

DIFFERENTIATING SKILL AND LUCK

SCOTT E. PAGE AND NATHAN SEEGER

Abstract: *In this paper, we construct a model in which two players compete for a payoff. The outcome of the contest depends on the differences in the skills of the agents as well as on luck. The skill of each player depends on her exogenous ability and her effort. We first show how equilibrium effort depends on the amount of luck in the game as well as the payoff and the difference in exogenous ability. We then derive levels of luck that maximize total effort. We then demonstrate show how the ability-skill distortion, the change between realized skill differences and innate ability differences depends on payoffs and levels of luck. that We then extend this game to allow the players to increase or decrease the amount of luck and find that the advantaged player always has an incentive to offset any additional luck created by the weaker player. Finally, we consider multi-stage games, in which the abilities of the player in later rounds depend on the outcomes of earlier rounds.*

Internal competition have become an important tool for many organizations attempting to increase the performance or effort of their members. For example, companies like 3M, IBM, Procter and Gamble, and GM increasingly use competition to reward quality and innovation (Marino and Zabojnuk, 2003; Eisenhardt and Gahmic, 2000). In addition, Barack Obama’s education policy of “race to the top” increasingly rewards education funding in a competition structure. In any of these competitions it is important to understand how the combination of skill and luck affect the incentives of the players. This paper suggests that these competitions are able to increase performance if they are structured correctly however if the game is set up incorrectly these competitions may do more harm than good. If the prize or luck in the game is set incorrectly the game may cause players to decrease their effort, play extreme strategies, or randomly quit.

Untangling skill and luck is also important for individuals to make better decisions, such as which mutual fund to invest in or which CEO to hire. If investors overestimate the role of skill in determining mutual fund returns, they would base their investment decision solely on past performance, investing in funds that recently outperformed the market but whose returns would decrease quickly as the fund reverts to the mean. Similarly, a board of directors would like to hire the CEO with the most ability but they only observe past performance. Correctly identifying the amount of skill and luck that went into past performance informs the board of directors how much weight they should place on past performance of a job candidate. If CEO past performance is similar in nature to running a 200 meter dash (i.e., mostly skill), then observing one candidate outperform the other for five consecutive years provides a lot of information about the candidates

relative skill. If instead CEO performance is similar in nature to at bats in major league baseball (i.e. mostly luck) even a hundred observations may not be very informative.¹

The innovation of this paper is to unpack skill and luck into its exogenous and endogenous fundamentals. The model in the paper accounts for how aspects of the competition, such as the prize level, amount of luck in the game, and the ability difference between competitors, affect the endogenous choices of the agents. The results of the model demonstrate that there are optimal prize and luck levels for maximizing effort of the players and minimizing bias.

Three important effects that can occur in competitions are characterized by the model. The first is the “superstar effect” which demonstrates that ability differences between competitors can cause adverse effects and can decrease effort and performance. While adverse effects caused by ability differences are present in the literature (Brown, 2008) the innovation of this paper is to demonstrate that the ability differences must be large to cause the adverse effects. Adverse effects only occur if the best competitor is truly a “superstar”.

The “calling it in” effect demonstrates that although increasing the prize increases the incentives for players to put in more effort, this effect is finite. The prize eventually becomes so large that the optimal strategy for the disadvantaged player is to randomly choose between putting in effort and putting quitting, thus “calling it in”. As the prize level increases, the effort of the competitors increase until abruptly total effort discontinuously decreases. The prize level that maximizes total effort is the level on the “edge of chaos” where if the prize increases slightly more total effort would discontinuously decrease.

Finally the “extreme-strategies” effect demonstrates that when individuals are able to choose different strategies in situations with different levels of luck, there exists a range of ability differences that cause competitors to choose extreme strategies. When differences in ability are outside of this range the competitors choose moderate strategies and compete in effort. However, when the ability difference is in this range the competitors choose extreme strategies, which is wasteful and cause all utility levels to decrease.

The characterization of these effects and the other results in this paper extend a large literature on tournaments. Lazear and Rosen (1981) wrote the seminal paper on incentives in tournaments, which lead to many extensions, most notably Green and Stokey (1983), Nalebuff and Stiglitz (1983), Dixit (1987), and Moldovanu and Sela (2006). Moldovanu and Sela (2001) is the most recent to focus on the optimal prize, building on a long literature that encompasses complete-information contests with multiple prizes

¹Jim Albert, a professor of math and statistics, estimates that luck accounts for 80 percent of the resulting batting average for 100 at bats.

(Broecker, 1990; Wilson, 1979; Anton and Yao, 1992; Clark and Riis, 1998), best entry contests (Wright, 1983; Taylor, 1995; Fullerton and McAfee, 1999), incomplete information games (Weber, 1985; Hillman and Riley 1989; Krishna and Morgan, 1997), and endogenous number of prizes (Glazer and Hassin, 1988; Barut and Kovenock, 1998).

There is also a large literature focused on disentangling skill and luck in finance. For U.S. mutual funds there is little evidence of superior performance (Lakonishok et al., 1992; Grinblatt et al., 1995; Daniel et al., 1997; Carhart, 1997; Chevalier and Ellison, 1999; Wermers, 2000; Baks et al., 2001; Pastor and Stambaugh, 2002). Similar results have been found for U.K. data. These studies estimate skill and luck under the assumption that both are exogenous factors. This model extends these studies, demonstrating the bias that exists when endogenous effort and strategies are not accounted for.

The following section sets up the model that is used throughout the paper. The model is applied to a moral hazard context in section 2, a screening context 3, and a dynamic context 4.

1. SETUP

We consider a game in which two players compete in a winner take all game with a payoff of v . The winner of the game will be determined by the differences in the skills of the two players as well as a luck component. The *skill* for player i can be written as a function of effort and ability: $s_i = s(e_i, \alpha_i)$, which is increasing and concave in e_i and increasing in α_i . Throughout, we assume $\alpha_1 \geq \alpha_2$, and we will refer to the first player as the advantaged player and the second player as the disadvantaged player. The amount of luck involved in the game, L , comes from two sources: exogenous luck, ℓ_0 , and the endogenous contributions to luck ℓ_1 and ℓ_2 from the two players.

We let $S \in R^+$ denote the difference in skill of the two players, $A \in R^+$ denote the difference in ability of the two players, and we let L denote the total amount of luck. The probability that the advantaged player wins is given by a function $P(S, L) \in [0, 1]$. We make the following assumptions about the function P .

A1: $P(0, L) = \frac{1}{2}$ for all L

A2: $P(S, 0) = 1$ for all $S > 0$

A3: $\frac{\partial P}{\partial S} > 0$ and $\frac{\partial^2 P}{\partial S^2} < 0$ if $S > 0$

A4: $\frac{\partial P}{\partial L} < 0$ if $S > 0$

A5 $\frac{\partial P}{\partial S}$ is a singled peaked function in L

Given our assumptions, player 1 would like to decrease the amount of luck and player 2 would like to increase the amount of luck in the game. The cost of effort for player i equals $c_i e_i^2$ and the cost of increasing (decreasing) luck by an amount ℓ_i equals $k\ell_i^2$. Given these assumptions, the payoff to player 1 can be written as

$$\pi_1 = vP(s_1(e_1, \alpha_1) - s_2(e_2, \alpha_2), \ell - \ell_1 + \ell_2) - c_1 e_1^2 - k\ell_1^2$$

Similarly, the payoff to player 2 can be written as

$$\pi_2 = v[1 - P(s_1(e_1, \alpha_1) - s_2(e_2, \alpha_2), \ell - \ell_1 + \ell_2)] - c_2 e_2^2 - k\ell_2^2$$

In equilibrium, players must each choose effort and endogenous luck. The FONC for player i are given by

$$v \frac{\partial P}{\partial S} s'_i - 2c_i e_i = 0 \qquad e_i = \frac{v P_S s'_i}{2c_i}$$

(FONC)

$$-v \frac{\partial P}{\partial L} - 2k\ell_i = 0 \qquad \ell_i = \frac{v P_L}{2k}$$

2. MORAL HAZARD GAME

A manager has two employees whose efforts are unobservable. The manager's objective is to create incentives for the employees to induce the highest level of total effort. The manager must consider three important factors: the amount of luck in the game, the payoff structure, and the difference in abilities of the two employees.

2.1. Endogenous Luck. First, this section considers the endogenous luck chosen by each player simultaneously in the game. Second, this section considers how luck in the game would be chosen by the manager, the advantaged, and the disadvantaged player if they could choose the level before the game started and players could not change the level during the game.

Claim 1. *Both players choose identical levels of endogenous luck in equilibrium.*

pf. The FONC are identical for the two players. These are maxima given our assumptions on the second derivative.

When luck is endogenous, total effort is maximized with the same prize level and level of luck as when luck is exogenous because the resulting game has the same level of luck even though the agents choose costly strategies. When agents are able to choose their strategy, their utility levels are decreasing in the absolute magnitudes of their strategies. If the probability function is given by $P(S, L) = \int_{-\infty}^{\infty} G_2(x_1)g_1(x_1)dx_1$ where $g(x)$ is a uniform distribution on the interval $[S(e_2, \alpha_2) - L, S(e_1, \alpha_1) + L]$ the following proposition holds.

Proposition 1. *Extreme Strategies: There exists a difference in ability that causes the equilibrium strategies for both players to go to their extremities, $A = A^{\text{extreme-strategy}}$*

pf. In the appendix this model is solved explicitly producing a closed form solution for ℓ^* . The ability difference that causes the agents to choose the most extreme strategies, given in equation 1, is increasing with the cost of effort and exogenous luck but decreasing with the prize level.

$$(1) \quad A^{\text{extreme}} = \sqrt{\frac{8cl_0^2 + v}{v}} - 1$$

If the difference in ability is either small or large, both agents choose moderate strategies. However, if the difference in abilities is somewhere in the middle both agents attempt extreme strategies. Extreme strategies decreases the utility of both players and should be avoided if possible. For example, basketball teams that trail in the second half attempt to speed the game up, allowing for more but lower quality shots by both teams, thus increasing the role luck plays in the game. In response, teams with the lead try to slow the game down and take as much time off the clock as possible. The extreme case of this is the four corners offense created by John McClendon and popularized by North Carolina coach Dean Smith. In the 1982 ACC championship game North Carolina won by holding the ball for almost twelve minutes at the end of the

TABLE 1. Optimal Level of Luck

		Total Effort	Advantaged Player	Disadvantaged Player	Conditions
L^*	Without Cost:	Positive and Finite	0	∞	A big
		Positive and Finite	∞	∞	A small
	With Cost:	Smallest	Middle > 0	Largest	A small
		Middle	Smallest = 0	Largest	A big

game. The four corners offense, and specifically the 1982 ACC championship game, led to the adoption of the shot clock in basketball. The shot clock disallows teams from holding the ball without making a shot.

The manager's objective of maximizing the total level of effort exerted by both agents may be in conflict with the objectives of the agents themselves. Table 1 depicts the level of luck the manager and each of the agents would choose. Comparisons across columns demonstrate the conflicting incentives. The manager and disadvantaged player prefer to set large levels of luck when there is no cost of doing so. In contrast, the advantaged player prefers zero luck when the difference in ability is large and infinite luck when the difference in ability is small. When there is a cost of increasing the luck the disadvantaged player's preferred level of luck is the largest. When the ability difference is small the advantaged player's preferred level of luck is nonzero.

The advantaged player's probability of winning is strictly decreasing in the amount of luck in the game. However, both players put in a lot of costly effort when the difference in ability is small. By increasing the level of luck in the game the benefit of added effort decreases. The advantaged player can be better off if the decrease in the probability of winning is more than offset by the decrease in the cost of added effort. Therefore, if the difference in abilities is small and there is no cost to increasing luck, both players will have an incentive to choose the winner by flipping a coin rather than exerting costly effort in the game.

The following sections drop the endogenous luck element of the game because under mild assumptions the level of luck in the game reverts to the exogenous luck in the game.

2.2. Linear Effort. Consider a case in which skill is the sum of ability and effort and the cost function is homogeneous across players.

Claim 2. *If $s_i(e_i, \alpha_i) = \alpha_i + e_i$, and $c_1 = c_2$ then both players choose identical levels of effort in equilibrium*

pf. This follows directly from the first order conditions above.

The equilibrium effort levels are given in equations LM when the probability function is given by $P(S, L) = \int_{-\infty}^{\infty} G_2(x_1)g_1(x_1)dx_1$ where $g(x)$ is a uniform distribution on the interval $[S(e_2, \alpha_2) - L, S(e_1, \alpha_1) + L]$.

$$e_1^{\text{Linear}} = \frac{c_2(2L - \alpha_1 + \alpha_2)v}{(c_2 - c_1)v + 8c_1c_2L^2}$$

(LM)

$$e_2^{\text{Linear}} = \frac{c_1(2L - \alpha_1 + \alpha_2)v}{(c_2 - c_1)v + 8c_1c_2L^2}$$

Three main observations become apparent when the above expression is examined. The first observation is that when $c_1 = c_2$ we get the common result that $e_1^* = e_2^*$ (Brown, 2008). This result occurs in any game where the marginal benefits are equal across players and the cost functions are homogenous. The marginal benefits are equal in this game because any additional benefit by one player is a marginal loss for the other and because efforts enter the function linearly. We demonstrate above that this result is not robust to heterogeneous costs nor is it robust when effort does not enter linearly, as shown below. The second observation generated from the above expression is that as the difference in abilities increases the effort of both players decreases. This is the adverse effect described by Brown (2008).

The third observation is that the difference in skill does not represent the difference in ability when cost functions are heterogeneous. Naturally, if the cost of effort is lower for the advantaged player, the advantaged player will exert more effort and the resulting level of skill will be larger than the difference in abilities. Similarly, if the cost of effort is lower for the disadvantaged player he will exert more effort and catch-up to the advantaged player, which results in a skill difference that is smaller than the difference in abilities.

2.3. Optimal Prize and Luck. We next consider the case in which the marginal contribution to skill from effort is greater for the advantaged player. This includes the cases where $s_i(e_i, \alpha_i) = e_i * \alpha_i$ and $s_i(e_i, \alpha_i) = e_i + e_i * \alpha_i$

Claim 3. *Assume $\frac{\partial s_i}{\alpha_i} > 0$ and $c_1 = c_2$, then the advantaged player puts forth more effort*

pf. Let $c_1 = c_2 = c$. The FONC can then be written $v \frac{\partial P}{\partial S} s'_1 = 2ce_1$ and $v \frac{\partial P}{\partial S} s'_2 = 2ce_2$. By assumption $s'_1 > s'_2$, therefore $e_1 > e_2$.

Claim 4. *Total effort is single peaked in L*

pf. Totally differentiate the FONC $v \frac{\partial P}{\partial S} s'_i - 2c_i e_i = 0$ where \pm is positive for the advantaged player and negative for the disadvantaged player.

$$v s'_i P_{SL} dL + \underbrace{[v P_S s''_i \pm v (s'_i)^2 P_{SS} - 2c]}_{\text{Negative by SOC}} de_i = 0$$

$$(2) \quad \frac{de_i}{dL} = \frac{P_{SL}}{-[v P_S s''_i \pm v (s'_i)^2 P_{SS} - 2c]} v s'_i$$

This implies that effort for player i is single peaked in luck, increasing when the cross partial is positive, small L , and decreasing when the cross partial is negative, large L . Equation 2 implies that the level of luck that maximizes effort is the same for both players. Therefore total effort is maximized at the same level of luck and total effort is single peaked.

Proposition 2. *There exists a level of luck that maximizes total skill when competing for a fixed prize.*

pf. Total skill and effort are maximized by the same level of luck because skill is a nondecreasing function in effort and effort for both players is maximized at the same level of luck.

Proposition 2 is informative in examples where there exists a fixed prize, such as elections or scholarships, where the manager is only able to change the luck in the game. The optimal level of luck when the probability function is given by $P(S, L) = \int_{-\infty}^{\infty} G_2(x_1) g_1(x_1) dx_1$ where $g(x)$ is a uniform distribution on the interval $[S(e_2, \alpha_2) - L, S(e_1, \alpha_1) + L]$ is given in equation (3) and demonstrates the more general result that the optimal level of luck in a game increases with the prize level and the difference in abilities, where α_2 has been normalized to 1.

$$(3) \quad L^* = \sqrt{\frac{(\alpha_1^2 - 1)v}{8c}}$$

Claim 5. *Effort is increasing in the prize level v .*

pf. Totally differentiate the FONC $v \frac{\partial P}{\partial S} s'_i - 2c_i e_i = 0$ where \pm is positive for the advantaged player and negative for the disadvantaged player.

$$s'_i P_S dv + \underbrace{[v P_S s''_i \pm v (s'_i)^2 P_{SS} - 2c]}_{\text{Negative by SOC}} de_i = 0$$

$$(4) \quad \frac{de_i}{dv} = \frac{s'_i P_S}{-[v P_S s''_i \pm v (s'_i)^2 P_{SS} - 2c]} > 0$$

For an interior solution equation 4 is positive by A3.

Claim 6. *The second order condition for the disadvantaged player may fail when the prize is large.*

pf. Rearrange the second order condition $[v P_S s''_i \pm v (s'_i)^2 P_{SS} - 2c]$.

$$(5) \quad v [\underbrace{P_S s''_i}_{(+)} - \underbrace{(s'_i)^2 P_{SS}}_{(-)}] > 2c$$

When the second term is larger in magnitude than the first term, for example when $s''_i = 0$, the coefficient for the prize level is positive implying that for large levels of v the second order condition fails. When the inequality in (5) holds, the second order condition fails, implying that there is not an interior solution. A pure-strategy equilibrium does not exist when the second order condition fails. However, there does exist a mixed-strategy equilibrium where the disadvantaged player switches between putting in and not putting in effort. Total effort with respect to the prize level drops discontinuously at the point where the second order condition fails.²

²Note that the second order condition always holds for the advantaged player.

Proposition 3. *“Calling It In:” For a fixed level of luck there exists an optimal prize level at the point where the disadvantaged player’s second order condition fails.*

Proposition 3 implies managers able to adjust the prize level but not the amount of luck can increase total effort by increasing the prize. However, this increase is not indefinite; when the prize gets large enough the disadvantaged player will begin to “call it in”, sometimes putting in effort and sometimes not. Equation (6) is the optimal prize level when the probability function is given by $P(S, L) = \int_{-\infty}^{\infty} G_2(x_1)g_1(x_1)dx_1$ where $g(x)$ is a uniform distribution on the interval $[S(e_2, \alpha_2) - L, S(e_1, \alpha_1) + L]$ and demonstrates the more general result that the optimal prize level is increasing in the amount of luck in the game.

$$(6) \quad v = 8cL^2$$

In the case where the manager is able to change both the prize and luck levels in the game the optimal levels approach infinity. However, if there is a cost to the manager of increasing the prize or luck levels, the levels need not approach infinity.

2.4. Adverse and Superstar Effects. Choosing pairs of agents to compete in the game is as important as choosing the prize level or level of luck in the game.³

Claim 7. *The disadvantaged player’s effort is decreasing in the difference in abilities, A .*

pf. This follows from the total derivative of the equilibrium condition FONC given in equation 7.

$$(7) \quad \underbrace{[2c + P_{SS}v(s'_2)^2]}_{(+)\text{ Negative SOC}} de_2^* = \left[\underbrace{vP_{SS}S_A s'_2}_{(-)\text{ Strategic Effect}} \right] dA$$

Claim 8. *The advantaged player’s effort is single peaked in the difference in abilities.*

pf. Total differentiate the equilibrium condition FONC.

³Some boxing story where choosing the match up matters

$$(8) \quad [2c - P_{SS}v(s'_1)^2]de_1^* = [\underbrace{vP_S(s'_1)_A}_{(+)} \text{ Direct Effect} + \underbrace{vP_{SS}S_A s'_1}_{(-)} \text{ Strategic Effect}] dA$$

The right hand side of equation (8) is a linear combination of the direct and strategic effects that occur when differences in ability change. The direct effect creates an incentive for the advantaged player to increase effort because effort and ability are complements. The strategic effect is negative because as the difference in skills increases the additional benefit to increasing the difference in skills decreases. When there are small differences in ability, the effort exerted by the advantaged player increases because the direct effect outweighs the strategic effect. However, for large differences in ability the strategic effect outweighs the direct effect and the advantaged player's effort decreases. When skill is linear in effort and ability the direct effect is zero which implies that there is only a strategic effect and the advantaged player's effort is decreasing in her ability.

Proposition 4. *“Superstar” Effect: The adverse effect of differences in ability on total effort occurs only when the advantaged player is a true superstar, when differences in ability are large, $A > A^{Superstar}$.*

pf. This follows directly from claims 7 8.

The critical difference in ability where total effort starts to decrease is found by differentiating total effort and solving for α_1 . Equation (9) provides the superstar ability difference when the probability function is given by $P(S, L) = \int_{-\infty}^{\infty} G_2(x_1)g_1(x_1)dx_1$ where $g(x)$ is a uniform distribution on the interval $[S(e_2, \alpha_2) - L, S(e_1, \alpha_1) + L]$.

$$(9) \quad A^{Superstar} = \sqrt{\frac{8cL^2}{v}} - 1$$

Equation (9) demonstrates the superstar ability difference increases with luck and the cost of effort and decreases with the prize level. This implies that a manager with two employees with large differences in ability can mitigate the adverse effect by increasing the luck in the game or decreasing the prize level.

3. SCREENING GAME

In many contexts it is important to be able to separate the effects of ability and luck. Investors picking mutual funds only observe the returns of a fund, and from this need to infer the ability of the fund's manager to make good investment decisions. The same is true for a board of directors choosing a CEO, inferring ability from past performance.⁴ The observed performance in both of these cases is a noisy and biased estimate of the underlying ability. The level of luck in the game makes the observed performance noise. The incentives of the game produce equilibrium levels of effort that can bias the estimate of ability.

To account for the bias, the difference in skill resulting from our model is compared with the initial difference in abilities, our null model. Figure 1 plots the difference in abilities and skill with respect to the ability of the advantaged player. As the ability of the advantaged player increases, the difference in abilities increases with a slope of 1. In comparison, the difference in skill initially increases faster than the difference in ability. However, the increase in skill increases at a decreasing rate such that at some point the difference in ability is larger than the difference in skill. The skill-ability bias is defined as the difference between the difference in abilities and the difference in skills and is demonstrated graphically in Figure 1 and algebraically in equation 10. The absolute bias is the absolute value of the skill-ability bias defined in equation 11.

$$(10) \quad \text{Skill-Ability Bias} = (\text{Skill}_1 - \text{Skill}_2) - (\text{Ability}_1 - \text{Ability}_2)$$

$$(11) \quad \text{Absolute Bias} = |(\text{Skill}_1 - \text{Skill}_2) - (\text{Ability}_1 - \text{Ability}_2)|$$

Proposition 5. *The ability-skill bias overestimates ability differences for small ability differences and underestimates ability differences for large ability differences.*

This result demonstrates the ability-skill bias that stems from inferences of skill and luck without adjusting for endogenous effort. For example, consider two high-stakes games; the first among entrepreneurs with large differences in ability and the second among blackjack players with small differences in ability. The bias would

⁴In both of these examples ability, not skill, is the parameter of interest. Skill includes effort, which can change depending on the incentive structure but ability is constant. The goal of the game is to pick the person with the most ability. Once that person is chosen, an incentive structure can be set up to induce the most amount of effort from that person.

underestimate the difference in abilities for entrepreneurs and over estimate differences in ability for blackjack players.

The magnitude of the skill-ability bias depends on the prize structure. As the difference in prizes between first and second increases the game has higher stakes and becomes more winner-take-all. When the prize increases, the difference in skills increases which causes the skill-ability bias to increase and the absolute bias to increase or decrease. Table (2) and Figure 2 demonstrate how the bias changes as the stakes increase.

Claim 9. *Given an interior solution, increasing the stakes in a game increases the skill-ability bias.*

Increasing the stakes in a game always expands the difference in skills, but they expand more in situations where the difference in abilities is small. Therefore, as the stakes increase, the difference in skills between the blackjack players and the entrepreneurs shrinks. If differences in ability are inferred from differences in skill without taking into account endogenous effort we would infer, incorrectly, that the differences in abilities between blackjack players and between entrepreneurs were similar.

Claim 10. *The skill-ability bias is negative as the prize goes to zero and positive as the prize goes to infinity.*

The advantaged player's incentive to exert effort and separate from the disadvantaged player is small when the prize is small. However, when the prize is large the advantaged player will exert effort and separate from the disadvantaged player causing the skill-ability bias to be positive.

Proposition 6. *For a given small difference in ability there exists a prize v^* that minimizes the absolute bias. Thus for large levels of v increasing the stakes increases the absolute bias when differences in ability are small and decreases the absolute bias when differences in ability are large.*

When the stakes are small the disadvantaged player can catch up causing the difference in skills to be less than the difference in abilities. Raising the stakes increases the incentive for the advantaged player to pull away causing the difference in skills to expand. For small differences in ability, increasing the stakes can cause the difference in skills to become larger than the difference in abilities. This is not possible for large differences in abilities because the large difference dampens the response to the increased stakes.

Claim 11. *The skill-ability bias is negative as luck goes to zero and infinity.*

The advantaged player's incentive to separate from the disadvantaged player is small when there is a lot of luck in the game because the benefit of effort is small. When there is very little luck in the game the

TABLE 2. Change in Bias as Stakes Increases

	Small v_1	Large v_1
Small α_1	<i>Low Stakes Blackjack</i>	<i>High Stakes Blackjack</i>
	Skill-Ability Bias Changes Sign From Under to Over Estimation Ability	Larger Absolute Bias Increased Overestimation Ability
Large α_1	<i>Low Stakes Industry Entrepreneur</i>	<i>High Stakes Industry Entrepreneur</i>
	Smaller Absolute Bias Decreased Underestimation Ability	Smaller Absolute Bias Decreased Underestimation Ability

advantaged player wins with large probability and additional effort is unable to substantially increase the probability of winning.

Claim 12. *The skill-ability bias is single peaked with respect to luck.*

However, when luck is neither small nor large the game is competitive and the advantaged player exerts effort, resulting in a large difference in skill.

Proposition 7. *There exists an $L^* > 0$ that minimizes the absolute bias. Therefore, increasing luck in the game can decrease the absolute bias and does so when differences in ability are large and the level of luck in the game is small.*

This counter intuitive result implies exogenous shocks to luck in a game in some scenarios can be used to better estimate ability differences. For example, in tennis the large differences in ability and the small amount of luck in the game suppress effort by the disadvantaged player causing the skill-ability bias to be negative. If the tennis match was occurring outdoors and the wind increased, increasing the luck in the game, the disadvantaged player would have an incentive to exert more effort resulting in a decrease in the absolute bias. This result is demonstrated in Table 3 and Figure 3.

4. DYNAMIC GAMES

The previous model is a static one-shot game but it can easily be applied to dynamic settings with some simple reinterpretation. Consider the game of tennis and let the one-shot game be a single set.⁵ At the

⁵Tennis is scored using points that win games, games that win sets, and sets that win matches. Typically four points wins a game, six games win a set, and three sets win a match where the winner must win by two in the first two settings.

TABLE 3. Change in Bias as Luck Increases

	Small Luck	Large Luck
	<i>Olympic 200 Meter Dash</i>	<i>Blackjack</i>
Small α_1	Skill-Ability Bias Increases	Skill-Ability Bias Decreases
	Absolute Bias Increase	Absolute Bias Decreases Then Increases
	<i>Tennis</i>	<i>Entrepreneur</i>
Large α_1	Skill-Ability Bias Increases	Skill-Ability Bias Decreases
	Absolute Bias Decreases	Absolute Bias Increases

beginning of the second set the game score is reset to zero-zero however, the player that won the first set is one set closer to winning the match. This advantage can be reinterpreted as the player that won the first set now has a higher ability of winning the match. This increased ability occurs whether the player that won the first set is the disadvantaged player or the advantaged player. Therefore in the second set the abilities will be slightly changed and effort will change accordingly, but there will also be a new luck draw for each player.

This reinterpretation generalizes the model to games that are dynamic in nature. In addition, this reinterpretation rationalizes the result in Brown 2008 that the effect of Tiger Woods playing in a tournament is heterogeneous depending on whether he is in a streak or a slump. In the streak periods all other players perform worse but in slump periods other players perform better. Since Tiger Woods is the same person, we would not expect his ability to differ greatly across time. However, with this reinterpretation his effective ability may have changed. In tournaments where Tiger Woods received positive luck draws in the first round his effective ability increased, which caused other players to decrease their effort and perform worse. However, in tournaments where Tiger Woods received negative luck draws in the first round his effective ability decreased, which caused other players to increase their effort and perform better.

As the game progresses the ability differences are expected to diverge as one player, through a combination of skill and luck, gets closer to winning. As one player gets closer to winning their effective ability difference may increase to be larger than $A^{superstar}$ causing an adverse effect on total effort in the game. The incentives in the game should change to avoid an anticlimactic ending.

Proposition 8. *As a game progresses, previous outcomes should have less impact on the ability of a player to win the next stage.*

When racers, such as cross country skiers, or basketball teams get a large lead it becomes very difficult for an disadvantaged player to catch up and the end of the race or game can become anticlimactic. However, there are games that attempt to minimize previous outcomes such as tennis and bridge. In both of these games the score resets and it does not matter if the tennis player won the previous set 6 – 0 or the bridge partners just made game; the winner is always awarded with one game and the previous difference in score becomes irrelevant.

Proposition 9. *As a game progresses the scoring should change to allow disadvantaged players to quickly catch-up.*

In the game-show Jeopardy the point totals double in the second round, meaning that a player has to answer on average half as many questions correctly to narrow a deficit accrued during the first round. Similarly, in Rubber Bridge when a team wins a game they become vulnerable and if they lose the next game, the other team receives double the points for a normal win (i.e., when neither team is vulnerable). This rule minimizes ability differences between players and may be able to avoid adverse effects from large differences in ability that can occur during the progress of a game.

Proposition 10. *As a game progresses the luck in the game should increase.*

When luck in the game increases, the disadvantaged player’s incentive to exert effort increases. This result is an implication of the comparative statistic that $A^{superstar}$ increases with luck. According to PGA officials the pin placements become easier as the tournament progresses.⁶ Making the pin placements easier can increase the role of luck. When the pin is at the edge of the green, perhaps near a sand trap, a player’s approach to the green is less aggressive. The players aim to get the ball on the green but not necessarily as close to the pin as possible. In contrast, when the pin is in the center of the green, a player’s approach can be more aggressive, increasing the probability of a lucky roll into the hole.

5. CONCLUSION

This paper demonstrates that correctly structured tournament-style internal competitions may be able to improve worker performance. The main contribution of this paper is to demonstrate that there exist optimal prize and luck levels for varying differences in ability. However, if the competition is set up incorrectly there may be adverse effects caused by differences in abilities. When the advantaged player is a superstar,

⁶However, many players oppose this claim and assert that the pin placement become harder as the tournament progresses.

the difference in ability is large relative to the prize and luck levels, the disadvantaged player has very little incentive to put in effort. The advantaged player's incentive to put in effort decreases when the disadvantaged player puts in lower levels of effort. However, even if the difference in abilities is large if there is a large amount of luck in the game the disadvantaged player will have an incentive to put in effort and the adverse effects will be avoided.

When luck plays too small a role in a competition even a small difference in abilities can cause one player to become a superstar. However if the competition has too much luck the benefit to effort is small. In both cases the performance will be suboptimal. In addition, if the prize level is set too high the disadvantaged player may have an incentive to quit decreasing total performance in the game. Therefore there is an optimal prize and luck level for the manager to set to maximize total effort.

The implications of the superstar effect require firms to be cautious in how internal competitions are set up; however, the superstar effect should not cause firms to be cautious about hiring the "best athlete". This paper demonstrates that the prize and luck levels in the competition are able to alter the effective ability differences in ways that maximize total performance. Thus, the adverse effects caused by a superstar can be dampened to those of a mere advantaged player.

These results complement the peer-effect literature that investigates the effects of a student's peers on his or her performance. Previous studies have suggested that removing superstar pupils from a classroom (i.e., tracking) could be beneficial to all students. However, subsequent studies have suggested the unintended negative consequences of separating students by ability may outweigh the adverse effects of having superstar pupils in the classroom. The results of this paper suggest that superstar pupils need not be removed from a classroom for their adverse effects to be avoided. Instead, the game can be restructured to provide the optimal incentives, depending on the ability differences that exist within the classroom.

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6. APPENDIX

6.1. Endogenous Luck.

$$e_1^{\text{Strategies}} = \frac{2\alpha_1 \ell_0 v}{8c\ell_0^2 + (\alpha_1^2 - \alpha_2^2)v} \quad e_2^{\text{Strategies}} = \frac{2\alpha_2 \ell_0 v}{8c\ell_0^2 + (\alpha_1^2 - \alpha_2^2)v}$$

$$\ell_1^{\text{Strategies}} = \frac{4(\alpha_1^2 - \alpha_2^2)c\ell_0 v^2}{k(8c\ell_0^2 + (\alpha_1^2 - \alpha_2^2)v)^2} \quad \ell_2^{\text{Strategies}} = \frac{4(\alpha_1^2 - \alpha_2^2)c\ell_0 v^2}{k(8c\ell_0^2 + (\alpha_1^2 - \alpha_2^2)v)^2}$$

Taking the derivative of ℓ_i with respect to α_1 produces proposition 1

6.2. Example.

$$P_{\text{leader}}(\text{win}) = \int_b^a F_2(p_1) f_1(p_1) dp_1$$

$$P_{\text{underdog}}(\text{win}) = \int_d^c F_1(p_2) f_2(p_2) dp_2$$

Where

$$a = S(\alpha_1, e_1) + L$$

$$b = S(\alpha_1, e_1) - L$$

$$c = S(\alpha_2, e_2) + L$$

$$d = S(\alpha_2, e_2) - L$$

then

$$\begin{aligned} P_2(\text{win}) &= \int_b^a F_1(p_2) f_2(p_2) dp_2 \\ &= \int_d^b \underbrace{F_1(p_2)}_{=0} f_2(p_2) dp_2 + \int_b^c \frac{p_2 - b}{(a - b)(c - d)} dp_2 \\ &= \frac{(c - b)^2}{2(a - b)(c - d)} \end{aligned}$$

For $S(\alpha, e) = \alpha e$

$$P_2(\text{win}) = \frac{\alpha_2^2 e_2^2 + \alpha_1^2 e_1^2 + 4\alpha_2 e_2 L - 4\alpha_1 e_1 L + 4L^2 - 2\alpha_1 e_1 \alpha_2 e_2}{8L^2}$$

For $S(\alpha, e) = \alpha + 2$

$$P_2(\text{win}) = \frac{\alpha_2^2 + e_2^2 + 2\alpha_2 e_2 + \alpha_1^2 + e_1^2 + 2\alpha_1 e_1 + 4\alpha_2 L + 4e_2 L - 4\alpha_1 L - 4e_1 L - 2\alpha_1 \alpha_2 - 2e_1 e_2 - 2e_1 \alpha_2 - 2e_2 \alpha_1}{8L^2}$$

6.3. Results. Result 3: *There is an adverse effect on total effort of increasing the difference in abilities for some level of difference in abilities.*

First, take the derivative of total effort with respect to the ability of the leader, α_1 .

$$\frac{\partial TotalEffort}{\partial \alpha_1} = \frac{2L(8cL^2 - (\alpha_1 + \alpha_2)^2)(v_1 - v_2)}{(8cL^2 + (\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)(v_1 - v_2))^2}$$

Second, set the derivative of total effort with respect to the ability of the leader less than or equal to zero and solve for the ability of the leader.

$$\bar{\alpha}_1 > \sqrt{\frac{8cL^2}{v_1 - v_2}} - \alpha_2$$

Therefore, for $\alpha_1 > \bar{\alpha}_1$ there will be an adverse effect on total effort from increasing the difference in abilities.

Result 4: *In some cases the adverse effect on total effort of differences in abilities occurs only for large differences in abilities, the “Superstar” effect.*

First, take the derivative of total effort with respect to the ability of the leader, α_1 .

$$\frac{\partial TotalEffort}{\partial \alpha_1} = \frac{2L(8cL^2 - (\alpha_1 + \alpha_2)^2)(v_1 - v_2)}{(8cL^2 + (\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)(v_1 - v_2))^2}$$

Second, set the derivative of total effort with respect to the ability of the leader greater than or equal to zero and solve for the ability of the leader.

$$\bar{\alpha}_1 < \sqrt{\frac{8cL^2}{v_1 - v_2}} - \alpha_2$$

Third, check if the condition above holds for small levels of α_1 by setting $\alpha_1 = \alpha_2 = 1$. There will be a range of ability differences such that total effort actually increases with differences in ability when the following condition holds.

$$v_1 - v_2 < 2cL^2$$

When this condition holds there is an adverse effect on total effort only if differences in abilities are large. This implies that the leader must be a “superstar” to cause adverse effects on total effort.

Note that the “superstar” condition holds when the prize structure is not winner-take-all, there is a large amount of luck, or costs are high. Result 1 confirms this logic because luck diminishes the adverse effect.

6.4. **Bias.** From above:

$$P_2(win) = \frac{(c-b)^2}{2(a-b)(c-d)}$$

three cases:

No Effort

$$a1 = \alpha_1 + L$$

$$b1 = \alpha_1 - L$$

$$c1 = \alpha_2 + L$$

$$d1 = \alpha_2 - L$$

$$\Delta = \alpha_1 - \alpha_2$$

Linear Effort

$$a1 = \alpha_1 + e_1 + L$$

$$b1 = \alpha_1 + e_1 - L$$

$$c1 = \alpha_2 + e_2 + L$$

$$d1 = \alpha_2 + e_2 - L$$

$$\hat{\Delta} = \alpha_1 - \alpha_2 + e_1 - e_2$$

Multiplicative Effort

$$a1 = \alpha_1 e_1 + L$$

$$b1 = \alpha_1 e_1 - L$$

$$c1 = \alpha_2 e_2 + L$$

$$d1 = \alpha_2 e_2 - L$$

$$\tilde{\Delta} = \alpha_1 e_1 - \alpha_2 e_2$$

$$\begin{aligned} P(win) &= \frac{(c-b)^2}{2(a-b)(c-d)} \\ &= \frac{\bar{\Delta}^2 - 4L\bar{\Delta} + 4L^2}{8L^2} \end{aligned}$$

Where $\bar{\Delta}$ can equal any of the above Δ s.

$$\begin{aligned} \Delta &= \frac{4L \pm \sqrt{16L^2 - 16L^2(1 - 2P_2(win))}}{2} \\ &= 2L \pm 2L\sqrt{2P_2(win)} \end{aligned}$$

Where by assumption (built into the probabilities) that $P_2(win) \in [0, 1/2]$ which implies that $\Delta \in [0, 2L]$.

Now to show the bias if we thought that $\bar{\Delta} = \Delta$ but instead the real model has effort entering linearly.

$$\begin{aligned} \hat{\Delta} &= \alpha_1 - \alpha_2 + e_1 - e_2 = 2L - 2L\sqrt{2P_2(win)} \\ \Delta &= 2L - 2L\sqrt{2P_2(win)} + \underbrace{e_2 - e_1}_{\text{bias}} \end{aligned}$$

For the linear case when $c_1 > c_2$ this implies that $e_2 > e_1$ and this causes the bias to be positive leading the researcher to believe that the difference in skills is smaller than it actually is.

DEPARTMENTS OF ECONOMICS AND POLITICAL SCIENCE, CENTER FOR THE STUDY OF COMPLEX SYSTEMS, UNIVERSITY OF MICHIGAN

E-mail address: `spage@umich.edu`

DEPARTMENT OF ECONOMICS, UNIVERSITY OF MICHIGAN

E-mail address: `nathan.seegert@gmail.com`

FIGURE 1. Skill-Ability and Absolute Bias

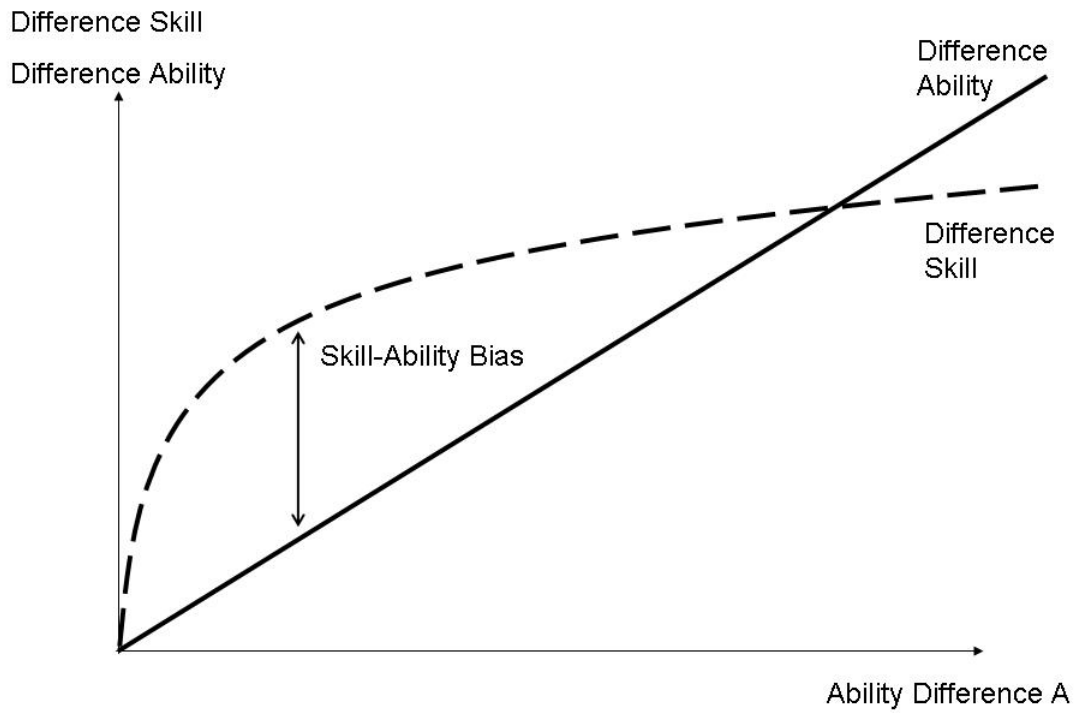


FIGURE 2. Changes in Prize

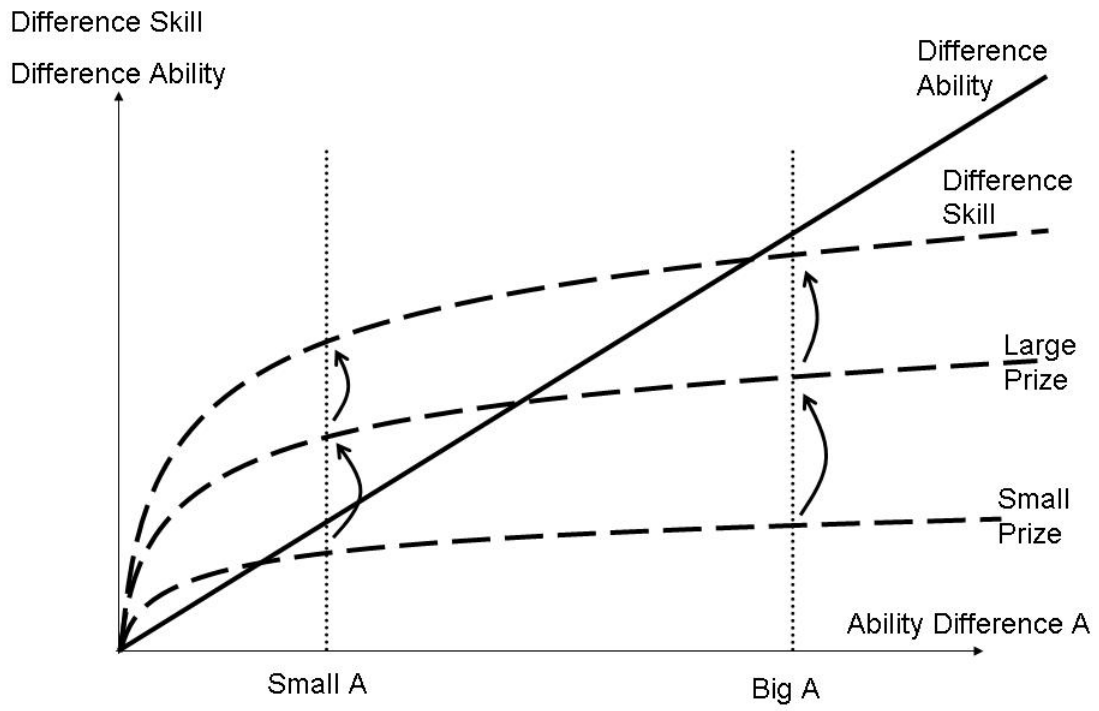


FIGURE 3. Changes in Luck

