ASYMPTOTICS FOR THE PARTITION FUNCTION
VIA THE CIRCLE METHOD

DAVID SCHWEIN

1. Introduction

A partition of a natural number \( n \) is a multiset of natural numbers that sum to \( n \). For instance,

\[ 7 = 1 + 2 + 4 = 2 + 4 + 1 \]

are two (identical) partitions of 7. The partition function \( p(n) \) counts the numbers of partitions of \( n \). The first few values of this function are tabulated below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( p(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>11</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>22</td>
</tr>
<tr>
<td>9</td>
<td>30</td>
</tr>
<tr>
<td>10</td>
<td>42</td>
</tr>
</tbody>
</table>

Our goal is to understand the asymptotic behavior of the partition function.

Let \( F(x) = \sum_{n \geq 0} p(n) x^n \) be the (power series) generating function of the partition function. We can turn information about \( F(x) \) into information about \( p(n) \) because the partition function can be recovered from its generating function:

\[
p(n) = \frac{1}{2\pi i} \int_{|x|=r} \frac{F(x)}{x^{n+1}} \, dx, \quad r < 1.\]

Perhaps the most important information about \( F(x) \) is that it factors as an infinite product of rational functions:

\[
F(x) = \prod_{k \geq 1} (1 - x^k)^{-1}.
\]

This identity follows from the definition of the partition function by expanding the factor \((1 - x^k)^{-1}\) as a power series, then multiplying out the product. Since the poles of the factor \((1 - x^k)^{-1}\) are the \( k \)th roots of unity, we can think of \( F(x) \) as having a pole (of “infinite order”) at every root of unity. In reality, of course, \( F(x) \) has no poles: it does not admit an analytic continuation to any connected open subset properly containing the unit disc.

To get a sense for how \( F(x) \) behaves, consider the partial product

\[
F_N(x) = \prod_{k=1}^{N} (1 - x^k)^{-1}.
\]

At each \( k \)th root of unity with \( k \leq N \), the rational function \( F_N(x) \) has a pole of multiplicity \([N/k]\); there are no other poles. When \( N \geq n \), the power series expansions of \( F(x) \) and \( F_N(x) \) have the same coefficient of \( x^n \), namely \( p(n) \). Consequently, replacing \( F \) by \( F_N \) in the integral (1) leaves its value unchanged.

After making this replacement we can dilate the contour of integration past the unit circle, increasing the radius \( r \) in (1) to be larger than 1. This has the effect of subtracting the residues of all poles of \( F_N(x)/x^{n+1} \) on the unit disc. The resulting contour integral actually vanishes, as can be seen by estimating its size in

\textit{Date: 15 November 2017.}
terms of \( r \) and letting \( r \) tend to infinity. We conclude that the partition function is related to the residues of \( F_N(x) \) by the equation

\[
p(n) + \sum_{k=1}^{N} \sum_{\omega \in \mu_k} \frac{\text{Res}(F_N, \omega)}{\omega^{n+1}} = 0,
\]

where \( \mu_k \subset \mathbb{C}^\times \) is the set of \( k \)th roots of unity. (Alternatively, this equation is a consequence of the fact that the residues of the poles of a meromorphic function on the Riemann sphere sum to zero, applied to the rational function \( F_N(x)/x^{n+1} \).

The obvious next step is to simplify the residues in the expression above, each the value of a certain rational function at a root of unity. Executing this strategy would (probably) require a great deal of algebraic manipulation; preferring analysis over algebra, we are forced to abandon it. But not all hope is lost. This failed plan has taught us that the behavior of \( F(x) \) near the roots of unity carries information about \( p(n) \).

Around 1918, these considerations led Hardy and Ramanujan to analyze the integral (1) using a contour of integration formed by arcs of circles approaching roots of unity. Their method produced the following asymptotic formula.

**Theorem 1** ([5], p. 79). \( p(n) \sim \frac{\exp(\pi \sqrt{2n/3})}{4n\sqrt{3}} \).

In accordance with Stigler’s law of eponymy, Hardy and Ramanujan’s method is known today as the *Hardy-Littlewood circle method*. (The name likely derives from a paper of Hardy and Littlewood published soon thereafter, in 1919, on Waring’s problem of representing integers as sums of powers [4].) The circle method has become a standard tool in the repertoire of analytic number theorists, and in Section 4 we will give an overview of its applications.

In the 1930s, Rademacher modified Hardy and Ramanujan’s argument to give a convergent infinite series expansion for \( p(n) \); earlier, Hardy and Ramanujan had only given a *divergent* asymptotic expansion (different from Theorem 1).

**Theorem 2** ([6], p. 422). For \( n \geq 1 \), the partition function \( p(n) \) is represented by the convergent series

\[
p(n) = \frac{1}{\pi \sqrt{2}} \sum_{k=1}^{\infty} A_k(n) \sqrt{k} \frac{d}{dn} \left( \left( n - \frac{1}{2k} \right)^{-1/2} \sinh \left( \frac{2\pi}{k\sqrt{6}} \left( n - \frac{1}{2k} \right)^{1/2} \right) \right),
\]

where \( A_k(n) \) is a certain sum of roots of unity.

Our goal in this paper is to derive Rademacher’s asymptotic expansion of the partition function and (in a future version of this paper) to use it to prove Hardy and Ramanujan’s simpler asymptotic formula.

To analyze the partition function using the circle method, instead of replacing \( F \) by its partial product \( F_N \) and deforming the contour of integration across the roots of unity, as in our first attempt at the problem, we will retain \( F \) and deform the contour of integration to approach the roots of unity of order at most \( N \). The deformed contour consists of arcs of circles, one arc for every such root of unity. Adding up the contributions from each arc and letting \( N \) tend to infinity gives Rademacher’s infinite sum decomposition of the partition function.

The arcs of Rademacher’s contour are arcs of Ford circles, a family of circles first studied by L. R. Ford in 1938 as a “visual representation of arithmetical results of diverse kinds” [3] (though not including the asymptotics of the partition function; it was Rademacher who synthesized these ideas). The Ford circles of order \( N \) are indexed by the roots of unity of order at most \( N \), in other words, by the fractions of denominator at most \( N \) in the interval \([0, 1]\). The fraction \( h/k \) corresponds to the circle of center \( h/k + i/(2k^2) \) and radius \( 1/(2k^2) \). Figure [1] depicts the Ford circles of order 4.

The set indexing the Ford circles of order \( N \), that is, the rational numbers in \([0, 1]\) with denominator at most \( N \), is known as the *Farey sequence of order \( N \)* after the British geologist John Farey, who reported them in an 1816 letter to the *Philosophical Magazine* [2]. Understanding Ford circles requires understanding basic properties of the Farey sequence, most notably that for any three consecutive fractions in the sequence, the middle term is the mediant of the outer terms (this observation was the content of Farey’s letter, although he could not prove it).

Constructing Rademacher’s contour is only part of the argument. Understanding the behavior of \( F(x) \) on each individual arc of the contour relies on a remarkable connection between \( F(x) \) and a certain modular
form $\eta(\tau)$, the Dedekind eta function. It is defined for $\tau$ in the upper half plane by the infinite product

$$\eta(\tau) = e(\tau/24) \prod_{k \geq 1} (1 - e(k))^{-1}, \quad \text{Re}(\tau) > 0, \quad e(\tau) := \exp(2\pi i \tau).$$

Much is known about the eta function, but for us, the key property is its functional equation

$$\eta(\gamma\tau) = \varepsilon(\gamma)(c\tau + d)^{1/2}\eta(\tau),$$

where $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z})$ and where $\varepsilon : \text{SL}_2(\mathbb{Z}) \to \mu_{24}$ is a certain function whose precise definition is not (yet) important. (Here $\text{SL}_2(\mathbb{Z})$ acts on the upper half plane in the usual way, by fractional linear transformations: $\gamma \tau = \frac{a\tau + b}{c\tau + d}.$) In other words, $\eta$ is a twisted modular form of weight one half.

Comparing the definition of $\eta$ with the infinite product for $F$, we find that

$$\eta(\tau) = e(\tau/24) F(e(\tau)).$$

Substituting this equation into the functional equation for $\eta$ gives a transformation formula for the function $F(e(\tau))$, which we write in the form

$$F(e(\tau)) = \varepsilon(\gamma)(c\tau + d)^{1/2}e((\tau - \gamma\tau)/24)F(e(\gamma\tau)).$$

When $x$ is close to zero, $F(x)$ is close to one; so when $\tau$ is close to $i\infty$, that is, when $\tau$ is large, $F(e(\tau))$ is close to one. Under this heuristic, the transformation formula for $F(e(\tau))$ implies that for $\tau$ close to the pole of $\gamma\tau$,

$$F(e(\tau)) \approx \varepsilon(\gamma)(c\tau + d)^{1/2}e((\tau - \gamma\tau)/24). \quad (2)$$

This approximation replaces the function $F(e(\tau))$ by elementary functions, which are much more amenable to analysis.

When $x = e(\tau)$ is a root of unity, $\tau = h/k$ is a rational number. Therefore, to focus our attention on the root of unity $e(h/k)$ we ought to choose a transformation $\gamma$ that sends $h/k$ to $\infty$. This forces the denominator of $\gamma\tau$ to be $k\tau - h$, and the requirement that $\gamma$ has determinant one forces the numerator of $\gamma\tau$ to be $H\tau - (hH + 1)/k$, where $H$ is any integer such that $hH + 1 = 0 \mod k$. So in summary, we should take

$$\gamma = \gamma_{h,k} = \begin{bmatrix} H & -(hH + 1)/k \\ k & -h \end{bmatrix}$$

in the functional equation for $F$.

The remainder of the analysis consists of manipulating contour integrals. The variable transformation $z = i(\tau - h/k)/k^2$ normalizes the contour for $h/k$, mapping it to an arc of the circle of radius and center $1/2$. The endpoints of this arc depend on $h$ and $k$, and to estimate error terms we will need to bound the
magnitude of the endpoints; these bounds ultimately depend on properties of the Farey sequence. Finally, after exploiting a certain integral representation of a Bessel function built up from the elementary function \(\text{Erf}(x)\), we will have Rademacher’s formula.

None of the core mathematical ideas presented here are new. My exposition follows Chapter 5 of Apostol’s second elementary analytic number theory text \([1]\), based in turn on Rademacher’s original paper \([5]\).

2. Preliminaries

2.1. The Transformation Formula.

Definition 3. The Dedekind eta function \(\eta : \mathbb{H} \to \mathbb{C}\) is the holomorphic function defined by the infinite product

\[
\eta(\tau) = e^{\pi i / 24} \prod_{k \geq 1} (1 - e(k\tau)), \quad e(\tau) := \exp(2\pi i \tau).
\]

Definition 4 (Dedekind Sum). Let \(h\) and \(k\) be integers with \(k \geq 1\). The Dedekind sum \(S(h, k)\) is

\[
S(h, k) = \sum_{r=0}^{k-1} \left( \left[ \frac{r}{k} \right] \left( \frac{hr}{k} \right) \right), \quad \text{where} \quad ((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}
\]

Proposition 5 (Functional Equation for \(\eta\)). For every \(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})\) and every \(\tau \in \mathbb{H}\),

\[
\eta(\gamma \tau) = \varepsilon(\gamma)(c\tau + d)^{1/2}\eta(\gamma),
\]

where \(\varepsilon : \text{SL}_2(\mathbb{Z}) \to \mu_{24}\) is the function defined by

\[
\varepsilon(\gamma) = e\left(\frac{a+d}{24c} + \frac{1}{2}S(-d, c) - \frac{1}{8}\right).
\]

Proof. See the supplement to Chapter 3 of Apostol’s book \([1]\). The strategy is to first prove the theorem in the special case where \(\gamma \tau = -1/\tau\), then deduce the general theorem by an inductive argument on the length of \(\gamma\) in terms of the standard generators for \(\text{SL}_2(\mathbb{Z})\), using a certain reciprocity law of Dedekind sums. An inductive proof of this sort makes clear that the codomain of \(\varepsilon\) is the 24th roots of unity; this claim is not at all clear from the definition of \(\varepsilon\).

Proposition 6 (Functional Equation for \(F(x)\)). Let \(h\) and \(k\) be integers with \((h, k) = 1\) and \(k \neq 0\), let \(H\) be any integer such that \(hH + 1 = 0 \mod k\), and let \(\gamma_{h,k} \in \text{SL}_2(\mathbb{Z})\) be the element acting on \(\mathbb{H}\) by \(\tau \mapsto (H\tau - (hH + 1)/k)/(k\tau - h)\). For every \(x \in \mathbb{D}\),

\[
F(e(\tau)) = \varepsilon(\gamma_{h,k})(k\tau - h)^{1/2}e\left(\frac{1}{24}(\tau - \gamma_{h,k}\tau)\right)F(e(\gamma_{h,k})).
\]

Proof. Substitute \(\eta(\tau) = e(\tau/24)/F(e(\tau))\) and \(\gamma = \gamma_{h,k}\) into the functional equation for \(\eta\).

2.2. Farey Fractions.

Definition 7. The Farey sequence of order \(N\) is the ordered set of rational numbers of denominator at most \(N\) in the interval \([0, 1]\).

Proposition 8. Let \(a/b < h/k < c/d\) be rational numbers with \(bc - ad = 1\). Then \(k \geq b + d\).

Proof. Start with the equation

\[
\frac{c}{d} - \frac{a}{b} = \left(\frac{c}{d} - \frac{h}{k}\right) + \left(\frac{h}{k} - \frac{a}{b}\right),
\]

which simplifies to

\[
\frac{1}{ab} = \frac{ck - dh}{dk} + \frac{bh - ak}{bk}.
\]

Let \(m = bh - ak\) and let \(n = ck - dh\). Multiplying the equation above by \(abk\) shows that \(k = bn + dm\). Then \(k \geq b + d\) because \(m\) and \(n\) are at least 1.

Definition 9. Let \(q = a/b\) and \(r = c/d\) be rational numbers with \((a, b) = (c, d) = 1\) and \(b, d \geq 0\). The mediant of \(q\) and \(r\) is the rational number

\[
\frac{a + c}{b + d}.
\]

Proposition 10. Let \(q < r\) be rational numbers and let \(s\) be their mediant.
If and only if $|z - w| = r + s$, then the point of tangency is a convex linear combination of $z$ and $w$, specifically, the point

$$sz + rw \over r + s.$$
Hence, Ford circles for the fractions adjacent to $h/k$ in the $N$th Farey sequence $F_N$, with $(h_0, k_0) = (h, k) = (h_1, k_1) = 1$, and let $z_j(h, k)$ denote the image of the intersection point of $C(h, k)$ and $C(h_j, k_j)$ under the linear transformation $z = -ik^2(\tau - h/k)$. Then

$$|z_j(h, k)| \leq \sqrt{2} \frac{k}{N}.$$

Proof. Proposition 12 and the easily-verified inequality $x^2 + y^2 \geq \frac{1}{2}(x + y)^2$ give the lower bound

$$k^2 + k_0^2 \geq \frac{1}{2}(k + k_0)^2 \geq \frac{1}{2}N^2.$$

By Proposition 14,

$$z_0(h, k) = \frac{kk_0}{k^2 + k_0^2} - \frac{ik^2}{k^2 + k_0^2}.$$

Hence

$$|z_0(h, k)| = \frac{k}{\sqrt{k^2 + k_0^2}} \leq \sqrt{2} \frac{k}{N}.$$

A similar (but not completely symmetric) argument using the other part of Proposition 14 gives the inequality for $|z_1(h, k)|$. \hfill \qed

2.4. Rademacher Contour. Fix a natural number $N$. The $h/k$th Ford circle tangentially intersects the Ford circles for the fractions adjacent to $h/k$ in the $N$th Farey sequence; let $\alpha(h, k)$ be the clockwise-oriented arc along $C(h, k)$ that connects these points and does not intersect the real axis. Figure 3 depicts the Rademacher contour for $N = 4$.

The careful reader will have noticed that this picture contains a slight lie: in the edge case $(h, k) = (0, 1)$, the Ford circle $C(0, 1)$ is divided into halves and one half of it is translated to the other side of the contour. This modification does change the contour, but it won’t change the integral in which the contour eventually appears, since the integrand will be periodic of period 1. (It might be even better to view the Ford circles...
2.5. **Bessel Functions.** The final steps of our analysis will use a small piece of the theory of Bessel functions, a special family of functions used widely in applied mathematics. I will cite the two results we need from Watson’s treatise on Bessel functions [10]. It would be nice to have a more conceptual explanation for the appearance of these formulas.

**Definition 16.** Let \( \nu \) be a complex number. The *Bessel function of order \( \nu \)*, denoted by \( J_\nu(z) \), is the meromorphic function defined by the power series expansion

\[
J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \cdot \Gamma(m + \nu + 1)} \left( \frac{z^2}{2} \right)^{2m+\nu}.
\]

The *modified Bessel function of order \( \nu \)*, denoted by \( I_\nu(z) \), is the meromorphic function

\[
I_\nu(z) = i^{-\nu}J_\nu(iz) = \sum_{m=0}^{\infty} \frac{1}{m! \cdot \Gamma(m + \nu + 1)} \left( \frac{z^2}{2} \right)^{2m+\nu}.
\]

**Proposition 17.** Suppose \( c > 0 \) and \( \Re(\nu) > 0 \). Then

\[
I_\nu(z) = \left( \frac{1}{2} \frac{z}{z} \right)^\nu \frac{\nu}{2\pi i} \int_{(c)} t^{-\nu-1} \exp(t + z^2/(4t)) \, dt,
\]

where \( \int_{(c)} \) is the contour integral over the infinite vertical line \( \Re t = c \) taken in the upward direction.

**Proof.** See [10, p. 181].

**Proposition 18.** \( I_{3/2}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz} \left( \frac{\sinh z}{z} \right) \).

**Proof.** The Bessel functions of half-integral order are elementary functions; see [10, Ch. III].

Putting these two propositions together gives a closed-form expression for the following integral.

**Proposition 19.** Suppose \( c > 0 \) and \( \Re(\nu) > 0 \). Then

\[
\int_{(c)} t^{-5/2} \exp(t + z^2/(4t)) \, dt = 8\sqrt{\pi} i z^{-1} \frac{d}{dz} \left( \frac{\sinh z}{z} \right).
\]
3. Rademacher’s Asymptotic Expansion of the Partition Function

In this section we will prove the following theorem.

**Theorem 20.** The partition function \( p(n) \) has the infinite series expansion

\[
p(n) = \frac{1}{\pi \sqrt{2}} \sum_{k \geq 1} A_k(n) \sqrt{k} \frac{d}{dn} \left( (n - \frac{1}{24})^{-1/2} \sinh \left( \frac{2\pi}{k \sqrt{6}} (n - \frac{1}{24})^{1/2} \right) \right),
\]

where

\[
A_k(n) = \sum_{\substack{(h,k) = 1 \\ 0 \leq h < k}} e \left( \frac{1}{2} S(h,k) \right) e(-nh/k)
\]

and \( S(h,k) \) is the Dedekind sum of Definition 4.

**Proof.** The partition function \( p(n) \) is given by the contour integral

\[
p(n) = \frac{1}{2\pi i} \int_{|x| = \epsilon(i)} \frac{F(x)}{x^{n+1}} \,dx.
\]

Change the domain to the upper half plane via the coordinate transformation \( x = e(\tau) \), giving the integral

\[
\int_{i}^{i+1} F(e(\tau))e(-n\tau) \,d\tau
\]

taken along a straight-line contour. Fix a large integer \( N \) and let \( \alpha(h,k) \) be the \( h/k \)th arc of the Rademacher contour, with \( (h,k) = 1 \). (Note: \( \alpha \) depends on \( N \).) Decompose the integral operator above as

\[
\int_{i}^{i+1} = \sum_{k=1}^{N} \sum_{h=1}^{k} \int_{\alpha(h,k)} F(e(\tau))e(-n\tau) \,d\tau
\]

so that

\[
p(n) = \sum_{k=1}^{N} \sum_{h=1}^{k} \int_{\alpha(h,k)} F(e(\tau))e(-n\tau) \,d\tau, \quad \sum_{h=1}^{k} = \sum_{(h,k) = 1 \atop 0 \leq h < k}
\]

Using the transformation formula of Proposition 6 this becomes

\[
p(n) = \sum_{k=1}^{N} \sum_{h=1}^{k} \int_{\alpha(h,k)} e^\left( -\frac{\gamma_{h,k}}{2} (k \tau - h)^{1/2} \right) e\left( \frac{1}{24} \frac{1}{\gamma_{h,k}} (\tau - \gamma_{h,k}) \right) F(e(\gamma_{h,k})e(-n\tau) \,d\tau
\]

Next, consider the \((h,k)\)th integral above. Change variables to move the Ford circle \( C(h,k) \) to the circle of radius and center \( 1/2 \), using the linear transformation \( \tau = h/k + iz/k^2 \). A calculation shows that \( \gamma_{h,k} = H/k + i/z \). Then

\[
e\left( \frac{1}{24} (h - H)/k \right) \exp \left( -\frac{2\pi}{24} z/k^2 \right) \exp \left( 2\pi z^{-1}/24 \right)
\]

and

\[
e(-n\tau) = e(-nh/k) \exp(2\pi n z/k^2),
\]

so that the integrand in the \((h,k)\)th integral becomes

\[
e(\gamma_{h,k}) \left( \frac{iz}{k} \right)^{1/2} e\left( \frac{h - H}{24k} \right) \exp \left( -\frac{2\pi}{24k^2} z \right) \exp \left( \frac{2\pi}{24} z^{-1} \right) (e(\gamma_{h,k})e(-n\tau) \,i \,dz/k^2
\]

Thus
\[ p(n) = i^{3/2} \sum_{k=1}^{N} k^{-5/2} \sum_{h} \varepsilon(h,k) e \left( \frac{h - H}{24k} \right) e \left( - \frac{nh}{k} \right) \int_{z_{0}(h,k)}^{z_{1}(h,k)} z^{1/2} \exp \left( \frac{2\pi}{k^2} \left( n - \frac{1}{24} \right) z \right) \exp \left( \frac{2\pi}{24} z^{-1} \right) (F \circ e) \left( \frac{H}{k} + iz^{-1} \right) dz, \]

where \( z_{0}(h,k) \) and \( z_{1}(h,k) \) are the images of the left and right endpoints of \( \alpha(h,k) \) under the change of variables. Since

\[ \varepsilon(h,k) = e \left( \frac{H - h}{24k} + \frac{1}{2} S(h,k) - \frac{1}{8} \right), \]

we can rewrite this as

\[ p(n) = i \sum_{k=1}^{N} k^{-5/2} \sum_{h} e \left( \frac{1}{2} S(h,k) \right) e \left( - \frac{nh}{k} \right) \int_{z_{0}(h,k)}^{z_{1}(h,k)} z^{1/2} \exp \left( \frac{2\pi}{k^2} \left( n - \frac{1}{24} \right) z \right) \exp \left( \frac{2\pi}{24} z^{-1} \right) (F \circ e) \left( \frac{H}{k} + iz^{-1} \right) dz, \]

Next, we will analyze and simplify the integral

\[ I(h,k) = \int_{z_{0}(h,k)}^{z_{1}(h,k)} z^{1/2} \exp \left( \frac{2\pi}{k^2} \left( n - \frac{1}{24} \right) z \right) \exp \left( \frac{2\pi}{24} z^{-1} \right) (F \circ e) \left( \frac{H}{k} + iz^{-1} \right) dz. \]

Decompose it as the sum \( I = I_{1} + I_{2} \) of the two integrals

\[ I_{1}(h,k) = \int_{z_{0}(h,k)}^{z_{1}(h,k)} z^{1/2} \exp \left( \frac{2\pi}{k^2} \left( n - \frac{1}{24} \right) z \right) \exp \left( \frac{2\pi}{24} z^{-1} \right) dz \]

\[ I_{2}(h,k) = \int_{z_{0}(h,k)}^{z_{1}(h,k)} z^{1/2} \exp \left( \frac{2\pi}{k^2} \left( n - \frac{1}{24} \right) z \right) \exp \left( \frac{2\pi}{24} z^{-1} \right) \left( (F \circ e) \left( \frac{H}{k} + iz^{-1} \right) - 1 \right) dz. \]

The first integral is close to a Bessel function and the second integral is close to zero. When we let the parameter \( N \) tend to infinity, these approximate statements will become exact.

**Claim 1.** Let \( K \) denote the contour traversing the circle \( \{|z - 1/2| = 1/2\} \) clockwise. Then

\[ I_{1}(h,k) = \int_{K} z^{1/2} \exp \left( \frac{2\pi}{k^2} \left( n - \frac{1}{24} \right) z \right) \exp \left( \frac{2\pi}{24} z^{-1} \right) dz + O((k/N)^{3/2}). \]

**Claim 2.** \( I_{2}(h,k) = O((k/N)^{3/2}). \)

Granted Claims 1 and 2 it follows that that

\[ \sum_{k=1}^{N} k^{-5/2} \sum_{h} e \left( \frac{1}{2} S(h,k) \right) e \left( - \frac{nh}{k} \right) I_{2}(h,k) = \sum_{k=1}^{N} k^{-5/2} \sum_{h} O((k/N)^{3/2}) \]

\[ = \sum_{k=1}^{N} k^{-5/2} O(k^{5/2} N^{-3/2}) = \sum_{k=1}^{N} O(N^{-3/2}) = O(N^{-1/2}). \]

A similar argument shows that

\[ \sum_{k=1}^{N} k^{-5/2} \sum_{h} e \left( \frac{1}{2} S(h,k) \right) e \left( - \frac{nh}{k} \right) I_{1}(h,k) = \]

\[ \sum_{k=1}^{N} k^{-5/2} \sum_{h} e \left( \frac{1}{2} S(h,k) \right) e \left( - \frac{nh}{k} \right) \int_{K} z^{1/2} \exp \left( \frac{2\pi}{k^2} \left( n - \frac{1}{24} \right) z \right) \exp \left( \frac{2\pi}{24} z^{-1} \right) dz + O(N^{-1}). \]

Putting these claims together gives the following expression for \( p(n) \):
\[
p(n) = i \sum_{k=1}^{N} \sum_{h} e^{\frac{\pi}{2} S(h, k)} e(- nh/k) \int_{K} z^{1/2} \exp \left( \frac{2\pi}{k^2} \left( n - \frac{1}{24} \right) z \right) \exp \left( \frac{2\pi}{24} z^{-1} \right) dz + O(N^{-1/2}).
\]

Letting \( N \to \infty \) gives
\[
\int_{K'} = \int_{K} - \int_{\ell_{0}(h,k)}.
\]
That is, \( K' \) is the clockwise-oriented arc passing through \( z_0, z_1, \) and 0. Since
\[
\left| \exp \left( \frac{2\pi}{k^2} \left( n - \frac{1}{24} \right) z \right) \right| \leq \exp \left( \frac{2\pi}{k^2} \left( n - \frac{1}{24} \right) \right) \leq \exp(2\pi n/24)
\]
and
\[
\left| \exp \left( \frac{2\pi}{24} z^{-1} \right) \right| = \exp \left( \frac{2\pi}{24} \Re(z^{-1}) \right) = \exp(\frac{2\pi}{24}),
\]
as \( \Re(z^{-1}) = 1 \) on \( K' \), and since, by Proposition 15,
\[
\text{length}(K') \leq C \frac{k}{N},
\]
it follows that the integral over \( K' \) is no more than
\[
\text{length}(K') \max_{z \in K'} C|z|^{1/2} = O((k/N)^{1/2}).
\]

**Proof of Claim 2.** Let \( \ell_{h,k} \) denote the line connecting \( z_0(h,k) \) and \( z_1(h,k) \); we may choose this to be the contour of the integral \( I_2(h,k) \).

Hence, for \( z \in \ell_{h,k} \),
\[
\left| \exp \left( \frac{2\pi}{k^2} \left( n - \frac{1}{24} \right) z \right) \right| \leq \exp(2\pi(n - \frac{1}{24})/k^2)
\]
and
\[
\left| \exp \left( \frac{2\pi}{24} z^{-1} \right) \left( (F \circ e) \left( \frac{H}{k} + iz^{-1} \right) - 1 \right) \right| = \left| \exp \left( \frac{2\pi}{24} z^{-1} \right) \sum_{m \geq 1} p(m) \exp \left( 2\pi i \frac{H}{k} - 2\pi z^{-1} \right) \right| \leq \sum_{m \geq 1} p(m) \exp \left( - 2\pi (m - \frac{1}{24}) \Re(z^{-1}) \right) \leq \sum_{m \geq 1} p(m) \exp \left( - 2\pi (m - \frac{1}{24}) \right) = \exp(\frac{2\pi}{24}) (F(\exp(-\frac{2\pi}{24})) - 1).
\]

It follows that the absolute value of the integral is at most
\[
|z_0(h,k) - z_1(h,k)| \max_{z \in \ell_{h,k}} C|z|^{1/2}.
\]

The estimate of Proposition 15 and the triangle inequality now show that the integral is \( O((k/N)^{3/2}) \).
It remains to analyze the integral in the expression for \( p(n) \),
\[
\int_K z^{1/2} \exp \left( \frac{2\pi}{k^2} (n - \frac{1}{24}) z + \frac{2\pi}{24} z^{-1} \right) \, dz.
\]
Under the change of variable \( w = z^{-1} \) this integral becomes
\[
-i \int_{(1)} w^{-5/2} \exp \left( \frac{\pi}{12} w + \frac{2\pi}{k^2} (n - \frac{1}{24}) w^{-1} \right) \, dw,
\]
and changing variables one last time to \( t = \frac{\pi}{12} w \) yields
\[
-i \left( \frac{\pi}{12} \right)^{3/2} \int_{(\pi/12)} t^{5/2} \exp \left( t + \frac{\pi^2}{6k^2} (n - \frac{1}{24}) t^{-1} \right) \, dt.
\]
Proposition 19 with
\[
z = \frac{2\pi}{k\sqrt{6}} (n - \frac{1}{24})^{1/2}
\]
shows this integral to be
\[
-i \left( \frac{\pi}{12} \right)^{3/2} 8\sqrt{\pi} i z^{-1/2} \frac{d}{dz} \left( \frac{\sinh z}{z} \right) = \frac{\pi^2}{3^{3/2} 2} z^{-1/2} \frac{d}{dz} \left( \sinh \frac{z}{z} \right).
\]
Since
\[
z^{-1} \frac{d}{dz} \left( \frac{\sinh z}{z} \right) = \left( \frac{dz}{dn} \right)^{-1} \frac{d}{dn} = \frac{3k^2}{\pi^2} \frac{d}{dn}
\]
as differential operators, this expression is
\[
\frac{\pi^2}{3^{3/2} 2} z^{-1/2} \frac{d}{dz} \left( \frac{\sinh z}{z} \right) = \frac{k^2}{\sqrt{3} \pi^2} \frac{d}{dn} \left( \frac{\sinh \left( \frac{2\pi}{k\sqrt{6}} (n - \frac{1}{24})^{1/2} \right)}{n - \frac{1}{24}} \right)
\]
\[
= \frac{k^2}{\sqrt{3} \pi^2} \frac{d}{dn} \left( n - \frac{1}{24} \right)^{-1/2} \sinh \left( \frac{2\pi}{k\sqrt{6}} (n - \frac{1}{24})^{1/2} \right)
\]
Putting everything together finally gives Rademacher’s asymptotic formula for the partition function:
\[
p(n) = \frac{1}{\pi^{3/2}} \sum_{k \geq 1} k^{1/2} \sum_{h} e \left( \frac{1}{12} S(h, k) \right) e \left( -nh/k \right) \frac{d}{dn} \left( n - \frac{1}{24} \right)^{-1/2} \sinh \left( \frac{2\pi}{k\sqrt{6}} (n - \frac{1}{24})^{1/2} \right).
\]

4. Epilogue: Further Applications of the Circle Method in Additive Number Theory

In addition to the problem of estimating the partition function, the circle method has been used to attack some of the most outstanding problems in analytic number theory, such as

**Waring’s Problem:** Is there a number \( g(k) \) such that every integer is the sum of at most \( g(k) \) \( k \)th powers?

**(Weak) Goldbach Conjecture:** Is every odd number the sum of at most two (three) primes?

Both of these problems fall under the rubric of **additive number theory**, which at its core investigates the summation operation on sets of integers:
\[
A + B := \{ a + b : a \in A, b \in B \}, \quad nA := \sum_{\text{\( n \) terms}} A
\]
(The set \( A + B \) is called the *sumset* of the sets \( A \) and \( B \).) In this language, we can rephrase the two problems above, as well as the problem of this paper, in the following way.

**Partition Number Problem:** Let \( \mathbb{N}^n / S_n \) be the set of unordered \( n \)-tuples of positive integers. How big are the fibers of the addition map \( \mathbb{N}^n / S_n \to \mathbb{N} \)?

**Waring’s Problem:** Let \( \mathbb{Z}^k \) be the set of \( k \)th powers of integers. Is \( n\mathbb{Z}^k = \mathbb{Z} \) for \( n \) sufficiently large?

**(Weak) Goldbach Conjecture:** Let \( \mathcal{P} \) be the set of prime numbers. What is \( 2\mathcal{P} \) (3\( \mathcal{P} \))?
I have phrased the partition number problem in rather baroque language if only to show the natural progression of the questions one asks in additive number theory. One of the first questions is to determine explicitly the sumset $A + B$ (or $nA$). For the Goldbach conjecture this question is extremely difficult; it was only recently that the weak Goldbach conjecture was shown to hold, and the (strong) Goldbach conjecture itself remains open.

Once one has determined the set $A + B$ (or $nA$), the next question to investigate is the number of decompositions, that is, the size of the fibers of the addition map $A \times B \to A + B$ (or $A^n/S_n \to nA$). Since it is patently obvious that every natural number is a sum of natural numbers, we can immediately promote the partition number problem into this second category of questions.

Waring’s problem straddles these two extremes. Hilbert was the first to solve the problem in the affirmative: the number $g(k)$ does exist. Attention has since turned to the related number $G(k)$ such that every sufficiently large natural number is the sum of $G(k)$ $k$th powers. As for the number of decompositions, in the case $k = 2$ (where $g(2) = 4$) there is a classical explicit formula, a certain divisor sum due to Jacobi, for the exact number of decompositions of an integer as a sum of four squares. Jacobi’s formula is more an algebraic result than an analytic one, and to my knowledge, there is not yet a significant research focus on the number-of-decompositions problem for $k$th powers. Nonetheless, one could imagine tackling this question, or even the question of the number of decompositions as a sum of primes, in a distant future when additive number theory technology is more advanced.

Setting aside these generalities about additive number theory, how does the circle method help us to investigate such questions? What do we mean by the circle method? Let $A$ be a set of natural numbers. Consider the generating function $F_A(x)$ for the indicator function of $A$:

$$F_A(x) = \sum_{a \in A} x^a.$$  

The $d$th power $F_A(x)^d$ is the generating function for the number $f_{A,d}(n)$ of (ordered) decompositions of a natural number $n$ as a sum of $d$ elements of $A$:

$$F_A(x)^d = \sum_{n \geq 0} f_{A,d}(n) x^n, \quad f_{A,d}(n) = \# \{(a_1, \ldots, a_d) \in A^d : a_1 + \cdots + a_d = n\}.$$  

As in our analysis of the partition function, we can extract the function $f_{A,d}(n)$ via the contour integral

$$f_{A,d}(n) = \frac{1}{2\pi i} \int_{|x|=r} \frac{F_A(x)^d}{x^{n+1}} dx, \quad r < 1.$$  

We could then hope to estimate $f_{A,d}(n)$ in the same way as the partition function, by analyzing the behavior of $F_A \circ e$ near the rational numbers and decomposing the contour of integration accordingly.

The modern formulation of the circle method is a variation on this idea, replacing contour integrals with Fourier series. Starting with the sum

$$S_A(x, \theta) = \sum_{A \in \mathbb{A} \leq x} e(a\theta),$$  

so that

$$S_A(x, \theta)^d = \sum_{n \leq x} f_{A,d}(n) e(n\theta),$$  

we can recover $f_{A,d}(n)$ using the orthogonality of the exponential characters:

$$f_{A,d}(n) = \int_0^1 S_A(x, \theta)^d e(-n\theta) d\theta.$$  

The strategy from here is to obtain estimates on the sums $S_A(x, \theta)$, then use these estimates to try to show, for instance, that $f_{a,d}(n) > 0$ for all $n$ (or for all odd $n$). This would mean that $nA = \mathbb{N}$ (or $2 \cdot \mathbb{N} + 1$), solving Waring’s problem (or Goldbach’s conjecture).

To estimate the sum $S_A(x, \theta)$, as in our analysis of the partition function, we will focus on rational $\theta$; since the rational numbers are a dense subset of the reals, we can often use the behavior of $S_A(x, \theta)$ at rational $\theta$
to understand the behavior at arbitrary $\theta$. When $\theta = h/q$ is rational, the function $x \mapsto e((h/q)x)$ is periodic of period $q$ and the sum splits up into pieces depending on the number of residues of elements of $A$ modulo $q$:

$$S_A(x, h/q) = \sum_{r=0}^{q-1} g_{A,r,q}(x) e((h/q)x) \quad g_{A,r,q}(x) = \# \{ a \in A : a \leq x \text{ and } a \equiv r \mod q \}.$$ 

In the case where $A = \mathbb{N}^k$ is the set of $k$th powers, for Waring’s problem, it is easy to determine a formula for $g_{A,r,q}(x)$. In the case where $A = \mathcal{P}$ is the set of primes, for Goldbach’s conjecture, the function $g_{\mathcal{P},r,q}(x)$ is usually written as $\pi(x; r, q)$, and its asymptotic behavior is known from the Siegel-Walfisz theorem.

There is much more to be said about the circle method. Terry Tao’s famous blog includes an introduction to the subject written for an analytic number theory class [8] as well as an explanation of why the circle method cannot (yet?) be used to prove the strong Goldbach conjecture [7]. For even more details on the circle method, the standard reference is Vaughan’s text [9].

References