterministic) functions $(\kappa_n(\cdot))$, with $\kappa_n(0) = 0$, such that, for all $n = 1, \ldots, N$, we have, $\mathbb{P}$-a.s.:

$$D_n(p_n^0 + p) = \kappa_n(p), \quad \sup_{p \in \mathbb{R}} \kappa^+_n(p) < \mu^{1,c}_n(\mathcal{A}), \quad \sup_{p \in \mathbb{R}} \kappa^-_n(p) < \mu^{2,c}_n(\mathcal{A})$$

Finally, we present the main result of this subsection. It shows that, under the above assumptions, there exists a non-degenerate LTC equilibrium with continuous LOB, provided the agents are market neutral and the continuous part of the empirical distribution does not degenerate.

**Theorem 3.** Consider a market model and an empirical distribution process $\mu$, as described in Section 2. Let Assumptions 1, 11, 12 and 17–19 hold for this model. Assume, in addition, that

- $\mu^{1,c}_n(\mathcal{A}) > 0$, $\mu^{2,c}_n(\mathcal{A}) > 0$, $\mathbb{P}$-a.s., for all $n = 0, \ldots, N - 1$;
- $p^0$ is a martingale under $\mathbb{P}$.

Then, in addition to the equilibria described in Theorem 2, the associated market model and $\mu$ admit a non-degenerate LTC equilibrium, in which, $\mathbb{P}$-a.s., at every step $n = 0, \ldots, N - 1$, the agents do not post market orders, and the LOB $\nu_n$ is continuous (i.e. has no mass points in $\mathbb{R}$).

**Proof:**

Within the scope of this proof, we use the Notational Convention 1, introduced in the proof of Proposition 1 (i.e. we shift the LOB, the expected execution prices and the demand by $p^0$, without changing the notation). The construction of the new equilibrium is similar to the one in the proof of Theorem 2. In this theorem, we are able to replace Assumption 13 with Assumption 17, because Assumption 19 imposes a stronger constraint on the size of the demand than Assumptions 14–16. In this proof, we construct two types of equilibria at the same time: the one in which all agents post limit orders at the bid and ask prices, and the one that has a continuous LOB. Notice that the former equilibrium is similar to the equilibrium constructed in the proof of Theorem 2. However, the choice of the bid and ask prices is different in the present construction. This choice, in particular, allows us to modify the construction and obtain an equilibrium with continuous LOB.

**Step 1.** As before, we begin with a single-period sub-game, for $n = N - 1, N$. First, we construct an equilibrium in which all agents post limit orders at the bid and ask prices. Namely, we would like to find $\mathcal{F}_{N-1}$-measurable random variables $p^b < 0 < p^a$, such that:

$$\hat{p}(-1) = p^b = \lambda^a < 0 < \lambda^b = p^a = \hat{p}(1)$$

Notice that we require $p^b = \lambda^a$ and $p^a = \lambda^b$. Recall that the incremental demand never exceeds $\mu^{1,c}(\mathcal{A}) \leq \mu^1(\mathcal{A})$ and $\mu^{2,c}(\mathcal{A}) \leq \mu^2(\mathcal{A})$. Then, due to Corollary 2, it suffices to find $\mathcal{F}_{N-1}$-measurable $p^b$ and $p^a$, such that, $\mathbb{P}$-a.s., we have:

$$\begin{cases} p^b < 0 < p^a, \\
 p^a = \inf \{ p \in \mathbb{R} \mid E_{N-1} [(p - p^b - \xi_N)1_{(\xi_N > p)}] = 0 \}, \\
 p^b = \sup \{ p \in \mathbb{R} \mid E_{N-1} [(p^a - p + \xi_N)1_{(\xi_N < p)}] = 0 \}, \end{cases}$$

(29)
Assumption 17 guarantees that such pair \((p^a, p^b)\) exists and satisfies:

\[ p^b = \bar{p}_{N-1}^b \leq -\bar{\varepsilon} < 0 < \bar{\varepsilon} \leq \bar{p}_{N-1}^a = p^a, \]

with \(\bar{p}_{N-1}^a\) and \(\bar{p}_{N-1}^b\) appearing in Assumption 17. Notice also that

\[
\lambda^a = p^b + E_{N-1} \left[ (p^a - p^b - \xi_N)1_{\{\xi_N > p^a\}} \right] = p^b \leq -\bar{\varepsilon} < 0, \\
\lambda^b = p^a + E_{N-1} \left[ (p^a - p^b + \xi_N)1_{\{\xi_N < p^b\}} \right] = p^a \geq \bar{\varepsilon} > 0
\]

Due to Corollary 2 and the properties of \(\lambda^a\), \(\lambda^b\), \(p^a\) and \(p^b\), it is optimal for all agents who are long the asset to post limit sell orders at the price level \(p^a\), of the sizes equal to their inventories. Then, the total size of the limit sell orders at \(p^a\) is \(\mu^1(\mathbb{A})\). It is easy to see, due to the above constructions, that such actions are optimal and produce an equilibrium with the LOB being a combination of two Dirac delta-functions (cf. (26)).

Let us, now, modify the above construction, to obtain an equilibrium with continuous LOB, in a single-period sub-game. For every \(p \geq p^a\), let \(x = x(p) \geq 0\) be the smallest nonnegative solution to

\[
E_{N-1} \left[ \xi_N \mid \kappa_N(p - \xi_N) > x \right] - p = -p^b
\]

(30)

Notice that, for \(p = p^a\), we have \(x(p) = 0\), due to the choice of \(p^a\). Notice also that the left hand side of (30) is jointly continuous in \((p, x)\), due to Assumptions 12 and 19. In addition, the value of the left hand side of (30) converges to infinity, as \(x \to \infty\), and, due to Assumption 18, for all \(p \geq p^a\), we have:

\[
E_{N-1} \left[ \xi_N \mid \kappa_N(p - \xi_N) > 0 \right] - p = E_{N-1} \left[ \xi_N \mid \xi_N > p \right] - p \\
\leq E_{N-1} \left[ \xi_N \mid \xi_N > p^a \right] + p - p^a - p = -p^b
\]

Thus, a nonnegative solution to (30), denoted \(x = x(p)\), exists for any \(p \geq p^a\). Due to the maximum theorem, \(x(\cdot)\) is continuous. Let us show that this function is also non-decreasing. For any \(\Delta x, \Delta p > 0\), consider

\[
E_{N-1} \left[ \xi_N \mid \xi_N > p + \Delta p - \kappa_N^{-1}(x - \Delta x) \right] - p - \Delta p \\
\leq E_{N-1} \left[ \xi_N \mid \xi_N > p - \kappa_N^{-1}(x - \Delta x) \right] - p
\]

If the right hand side of the above is equal to \(-p^b\), for some \(\Delta x > 0\), then, we have a contradiction with the minimality of \(x\) as a solution to (30). Therefore, there exists no such \(\Delta x\) and, in turn, \(x(p + \Delta p) \geq x(p)\).

Finally, we define the LOB \(\nu_{N-1}^+\) via

\[
\nu_{N-1}^+((-\infty, p]) = x(p) \land \mu_{N-1}^1(\mathbb{A}), \quad p \in \mathbb{R}
\]

Analogous construction is used to obtain \(\nu_{N-1}^-\). It is easy to check that, with such LOB, the optimal action for every agent who is long the asset is to post a limit sell order at or above \(p^a\), or wait, and the agents are indifferent between these choices. Similar conclusion applies to the agents who are short the asset. It only remains to show that such LOB can, indeed, result from the aggregated actions of the agents. First, consider the agents who are long the asset. Let us choose \(\tilde{s} \leq \infty\), such that

\[
\int_{(0, \tilde{s})} s \mu_{N-1}^\kappa(\alpha \times ds) = \nu_{N-1}^+(\mathbb{R})
\]

Notice that such \(\tilde{s}\) exists, due to the choice of \(\nu^+\). We assume that the agents whose states \((s, \alpha)\) correspond to the atoms of \(\mu_{N-1}\), and those with \(s > \tilde{s}\), choose to wait. Every remaining agent who is long the asset and is
at state \((s, \alpha)\), posts a limit sell order at the price level \(\hat{p}(s)\), for \(s \in [0, \tilde{s}]\). We choose \(\hat{p}\) to be nondecreasing, starting from \(\hat{p}(0) = p^a\), ending at \(\hat{p}(\tilde{s})\), and defined for all other values of \(s\) via

\[
\hat{p}(s) = \sup \left\{ p \in \mathbb{R} \mid \int_0^s s' \mu^c_{N-1}(\alpha \times ds') = x(p) \right\}
\]

Notice that such \(\hat{p}\) satisfies

\[
\int_0^s s' \mu^c_{N-1}(\alpha \times ds') = x(\hat{p}(s)), \quad s \in (0, \tilde{s}],
\]

which implies that (7) holds. The actions of agents who are short the asset are defined analogously. Thus, we ensure that (7)–(8) are satisfied. Notice that, in the new equilibrium, the bid and ask prices, as well as the expected execution prices, remain the same as in the original equilibrium (constructed in this proof), in which the LOB is a combination of two Dirac measures.

**Step 2.** Here, we use the induction argument. Assume that we have constructed an equilibrium with the LOB \(\nu\), for the sub-games with times \(n, n+1, \ldots, N\), for all \(m = n + 1, \ldots, N - 1\), such that the value function and the LOB satisfy, \(\mathbb{P}\)-a.s.: \(\lambda^a_m = \tilde{p}^a_m \leq -\tilde{\varepsilon} < 0\) and \(\lambda^b_m = \tilde{p}^b_m \geq \tilde{\varepsilon} > 0\). Step 1 shows that the above assumptions are satisfied for \(n = N - 2\). We need to show that there is an equilibrium in the sub-game with time steps \(n, n + 1, \ldots, N\).

First, we construct an equilibrium in which the LOB is a combination of two Dirac measures. This construction is similar to the one in the proof of Theorem 2. First, notice that, if all agents who are long the asset post limit sell orders at the price level

\[
p^a_n = \inf \left\{ p \in \mathbb{R} \mid \mathbb{E}_n \left[ (p - \lambda^a_{n+1} - \xi_{n+1})1_{\{\xi_{n+1} > p\}} \right] = 0 \right\},
\]

then an agent never benefits from posting a limit sell order at \(p \neq p^a_n\). This follows from the fact that the incremental demand never exceeds \(\mu^c(\mathcal{A}) \leq \mu^1(\mathcal{A})\). The above \(p^a_n\) is finite, due to Assumption 12 and the fact that \(\lambda^a_{n+1} \leq -\tilde{\varepsilon} < 0\). Due to Assumption 18, we have \(p^a_n \geq \tilde{p}^a_n\). The expected execution price is then given by

\[
\lambda^a_n = \mathbb{E}_n \lambda^a_{n+1} + \mathbb{E}_n \left[ (p^a_n - \lambda^a_{n+1} - \xi_{n+1})1_{\{\xi_{n+1} > p^a_n\}} \right] = \mathbb{E}_n \lambda^a_{n+1} = \mathbb{E}_n \tilde{p}^b_{n+1} = \tilde{p}^b_n \leq -\tilde{\varepsilon} < 0
\]

Similarly, we define the bid price

\[
p^b_n = \sup \left\{ p \in \mathbb{R} \mid \mathbb{E}_n \left[ (\lambda^b_{n+1} - p + \xi_{n+1})1_{\{\xi_{n+1} < p\}} \right] = 0 \right\},
\]

and notice that \(p^b_n \leq \tilde{p}^b_n\) and \(\lambda^b_n = \tilde{p}^b_n \geq \tilde{\varepsilon} > 0\). In particular, we have: \(\lambda^a_n = \tilde{p}^a_n \geq p^a_n\) and \(\lambda^b_n = \tilde{p}^b_n \leq p^b_n\).

According to Corollary 2, we have constructed an equilibrium, in which, at time \(n\), all agents who are long the asset post limit sell orders at \(p^a_n\), while all agents who are short the asset post limit buy orders at \(p^b_n\), and the sizes of all orders are equal to the agents’ inventories. The induction assumptions are verified above.

The modification of this equilibrium, which leads to a continuous LOB at time \(n\), is exactly the same as in Step 1. As the bid and ask prices, as well as the expected execution prices, remain the same in the new equilibrium, we conclude that the induction assumptions are verified for the new equilibrium as well.

Finally, the following corollary shows the existence of an equilibrium with endogenous \(\mu\) (i.e. satisfying (9)).

**Corollary 6.** Consider a market model, as described in Section 2, and an initial empirical distribution \(\mu_0\) (which is a finite sigma-additive measure on \(\mathcal{S}\)). Assume, in addition, that
\begin{itemize}
    \item $\mu_0^{1,c}(\mathcal{A}) > \sum_{n=1}^{N-1} \sup_{p \in \mathcal{R}} D^+_n(p)$ and $\mu_0^{2,c}(\mathcal{A}) > \sum_{n=1}^{N-1} \sup_{p \in \mathcal{R}} D^-_n(p)$, $\mathbb{P}$-a.s.;
    \item $p^0$ is a martingale.
\end{itemize}

Then, there exists an empirical distribution process $\mu$, having the prescribed initial value, such that the associated market model and $\mu$ admit a non-degenerate LTC equilibrium satisfying (9).

\textbf{Proof:}

The proof follows easily from the proof of Theorem 3. This is due to the fact that, in the proof of Theorem 3, we have constructed the LOB and the optimal controls explicitly, and with a very simple dependence on $\mu_{\text{em}}$, as long as the incremental demand never exceeds $\mu_1^{1,c}(\mathcal{A})$ and $\mu_2^{2,c}(\mathcal{A})$. Indeed, for a fixed $n$ and a fixed random outcome, each side of the LOB constructed in the proof of Theorem 3 is the same measure, cut off at a price level that depends only on $\mu_1^{1,c}(\mathcal{A})$ and $\mu_2^{2,c}(\mathcal{A})$. If we construct $\mu$ so that the incremental demand never exceeds these values, the amount of executed orders does not depend on $\mu$. More precisely, let us denote the amount of limit sell orders executed in the time period $[n, n+1]$ by $D^+_{n+1} = \nu^+\left((-\infty, p^0_{n+1})\right)$, where

$$p^0_{n+1} = \sup \{ p \in \mathcal{R} | \nu^+_n((-\infty, p)) < D^+_n(p) \}$$

Neither $p^0_{n+1}$ nor $D^+_{n+1}$ depend on $\mu$, as long as $\mu_1^{1,c}(\mathcal{A}) > \sup_{p \in \mathcal{R}} D^+_n(p)$ and $\mu_2^{2,c}(\mathcal{A}) > \sup_{p \in \mathcal{R}} D^-_n(p)$. Similar conclusions hold for the limit buy orders. The optimal controls do depend on $\mu$, but in a very simple way: the agents with lower inventory post limit orders closer to the opposite side of the book. It follows from the above observations that, in the equilibrium constructed in the proof of Theorem 3, for any given $\mu$, satisfying $\mu_1^{1,c}(\mathcal{A}) > \sup_{p \in \mathcal{R}} D^+_n(p)$ and $\mu_2^{2,c}(\mathcal{A}) > \sup_{p \in \mathcal{R}} D^-_n(p)$, the endogenous empirical distribution of the states, defined by

$$\mu_n = \mu_0 \circ (s, \alpha) \mapsto \left(\mathcal{S}_{n,s}^{0,s,(p,q,r)}, \alpha\right)^{-1},$$

changes as follows: for the states with nonnegative inventory ($s \geq 0$), the continuous part of $\mu$ satisfies

$$\mu_{n+1}^c(ds) = 1_{\{D^+_{n+1} < \infty\}} \mu_n^c(ds), \quad \mu_0^c = \mu_0^c,$$

while its purely atomic part remains unchanged (except for the atom at zero – it keeps growing). Similar dynamics hold for the states with negative inventory. Notice that these dynamics are independent of $\mu$, which is used to construct the LOB and the optimal controls. To summarize: any choice of $\mu$, having fixed initial value $\mu_0$ and satisfying $\mu_1^{1,c}(\mathcal{A}) > \sup_{p \in \mathcal{R}} D^+_n(p)$ and $\mu_2^{2,c}(\mathcal{A}) > \sup_{p \in \mathcal{R}} D^-_n(p)$, leads to the same empirical measure $\mu$, in the equilibrium constructed in the proof of Theorem 3. In addition, due to the bounds on the total demand size (see the assumptions of the corollary), we have: $\mu_1^{1,c}(\mathcal{A}) > \sup_{p \in \mathcal{R}} D^+_n(p)$ and $\mu_2^{2,c}(\mathcal{A}) > \sup_{p \in \mathcal{R}} D^-_n(p)$. Thus, it only remains to set $\mu = \tilde{\mu}$, to obtain an equilibrium satisfying (9).

\section{Example: a Gaussian random walk model for $p^0$}

In this section, we consider a specific market model satisfying Assumptions 1–11. Namely, we assume that the set of agents’ beliefs is a singleton $\mathcal{A} = \{\alpha\} \subset \mathbb{R}$. We also assume that the continuous time fundamental price process $(\tilde{p}^0_t)_{t \in [0,T]}$ is given by:

$$\tilde{p}^0_t = p_0^0 + \alpha t + \sigma W_t, \quad p_0^0 \in \mathbb{R},$$

where $\sigma > 0$ is constant and $W$ is a Brownian motion under $\mathbb{P}^0 = \mathbb{P}$. The discrete time models are defined on the same stochastic basis, with the filtration observed at discrete times only. In any such model, with the
partition diameter $\Delta t > 0$, the fundamental price is given by $p^0_n = \hat{p}^0_{n\Delta t}$, for $n = 0, \ldots, N = T/\Delta t$. The price increments are given by $\xi_n = p^0_n - p^0_{n-1} \sim \mathcal{N}(\alpha \Delta t, \sigma \sqrt{\Delta t})$. Note that every $\xi_n$ is independent of $\mathcal{F}_{n-1}$. Thus, we obtain a random walk model for $p^0$. Due to Lemmas 9 and 10, Assumptions 12 and 13 hold for all small enough $\Delta t > 0$. Finally, we let Assumption 15 hold for all $n = 1, \ldots, N$ (in which case, Assumption 14 becomes redundant), and assume that the empirical distribution satisfies: $\mu_n(\mathbb{R}^+_+ \times \mathcal{A}) > 0$, $\mu_n(\mathbb{R}^-_+ \times \mathcal{A}) > 0$. Then, the conclusions of Proposition 1 and Theorems 1 and 2 are valid in the present setting, provided $\Delta t$ is small enough.

In the remainder of this section, we illustrate numerically these theoretical conclusions. Namely, we identify a certain type of equilibrium in the proposed random walk model, and implement numerically an algorithm for computing the associated value function and the LOB, whenever such equilibrium exists. We will demonstrate that:

- if $\alpha = 0$, an equilibrium of the chosen type exists and is non-degenerate, for any partition diameter $\Delta t > 0$,
- if $\alpha = 0$, the resulting bid-ask spread converges to zero as frequency goes to infinity,
- for any fixed $\Delta t > 0$, the equilibrium of this type either fails to exist or becomes degenerate, as $|\alpha|$ exceeds a certain threshold.
- and, finally, the smaller is $\Delta t$, the smaller is the aforementioned threshold.

For every partition diameter $\Delta t > 0$, we attempt to construct a non-degenerate equilibrium such that, at every step $n$, all agents who are long the asset post limit sell orders at the ask price, while all agents who are short the asset post limit buy orders at the bid price, and the size of every order is equal to the agent’s inventory. Theorem 2 implies that it is possible when $\alpha = 0$, while Theorem 1 implies that, if $\alpha \neq 0$, this construction will fail for all small enough $\Delta t$. Besides presenting a tractable numerical algorithm for computing the equilibrium, this analysis also demonstrates what exactly goes wrong when the agents’ beliefs are not market-neutral. In the remainder of this section, we use the Notational Convention 1, introduced in the proof of Proposition 1. Namely, we shift the LOB, the expected execution prices, and the demand, by remainder of this section, we use the Notational Convention 1, introduced in the proof of Proposition 1. Namely, we substitute the conditional expectations to the unconditional ones, as $\xi_N$ is independent of $\mathcal{F}_{N-1}$. We also drop the subscript “$N$”, to streamline the notation. It is shown in Lemma 10 and in the proof of Theorem 2 that, for all small enough $\Delta t$, there exists a solution to the above system, for all small enough $\Delta t > 0$. However, such solution may not be unique. For numerical purposes, it may be more convenient to solve the following system instead:

\[
\begin{align*}
    p^a & = \arg \max_{p \leq p^b} \mathbb{E} \left[ (p - p^b - \xi) 1_{\{\xi > p\}} \right], \\
    \lambda^a & = p^a + \mathbb{E} \left[ (p^a - p^b - \xi) 1_{\{\xi > p^a\}} \right], \\
    p^b & = \arg \max_{p \geq p^a} \mathbb{E} \left[ (p^a + \xi - p) 1_{\{\xi < p\}} \right], \\
    \lambda^b & = p^b - \mathbb{E} \left[ (p^a - p^b + \xi) 1_{\{\xi < p^b\}} \right], \\
    p^b & \leq \lambda^a \leq p^a, \quad p^b \leq \lambda^b \leq p^a,
\end{align*}
\]

where we substitute the conditional expectations to the unconditional ones, as $\xi_N$ is independent of $\mathcal{F}_{N-1}$. We also drop the subscript “$N$”, to streamline the notation. It is shown in Lemma 10 and in the proof of Theorem 2 that, for all small enough $\Delta t$, there exists a solution to the above system, for all small enough $\Delta t > 0$. However, such solution may not be unique. For numerical purposes, it may be more convenient to solve the following system instead:

\[
\begin{align*}
    p^a = \arg \max_{p \leq R} \mathbb{E} \left[ (p - p^b - \xi) 1_{\{\xi > p\}} \right], \\
    p^b = \arg \max_{p \geq R} \mathbb{E} \left[ (p^a + \xi - p) 1_{\{\xi < p\}} \right].
\end{align*}
\]
It is easy to see that such \((p^a, p^b)\), along with the associated \(\lambda^a\) and \(\lambda^b\), provide a particular solution to (32). In fact, if \(\alpha = 0\), such solution produces the largest ask and the smallest bid prices, among all possible equilibria at time \(N - 1\). Let us show that, for all small enough \(\Delta t > 0\), there exists a unique solution to (33), and find an algorithm to compute it. First, it is clear that the solution \((p^a, p^b)\) scales proportionally, once we multiply \(\xi\) by a constant. Therefore, without loss of generality, we can replace \(\sigma\) by \(1/\sqrt{\Delta t}\) and \(\alpha\) by \(\beta/\sqrt{\Delta t}\), with some constant \(\beta \in \mathbb{R}\). Consider the following functions:

\[
\hat{\hat{p}}^b : 0 < p^a \mapsto \arg \max_{p \in \mathbb{R}} \mathbb{E}^{\alpha, \Delta t} \left[ (p^a + \xi - p) \mathbf{1}_{\{\xi < p\}} \right], \\
\hat{\hat{p}}^a : 0 > p^b \mapsto \arg \max_{p \in \mathbb{R}} \mathbb{E}^{\alpha, \Delta t} \left[ (p - p^b - \xi) \mathbf{1}_{\{\xi < p\}} \right],
\]

and notice that

\[
\hat{\hat{p}}^a(x) = G^{-1}(-x) + \beta \sqrt{\Delta t}, \quad \hat{\hat{p}}^b(x) = H^{-1}(x) + \beta \sqrt{\Delta t},
\]

with

\[
G(x) = \frac{1 - F(x)}{\phi(x)}, \quad H(x) = \frac{F(x)}{\phi(x)},
\]

where \(\phi\) is the density of a standard normal, and \(F\) is its cumulative distribution function. It is clear that we need to find \(p^a > 0\), such that \(\hat{\hat{p}}^a(p^a) < 0\) and \(\hat{\hat{p}}^a \circ \hat{\hat{p}}^b(p^a) = p^a\).

**Lemma 12.** The following holds for all small enough \(\Delta t > 0\). There exists \(p^{a,*} > 0\), such that \(H^{-1}(p^a + \alpha \Delta t) = 0\). The function \(P = \hat{\hat{p}}^a \circ \hat{\hat{p}}^b\) is well defined and continuous in \((0, p^{a,*})\), with \(P(0) = -\infty\) and \(P(p^{a,*}) = \infty\). The function \(x \mapsto P(x) - x\) is strictly increasing in \(x \in (0, p^{a,*} - \varepsilon)\), where \(\varepsilon\) is the smallest nonnegative number such that \(H^{-1}(p^{a,*} - \varepsilon) \leq 0\). Moreover, \(\varepsilon \downarrow 0\), as \(\Delta t \downarrow 0\).

**Proof:**

The first statement is obvious from the continuity and monotonicity of \(H\) (the latter can be verified by a direct computation of \(H'\)). The second statement follows from the fact that, as \(x \downarrow 0\): \(H^{-1}(x) \to -\infty\) and \(G^{-1} \circ H^{-1}(x) \to -\infty\). To show the last statement, we compute the derivatives of \(H\) and \(G\), to obtain:

\[
-1 < G'(x) < 0, \quad x > 0,
\]

\[
0 < H'(x) < 1, \quad x < 0.
\]

The above implies

\[
\left| (G^{-1})'(p) \right| > 1, \quad G(p) > 0,
\]

\[
\left| (G^{-1})'(p) \right| < 1, \quad H(p) < 0.
\]

Finally, we choose \(\varepsilon > 0\), such that \(H^{-1}(p) < 0\), for all \(p \in (0, p^{a,*}]\). This is possible, for all small enough \(\Delta t > 0\), due to the continuity of \(H^{-1}\) and the fact that \(H^{-1}(p^a) = -\alpha \Delta t\). It is also clear that \(\varepsilon\) vanishes as \(\Delta t \downarrow 0\).

The above lemma not only shows that there exists a unique \(p^a\) satisfying \(\hat{\hat{p}}^a \circ \hat{\hat{p}}^b(p^a) = p^a\), but it also provides a numerical algorithm for computing it. Indeed, if \(\alpha \geq 0\), then we can set \(\varepsilon = 0\) and choose \(p^{a,*}\) as the unique root of the equation \(H^{-1}(x) + \alpha \Delta t = 0\), which can be solved via the bisection algorithm. If \(\alpha < 0\), then we solve \(H^{-1}(x) = 0\) and set \(p^{a,*} - \varepsilon = x\). When \(\Delta t > 0\) becomes small enough, we know that \(P(p^{a,*} - \varepsilon) > 0\). Then, it suffices to find the unique root of \(P(x) = 0\) on the interval \(x \in (0, p^{a,*} - \varepsilon)\), which, again, can be done via the bisection algorithm. We implement this algorithm in MatLab, and the results can be seen as the right-most points on the left graph in Figure 1. It is worth mentioning that, in the present work, we didn’t manage to prove that the time zero bid-ask spreads of non-degenerate equilibria converge to zero as
frequency increases. However, the left part of Figure 3 confirms this conjecture, at least, in the context of the
Gaussian random walk example.

Notice that the above construction yields a non-degenerate equilibrium in any single-step model, regardless
of the drift, as long as $\Delta t > 0$ is small enough. However, we do know from Theorem 1 that, when $\alpha \neq 0$, this
construction has to fail. Clearly, this has to happen during the recursive procedure – going from $n = N - 1$ to
$n = 0$. Let us illustrate what exactly fails during this recursive procedure. Notice, first, that any solution to
(32) or (33) must satisfy

$$\lambda^a < 0 < \lambda^b$$

The above inequality can be interpreted as follows: the agents trying to sell the asset expect to sell it at a
discount relative to the fundamental price. Similarly, the agents trying to buy the asset expect to pay more than
the fundamental price. This may raise the following question: why do the agents even participate in the game,
if they are better off trading at the fundamental price? However, this seemingly contradictory behavior can
be explained by the fact that, in our framework, the fundamental price does not have a meaning of an actual
price level at which the transactions occur. Rather, it is an abstract quantity that represents the price level at
which the demand is balanced. Recall that the external investors trying to purchase the asset have to pay at least
the ask price, which is above the fundamental price, and, typically, above $\lambda^b$. Thus, the strategic players (aka
agents) still make profits relative to the external investors. Notice that we have also discovered another reason
to view the values of $|\lambda^a|$ and $|\lambda^b|$ as the measures of “market inefficiency”: they can be viewed as the cost to
“discover” the fundamental price $p^0$.

It turns out that the signs of $\lambda^a$ and $\lambda^b$ are exactly what determines whether the market degenerates or not!
To see this, consider, for example, the recursive formula for $p^a$ and $\lambda^a$:

$$\begin{align*}
\{ & p^a_n \in \arg \max_{p \leq p_n^a} \mathbb{E} \left[ (p - \lambda^a_n - \xi) \mathbf{1}_{\{\xi > p\}} \right], \\
& \lambda^a_n = \lambda^a_{n+1} + \alpha \Delta t \mathbb{E} \left[ (p^a_n - \lambda^a_{n+1} - \xi) \mathbf{1}_{\{\xi > p^a_n\}} \right],
\end{align*}$$

(34)

There may exist multiple solutions to the above system. However, we can identify a particular special solution.
One option is to find $p^a_n$ that satisfies

$$p^a_n \in \arg \max_{p \in \mathbb{R}} \mathbb{E} \left[ (p - \lambda^a_{n+1} - \xi) \mathbf{1}_{\{\xi > p\}} \right]$$

Similarly, we can identify the bid price:

$$p^b_n \in \arg \max_{p \in \mathbb{R}} \mathbb{E} \left[ (\lambda^b_{n+1} - p + \xi) \mathbf{1}_{\{\xi < p\}} \right]$$

In fact, the above choice of $p^a_n$ and $p^b_n$, for $n = N - 2, \ldots, 0$, produces the largest and the smallest possible
ask and bid prices, respectively, among all equilibria with a fixed pair $(p^a_{N-1}, p^b_{N-1})$. Hence, the resulting
equilibrium can be called a “large-spread” equilibrium. Figure 1 shows the bid and ask prices, as well the
expected execution prices, corresponding to the large-spread equilibrium described above.

Next, we notice that, whether we choose the ask price as shown above or via any other solution to (34), we
must have $\lambda^a_{n+1} < 0$: if this condition is violated, the additional expected profit an agent obtains from posting
a limit sell order at $p^0_n$ (as compared to waiting) is negative:

$$\mathbb{E} \left[ (p^a_n - \lambda^a_{n+1} - \xi) \mathbf{1}_{\{\xi > p^b_n\}} \right] < 0,$$

regardless of the value of $p^a_n$. This implies that the agents will never actually post any limit sell orders and one
side of the LOB degenerates. The fact that negative $\lambda^a_{n+1}$ prevents the agents to post limit sell orders at time
$n$ reflects the adverse selection phenomenon: in such a case, if a sell order is exercised, the fundamental price
will move above the price level of this order, increasing the expected execution price at the next time step so
Figure 1: Bid and ask prices (left) and the associated expected execution prices (right), as functions of time horizon. Different curves correspond to different trading frequencies. Zero drift case.

much that the agent will regret having exercised the order at the previous time step. This may, indeed, happen when \( \alpha > 0 \), as illustrated in Figure 2. If \( \alpha = 0 \), it is easy to see that \( \lambda_{n}^{a} \) remains negative:

\[
\lambda_{n}^{a} = E \left[ \lambda_{n+1}^{a} 1_{\{\xi \leq p_{n}^{a}\}} \right] + E \left[ (p_{n}^{a} - \xi) 1_{\{\xi > p_{n}^{a}\}} \right] < 0,
\]

as long as \( \lambda_{n+1}^{a} \) is negative. Thus, whenever \( \lambda_{n}^{a} \) or \( \lambda_{n}^{b} \) gets a wrong sign, the corresponding agents choose not to provide (and may even choose to consume) the liquidity in the equilibrium. This is exactly the kind of behavior that causes liquidity crises which are due to the market microstructure itself, rather than any real shortage of the asset: notice that the external demand may be large during such times. The likelihood of such liquidity crises becomes higher as the trading frequency increases. Indeed, for a fixed \( \Delta t > 0 \), if \( |\alpha| \) is small enough, a non-degenerate equilibrium may still exist. However, the market does degenerate as \( |\alpha| \) increases beyond a certain tolerance level, and the higher is the trading frequency, the lower is the tolerance level, for which a non-degenerate equilibrium exists. This is shown in the right part of Figure 3.

7 Summary and future work

In this paper, we have presented a new framework for modeling the market microstructure, which does not assume the existence of a designate market maker, and in which the LOB arises endogenously, as a result of equilibrium in a game between multiple strategic players (aka agents). To the best of our knowledge, this is the first class of such models, which also reproduce the mechanics of the exchange closely enough, so that they can be used to analyze the liquidity role of the market participants, and its dependence on the rules of the exchange, such as the trading frequency. The proposed framework is based on the theory of mean field games, but it goes beyond any known classes of mean field game models, thus, potentially, contributing to this theory.

The main economic conclusions stemming form the present work are as follows. First, we show that, even in the absence of any real shortage of the asset (i.e. when the external demand is large), the agents may choose not to provide liquidity in equilibrium, causing the market to degenerate, or, in other words, creating a liquidity crisis. We also show that the occurrence or non-occurrence of such degeneracy (or, crisis) depends on the rules of the exchange, and, in particular, on the trading frequency. We show that increasing the trading frequency has a dual effect on the market. On the one hand, if the market does not degenerate in the high frequency regime, then, it becomes more efficient. On the other hand, the market does not degenerate in the high frequency regime “if and only” the agents are market-neutral. At the same time, the violation of market-neutrality does not always
cause degeneracy for a finite trading frequency. Thus, not surprisingly, a decision on trading frequency means a trade-off between market efficiency and stability.

The present work raises many questions for further research. First, to justify the proposed framework based on an infinite number of agents, one needs to show that it can be viewed as a limit of finite-player games. Second, a more general existence result is also needed. This result should also be strengthened to allow for the additional constraint of “endogenous $\mu$” (cf. Subsection 5.1), making this framework consistent with the standard mean field game theory. In order to obtain the latter result, one needs to model more carefully the market orders of the agents: i.e. admit that they do affect the external demand) and restrict the equilibria to the ones with continuous LOB only. Instead of imposing the latter constraint, one can modify the proposed models by taking into account the time priority of limit orders.

It is also interesting to include risk aversion of the agents, or limit their admissible inventory levels, in order to observe a predatory behavior in the equilibrium. Notice that, in the present setting, the agents with zero position never benefit from posting an order, in any equilibrium. Thus, the agents serve as optimal executors (or, brokers) themselves – any violation of this makes the equilibrium impossible. This phenomenon may, arguably, be what the actual state of the market is currently approaching: as the high-frequency trading becomes more competitive, an ever larger part of their business comes from executing the orders of external investors. However, if this effect is undesirable, one has to either introduce a risk aversion of the agents or a bound on their maximum and minimum inventory levels – this will make the predatory behavior possible in equilibrium. It is interesting to see how much of the results of Section 4 remain true in this modified setting.

Another possible extension of this research is the analysis of continuous time models. Notice that, in the present paper, we chose not to start with the continuous time setting for an obvious reason: we wanted to analyze the effects of trading frequency. However, another reason is that it is not a priori obvious how to
formulate a continuous time analogue of the models proposed herein. For example, it is easy to see that, a continuous time limit of (1) is simply not well defined for an arbitrary control and LOB. However, the structure of optimal controls in discrete time may allow one to restrict the space of admissible controls, without decreasing the maximum objective function, so that, for the new set of controls, the continuous time limit is well defined. In addition, Theorem 1 provides an a priori restriction on the possible meaningful continuous time models.

Finally, it is important to describe a specific class of models from the proposed family (or using the extensions outlined above), such that it can be calibrated to the market data and used for predicting future deviations of the LOB from its “normal behavior”.

8 Appendix A

Proof of Lemma 2.

The first lemma shows that the normalized price increments are close to Gaussian in the sense of the conditional $L^2$ norm.

Lemma 13. Let Assumptions 2, 3, 4 and 7 hold. Then, there exists a deterministic function $\epsilon(\cdot) \geq 0$, such that $\epsilon(\Delta t) \to 0$, as $\Delta t \to 0$, and, $\mathbb{P}$-a.s., for all $\alpha \in \mathcal{A}$ and all $n = 1, \ldots, T/\Delta t$, we have:

$$
\mathbb{E}_{\mathbb{P}}\left[ (\frac{\tilde{\xi}_n}{\sqrt{\Delta t}} - \frac{\sigma_{t_{n-1}}(W_{t_n}^{\alpha} - W_{t_{n-1}}^{\alpha})}{\sqrt{\Delta t}})^2 \right] \leq \epsilon(\Delta t)
$$

Proof:

Notice that

$$
\frac{\tilde{\xi}_n}{\sqrt{\Delta t}} - \frac{\sigma_{t_{n-1}}(W_{t_n}^{\alpha} - W_{t_{n-1}}^{\alpha})}{\sqrt{\Delta t}} = \frac{1}{\sqrt{\Delta t}} \int_{t_{n-1}}^{t_n} \mu_s^\alpha \, ds + \frac{1}{\sqrt{\Delta t}} \int_{t_{n-1}}^{t_n} (\sigma_s - \sigma_{t_{n-1}}) \, dW_s^{\alpha}
$$

Figure 3: Time zero bid-ask spread (left) and the maximum value of drift that admits a non-degenerate equilibrium (right), as functions of trading frequency (measured in the number of steps).
and denote
\[ a_n = \frac{1}{\sqrt{\Delta t}} \int_{t_{n-1}}^{t_n} \mu^\alpha_s ds, \quad b_n = \frac{1}{\sqrt{\Delta t}} \int_{t_{n-1}}^{t_n} (\sigma_s - \sigma_{t_{n-1}}) dW^\alpha_s, \]

Then, using Assumption 7 and Itô’s isometry, we obtain: \( \mathbb{E}_{t_{n-1}}[(a_n + b_n)^2] \leq \epsilon(\Delta t) \to 0, \) as \( \Delta t \to 0. \)

The next lemma connects the proximity in terms of \( L^2 \) norm and the proximity of expectations of certain functions of the random variables. This result would follow trivially from the classical theory, but, in the present case, we require additional uniformity – hence, a separate lemma is needed (whose proof is nevertheless quite simple).

**Lemma 14.** For any constant \( C > 1 \), there exists a deterministic function \( \gamma(\cdot) \geq 0, \) such that \( \gamma(\epsilon) \to 0, \) as \( \epsilon \to 0, \) and, for any constant \( \epsilon > 0 \) and any random variables \( \xi \) and \( \eta, \) defined on some probability space and satisfying the two conditions: (a) \( \mathbb{E}(\xi - \eta)^2 \leq \epsilon, \) and (b) \( \eta \sim \mathcal{N}(0, \sigma^2), \) with an arbitrary \( \sigma \in [\frac{1}{\epsilon}, C]. \) Then, the following holds for all \( p \in \mathbb{R}: \)

\[
\begin{align*}
(i) & \quad |\mathbb{P}[\xi > p] - \mathbb{P}[\eta > p]| \leq \gamma(\epsilon) \\
(ii) & \quad |\mathbb{E}[\xi 1_{(\xi > p)}] - \mathbb{E}[\eta 1_{(\eta > p)}]| \leq \gamma(\epsilon) \\
(iii) & \quad |\mathbb{E}[p 1_{(\xi > p)}] - \mathbb{E}[p 1_{(\eta > p)}]| \leq \gamma(\epsilon)
\end{align*}
\]

**Proof:**

(i) Note that \( |\mathbb{P}[\xi > p] - \mathbb{P}[\eta > p]| \leq \mathbb{P}[\xi > p, \eta \leq p] + \mathbb{P}[\xi \leq p, \eta > p], \) and, hence, it suffices to estimate each of these terms:

\[
\mathbb{P}[\xi > p, \eta \leq p] \leq \mathbb{P}[p \geq \eta \geq p - \sqrt{\epsilon}] + \mathbb{P}[|\xi - \eta| > \sqrt{\epsilon}] \leq M \sqrt{\epsilon} + \frac{\mathbb{E}(|\xi - \eta|^2)}{(\sqrt{\epsilon})^2} \leq (M + 1) \sqrt{\epsilon},
\]

In the second to last inequality above, we used the fact that \( \eta \) has a density bounded by a fixed constant \( M, \) which, in turn, follows from the condition (b). We can similarly show that \( \mathbb{P}[\xi \leq p, \eta > p] \leq (M + 1) \sqrt{\epsilon}. \)

(ii) Note that

\[
|\mathbb{E}[\xi 1_{(\xi > p)}] - \mathbb{E}[\eta 1_{(\eta > p)}]| \leq |\mathbb{E}[(\xi - \eta) 1_{(\xi > p)}]| + |\mathbb{E}[\eta(1_{(\xi > p)} - 1_{(\eta > p)})]|,
\]

and, hence, it suffices to bound each of the above two terms. The first one is bounded by \( \epsilon, \) and, for the second one, we have:

\[
|\mathbb{E}[\eta(1_{(\xi > p)} - 1_{(\eta > p)})]| \leq \mathbb{E}[|\eta| 1_{(\xi > p)}] 1_{(\eta > p)} - 1_{(\eta > p)}] \leq \|\eta\|_2 \|1_{(\xi > p)} - 1_{(\eta > p)}\|_2 \\
\leq \|\eta\|_2 (\mathbb{P}[\xi > p, \eta \leq p] + \mathbb{P}[\xi \leq p, \eta > p]) \leq \|\eta\|_2 2(M + 1) \sqrt{\epsilon}
\]

(iii) Follows easily from parts (i) and (ii), using the fact that the tails of \( \eta \) are can be estimated uniformly over all \( \sigma \in [1/C, C]. \)

Taking \( \epsilon(\Delta t) = \gamma(\epsilon(\Delta t)) \) and applying the above lemmas, we get the statement of Lemma 2 for \((W^\alpha_{t_n} - W^\alpha_{t_{n-1}})/\sqrt{\Delta t} \) in place of \( \eta_0. \) Finally, we note that the laws of the two random variables coincide under \( \mathbb{P}_{t_{n-1}}, \) and the statement depends only on these laws. The last statement of Lemma 2 follows from the fact that Lemma 14 is stable under analogous substitution.
9 Appendix B

Proof of Lemma 10.

First, let us replace $\xi_{n+1}$, in the objective functions appearing in Assumption 13, by $\xi$, such that, under every $P_n$, we have: $\xi \sim N(0, \sigma^2)$, with the same constant constant $\hat{\sigma} > 0$ for all $n$. Then, we can satisfy the last two statements of Assumption 13 by choosing $\tilde{p}^b = -\tilde{p}^a$ and finding $\tilde{p}^a$ which solves

$$\tilde{p}^a = \arg\max_{p \in \mathbb{R}} E_{N-1} \left[ (p + \tilde{p}^a - \xi) 1_{\{\xi > p\}} \right]$$

(35)

The solution to the above equation is well defined, as the right hand side is a continuous and decreasing function of $\tilde{p}^a$, converging to $\infty$ at zero, and taking negative values for large enough $\tilde{p}^a > 0$. The second statement of Assumption 13 is satisfied because the above maximum is strictly positive. The first statement of Assumption 13 is satisfied because the above expectation is bounded from above by $\tilde{p}^a$. Let us now see what happens to the objective functions and their maximizers when $\hat{\sigma}$ changes. Multiplying $\xi$ by $v > 0$, we consider the maximum point of the objective with $v\xi$ in place of $\xi$: namely, we define $\tilde{p}^{b,v} = -\tilde{p}^{a,v}$ and find $\tilde{p}^{a,v}$ that solves

$$\tilde{p}^{a,v} = \arg\max_{p \in \mathbb{R}} E_n \left[ (p + \tilde{p}^a - v\xi) 1_{\{v\xi > p\}} \right]$$

(36)

Clearly, the multiplication by $v$ simply changes the standard deviation of the random normal from $\hat{\sigma}$ to $v\hat{\sigma}$. It is easy to see that $\tilde{p}^{a,v}$ is strictly increasing in $v > 0$ and $\tilde{p}^{a,1} = \tilde{p}^a$. Due to the continuity of the above expression with respect to $v$, we conclude that there exist constants $\varepsilon, \bar{\varepsilon} > 0$, such that

$$\mathbb{E}_{N-1} \left[ (2\tilde{p}^a - v\xi) 1_{\{v\xi > \tilde{p}^a\}} \right] \geq \bar{\varepsilon},$$

(37)

for all $v \in (1, 1 + \varepsilon)$. Notice also that we can choose $\varepsilon$ to be the same for all $\hat{\sigma} > 0$ ($\bar{\varepsilon}$ would depend on $\varepsilon$ and $\hat{\sigma}$, of course), because $\tilde{p}^a$ scales proportionally when $\hat{\sigma}$ is multiplied by a constant.

Finally, we prove the statements of Assumption 13. Notice that, due to scaling, it suffices to show that the statements of the assumption hold with $\xi_n$ replaced by $\xi_n/\sqrt{\Delta t}$, for all $n = 1, \ldots, N$. Next, let us set $\tilde{p}^b = -\tilde{p}^a$ and show that there exists a constant $\tilde{p}^a$ for which the statements of Assumption 13 hold. Lemma 2 implies that, as $\Delta t \to 0$, the objective functions appearing in Assumption 13 (after the aforementioned scaling) converge to (36) and (37), with $v\hat{\sigma}$ replaced by $\sigma_n\Delta t$, uniformly over all values of the argument and over all random outcomes. In addition, for $n = N - 1$, we have: $\sigma_n\Delta t \in [\hat{\sigma}, c\hat{\sigma}]$, with some constant $c > 1$ which we are free to choose. Let us choose $c \in (1, 1 + \varepsilon/2)$ and $\tilde{\sigma} = \tilde{\sigma}(1 + \varepsilon/2)/(1 + \varepsilon)$, so that

$$[\tilde{\sigma}, c\tilde{\sigma}] \subset (\hat{\sigma}, \hat{\sigma}(1 + \varepsilon)) = \{v\tilde{\sigma} \mid v \in (1, 1 + \varepsilon)\},$$

and construct $\tilde{p}^a$ as a solution to (35). With this choice, the aforementioned convergence implies that, for all small enough $\Delta t > 0$, we have, $p$-a.s.:

$$\mathbb{E}_{N-1} \left[ (2\tilde{p}^a - \xi_n/\sqrt{\Delta t}) 1_{\{\xi_n/\sqrt{\Delta t} > \tilde{p}^a\}} \right] \geq \bar{\varepsilon}/2.$$

As the above expectation is positive, it is clear that $\tilde{p}^a \geq \bar{\varepsilon}/2$ In addition, $\sigma_n\Delta t \geq \tilde{\sigma}$ implies

$$\arg\max_{p \in \mathbb{R}} \mathbb{E}_n \left[ (p + \tilde{p}^a - \xi_{n+1}/\sqrt{\Delta t}) 1_{\{\xi_{n+1}/\sqrt{\Delta t} > p\}} \right] \subset \{\tilde{p}^{a,v} \mid v > 1\} \geq \tilde{p}^a$$

Finally, we notice that $\tilde{p}^a \leq \tilde{p}^{a,1+\varepsilon} = \bar{\varepsilon}$. Clearly, analogous inequalities hold for $\tilde{p}^b = -\tilde{p}^a$. It only remains to notice that, after rescaling everything back by $\sqrt{\Delta t}$, we obtain the desired bounds on $\bar{\varepsilon}$ and $\tilde{\varepsilon}$, over all small enough $\Delta t$. 

}
Appendix C

Proof of Lemma 11.

Repeating the proof of Lemma 10, it is easy to see that Assumption 17 holds if we replace every $\xi_n$ with a zero-mean random normal (independent of $F_{n-1}$). Due to symmetry, the desired $\tilde{p}^b = -\tilde{p}^a$ can be easily computed as a solution to

$$\mathbb{E}_{N-1} \left[ (2\tilde{p}^a - \xi) 1_{\{\xi > \tilde{p}^a\}} \right] = 0$$

It is easy to see that there exists a unique root of the above equation, and it is strictly increasing in the variance of $\xi$. If $\sigma$ is deterministic and non-increasing on the entire interval $[0,T]$, the deterministic $\tilde{p}^b = -\tilde{p}^a$ can be constructed as above, using $\xi$ with the variance $\Delta t \sigma^2_{(N-1)\Delta t}$. Such $\tilde{p}^b = -\tilde{p}^a$ satisfy Assumption 17. If $\sigma$ is only deterministic on $[T-\tilde{\delta},T]$, then, we construct $\tilde{p}^b = -\tilde{p}^a$ in the same way and notice that they satisfy Assumption 17 for $n$ such that $n\Delta t \geq T - \tilde{\delta}$. For the other values of $n$, the assumptions of the lemma ensure that the possible values of $\sigma_{n\Delta t}$ are larger than $\sigma_{(N-1)\Delta t}$ plus a positive constant. Hence, the limiting argument presented in the proof of Lemma 10 implies that the roots of

$$\mathbb{E}_n \left[ (p + \tilde{p}^a_{n+1} - \xi_{n+1}) 1_{\{\xi_{n+1} > p\}} \right] = 0$$

are larger than $\tilde{p}^a$. The desired property of $\tilde{p}^b$ is verified analogously.

References


