Dynamic Stability of Journal Bearings

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(Submitted 1 April 1996 and in revised form 1 October 1996)

We investigate the hydrodynamic stability of a rotating journal translating inside a stationary bearing. A long (two-dimensional) journal bearing separated by Newtonian non-cavitating lubricant is studied for shaft stability. Spectral element methods, perturbation methods, and linear stability analyses are used. The influences of fluid inertia, eccentricity, ellipticity, shaft mass, and finite gap on hydrodynamic stability are explored. Lubrication theory using Reynolds equation ignoring fluid inertia leads to erroneous conclusions. Without fluid inertia, the shaft is always unstable. However, the journal is conditionally stable even in the limit Re → 0 if fluid inertia is included. Increasing eccentricity helps stabilize a whirling shaft. Non-circular shaft bearings, for example elliptical bearings, are observed to have better dynamic stability. Increasing the ellipticity leads to an improved anti-whirl capability. Comparisons are made between lubrication theory, spectral element methods, and perturbation methods.

1. Introduction

Journal bearings have rich applications in industry because they have high load capacity, low wear, and silent operation. The load capacity of a journal bearing supports a force including its weight and an external load applied perpendicular to the shaft axis. When the load increases, the shaft responds with an increase in the eccentricity. Theoretically, the load capacity of a journal bearing is infinite since pressure forces increase dramatically as the eccentricity increases and the journal approaches the bearing surface. The load capacity of a journal bearing is a function of shaft diameter length and rotating speed. A drawback of oil lubricated journal bearings is that the shaft center travels at low load or high operating speed. After the shaft deviates from its equilibrium position, the shaft may return to its original neutral position, keep spiraling outward and hit the bearing or may spiral outward but eventually reach a stable orbit. The third phenomenon, known as oil whirl or oil whip, is a source of machine vibration. Oil whirl is a self-excited instability due to an unbalanced pressure inside the lubricant generated by the rotating shaft. A high pressure zone exists in the converging side of the shaft with a low pressure region in the diverging side where cavitation or aeration may take place. The shaft center orbits at a fractional frequency of the spinning frequency. The frequency ratio nearly one-half for traditional journal bearings with small clearances is determined by the geometry and the bearing load.

Although anti-whirl techniques have been widely applied in industry for decades, research interest in dynamic analysis of oil whirl phenomenon remains strong. Osborne
Reynolds published his famous treatise based on experimental observations in 1886 that established lubrication theory. Most explorations of the oil whirl phenomenon have been based on traditional lubrication assumptions that ignore the sleeve curvature and fluid inertia and lead to restrictions on Reynolds number and geometry (small clearance). Using lubrication theory, Myers and Poritsky predicted that the journal is unconditionally unstable for a long bearing without cavitation and inertia effects. For no-load (vertical) journal bearing, Cole's experiment shows conditional stability for noncavitating bearings indicating the possible failure of lubrication theory. An experiment conducted by Simons revealed the journal is unconditionally stable for loaded journal bearing.

Subsequent research by Hollis and Taylor and Lund and Säbel showed the shaft is unconditionally stable when eccentricity $\varepsilon > 0.8$.

Applying a linear stability analysis to the solve unsteady two-dimensional boundary-layer equations, Collins and Taylor showed that fluid inertia is also an important stabilizing mechanism. Hollis et al. implemented a bifurcation analysis to show that limit cycles exist for systems above the critical speed. Nonetheless, a direct numerical simulation using a time-marching spectral element method to solve Navier-Stokes equations for loaded (eccentric) bearings by Gwynllyw Davies and Phillips reported that a constant viscosity noncavitating model is always unstable implying that fluid inertia by itself is not a stabilizing mechanism. In their time-marching calculations, a non-Newtonian lubricant could stabilize the whirling shaft. However, their concluding remarks about fluid inertia contradict Collins et al., Cole, Hollis et al., Lund et al., and Simons. Therefore, an exhaustive investigation of the fluid inertia effect by analytically and numerically solving the Navier-Stokes equations instead of the Reynolds equation is necessary.

Spectral domain decomposition methods, also known as spectral element methods, are hybrids of spectral algorithms and finite element schemes that combine high accuracy for fluid dynamic computations with flexibility in geometry. These techniques split an irregular fluid domain into regular subdomains while applying high degree polynomials in each local sub-domain. The accuracy of spectral element methods is improved by increasing the polynomial degree in each subdomain rather than the number of subdomains. Therefore, spectral element methods are ideal for complex geometry and are very similar to $p$-type finite element methods. A spectral domain decomposition algorithm is developed for solving the Navier-Stokes equations in a journal bearing. We investigate a general gap film journal bearing with only modest limitations on Reynolds number and geometry. The formulation using a Legendre expansion for pressure is shown to be slightly more accurate than the Lagrangian interpolant expansion used in the staggered grids of Ronquist and significantly more accurate for eigenvalue problems.

[THIS NEEDS WORK] The primary goal of this study is to resolve the differences between two different critical shaft masses obtained by our spectral element eigenvalue and spectral element time-marching algorithms. In the process, we hope to learn more about the physical mechanisms causing oil whirl. We will show that the spectral element eigenvalue results are consistent with a perturbation analysis that is developed here. Although we have made some improvements to the time-marching algorithm, continuing effort is required to identify the discrepancies in this method. In section 5, we use several perturbation methods to investigate flow for a vertical bearing with small eccentricity. Annular flow between a stationary bearing and a translating shaft is explored. Critical masses are obtained for concentric cylinders flow with small Reynolds number and different gap aspect ratios. A linear stability analysis using spectral elements is given in section 4. Under comparable conditions, the eigenvalue solution agrees with ??.

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Dynamic Stability of Pressurized Journal Bearings

2. Lubrication Theory

Journal bearings are traditionally explored by lubrication theory based on a partial differential equation for pressure first derived by Reynolds. Reynolds ignored fluid inertia and assumed that the gap is much smaller than the shaft radius. The axial direction is ignored in our analysis (the long bearing assumption). The dimensionless governing equations can be simplified to [JPB; yes this ignores the radial coordinate.]

\[
\frac{c^3}{R^3} \frac{\partial}{\partial \theta} \left( \frac{R^3}{12} \frac{\partial p}{\partial \theta} \right) = \frac{1}{2} \frac{\partial h}{\partial \theta} + \frac{\partial h}{\partial t},
\]

where pressure \( p \) is scaled by \( \mu \omega \) and \( h \) is the dimensionless local film thickness scaled by the averaged radial clearance \( c \). The left-hand side of Reynolds equation is a Poiseuille term driven by the pressure gradient. The first term on the right hand side is called the “physical wedge” and the second term is the “squeeze effect”. For concentric basic flow we assume the film thickness \( h \) can be expressed as

\[
h = 1 + x' \cos \theta + y' \sin \theta
\]

where \( x' \) and \( y' \) are small perturbations indicating the instantaneous center of the shaft. The basic state pressure for a concentric cylinder is a function of \( r \) only although this must be determined from the full equations of motion and not lubrication theory. Here we ignore this effect so the basic pressure is a constant. We assume a normal mode expansion in \( t \) for small perturbation quantities i.e.

\[
p' = \dot{p} e^{\alpha t}, \quad x' = \dot{x} e^{\alpha t}, \quad y' = \dot{y} e^{\alpha t}.
\]

Substituting (2.2) and (2.3) into the Reynolds equation (2.1) and ignoring higher-order terms gives

\[
\frac{d^2 \dot{p}}{d \theta^2} = 6 (\dot{x} \sin \theta + \dot{y} \cos \theta) \frac{R^2}{c^2} + 12 \sigma (\dot{x} \cos \theta + \dot{y} \sin \theta) \frac{R^2}{c^2}.
\]

Integrating twice with respect to \( \theta \) gives

\[
\dot{p} = 6 (\dot{x} \sin \theta - \dot{y} \cos \theta) \frac{R^2}{c^2} - 12 \sigma (\dot{x} \cos \theta + \dot{y} \sin \theta) \frac{R^2}{c^2}
\]

where one constant of integration is set to zero for periodicity and the other arbitrary constant is also set to zero.

The force exerted on the shaft surface is derived by integrating pressure along the shaft surface i.e.

\[
F_x = \frac{c}{R} M \dot{x}' = \frac{c}{R} M \sigma^2 \dot{x} e^{\alpha t} = \int_0^{2\pi} p' \cos \theta d\theta = -6 \pi (\dot{y} + 2 \sigma \dot{x}) e^{\alpha t} \frac{R^2}{c^2}
\]

\[
F_y = \frac{c}{R} M \dot{y}' = \frac{c}{R} M \sigma^2 \dot{y} e^{\alpha t} = \int_0^{2\pi} p' \sin \theta d\theta = -6 \pi (2 \sigma \dot{y} - \dot{x}) e^{\alpha t} \frac{R^2}{c^2}
\]

where unit length shaft mass is scaled by \( \mu \omega \). Here we have followed the usual lubrication approximation of neglecting viscous stresses in the force balance on the shaft. These two equations can be combined into one complex equation

\[
\frac{c^3}{R^3} M \sigma^2 + 12 \sigma \pi - 6i\pi = 0
\]

Solving the quadratic equation for the stability parameter \( \sigma \) gives

\[
\sigma = \frac{-1 \pm \sqrt{1 + 3\beta}}{\beta}
\]
where $\beta = c^3 M/(6\pi R^3)$ is a small positive number in the lubrication limit. Expanding in powers of $\beta$ yields

$$\sigma = \frac{i}{2} + \frac{\beta}{8} - \frac{i\beta^2}{16} + O(\beta^3).$$

The leading part is $i/2\Gamma$ showing neutral stability with the shaft whirling at half of the spinning frequency. The leading correction term $Mc^3/48\pi R^3$ agrees with section 3 for small gap journal bearings when the viscous stresses are ignored in evaluating the force exerted on the shaft surface. This term makes the problem unconditionally unstable for all $\beta > 0$ (CHECK!) However, since $\beta$ is normally a very small number, it is only marginally unstable. As such it may be stabilized by another small effect such as inertia. This will be included in the next section.

3. Linearized Perturbation Analysis

A two-dimensional model is considered to investigate the shaft dynamic stability problems in bearings. We assume a round shaft rotates inside an infinitely long cylindrical (usually circular) bearing separated by a Newtonian liquid. For simplicity, we assume the shaft is driven such that a constant angular velocity is maintained in the counterclockwise direction. The origin of the cylindrical coordinate system is located on the center of the stationary bearing. The behavior of the lubricant between the journal bearing is governed by the Navier-Stokes equations and the continuity equation. The nondimensionalized Navier-Stokes equations for two-dimensional flow in a cylindrical coordinate system are

$$\text{Re} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} - \frac{u^2}{r} \right) = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{u}{r^2} - \frac{2}{r} \frac{\partial v}{\partial \theta}$$

$$\text{Re} \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + uv \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} - \frac{v}{r^2} + \frac{2}{r^2} \frac{\partial u}{\partial \theta}$$

where $u$ and $v$ represent dimensionless velocity components in radial and azimuthal direction respectively $\Gamma$ is the dimensionless pressure and the Reynolds number is defined as $\text{Re} = R_i^2 \omega/\nu$. We select the radius of journal $R_i$ as the length scale $R_i \omega$ as the velocity scale $\mu \omega$ as the viscous pressure scale and the convective time scale $1/\omega$ where $\omega$ is the inner cylindrical angular velocity. The diffusive time scale $R_i^2/\nu$ would alter the kinematic boundary conditions to make the solution of the problem difficult in the limit $\text{Re} \rightarrow 0$. The continuity equation governing the conservation of mass for a two-dimensional cylindrical coordinate system is

$$r \frac{\partial u}{\partial r} + u + \frac{\partial v}{\partial \theta} = 0.$$  

3.1. The Linear Stability Analysis

We decompose the primitive variables $u \Gamma \epsilon \Gamma$ and $p$ into steady basic state solutions and time-dependent perturbations i.e.

$$u = U + u'; \quad v = V + v'; \quad p = P + p',$$

where capital quantities $U \Gamma \epsilon \Gamma P$ denote basic solutions and prime quantities $u' \Gamma \epsilon \Gamma p'$ indicate disturbances.

To satisfy the kinematic boundary conditions $V(1) = 1$ and $V(\gamma) = 0 \Gamma$ the basic state
velocities become

\[ U \equiv 0 \quad \text{and} \quad V(r) = \frac{\gamma^2 - r^2}{(\gamma^2 - 1)r}. \]  

(3.4)

Substituting (3.4) into (3.3) and integrating with respect to \( r \) gives

\[ P(r) = Re \int \frac{V^2}{r} \, dr = \frac{Re}{\left( \frac{\gamma^2}{2} - 2 \gamma^2 \ln r - \frac{\gamma^4}{2r^2} \right)}. \]  

(3.5)

The constant of integration is assigned to be zero since only gradients in pressure are of concern for incompressible flow. Using the simplifications \( U = 0 \) and \( V = V(r) \) the linearized disturbance Navier-Stokes equations for concentric flow become

\[ \text{Re} \left( \frac{\partial u'}{\partial t} + \frac{V}{r} \frac{\partial u'}{\partial \theta} - \frac{2Vr'}{r} \right) = -\frac{\partial \psi'}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\partial r} \frac{\partial u'}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u'}{\partial \theta^2} - \frac{u'}{r^2} - 2 \frac{\partial v'}{\partial \theta}, \]  

(3.6a)

\[ \text{Re} \left( \frac{\partial v'}{\partial t} + \frac{V}{r} \frac{\partial v'}{\partial \theta} + \frac{u'}{r} \right) = -\frac{1}{r} \frac{\partial \psi'}{\partial \theta} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\partial r} \frac{\partial v'}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v'}{\partial \theta^2} - \frac{v'}{r^2} + 2 \frac{\partial u'}{\partial \theta}. \]  

(3.6b)

The vorticity equation is obtained by taking the curl of Navier-Stokes equation and replacing the velocity components by the stream function \( \psi \). The two-dimensional stream function defined such that

\[ u \equiv \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v \equiv -\frac{\partial \psi}{\partial r}, \]  

(3.7)

automatically satisfying the continuity equation. This leads to

\[ \text{Re} \left( \frac{\partial \zeta'}{\partial t} + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \frac{\partial \zeta'}{\partial r} - \frac{1}{r} \frac{\partial \psi}{\partial r} \frac{\partial \zeta'}{\partial \theta} \right) = \nabla^2 \zeta = \frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r} \frac{\partial \zeta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \zeta}{\partial \theta^2}, \]  

(3.8)

where \( \zeta = -\nabla^2 \psi \) is defined as the vorticity. The stream function \( \psi \) can be decomposed into a basic state stream function \( \Psi \) and a small disturbance \( \psi' \) such that

\[ U = \frac{1}{r} \frac{\partial \Psi}{\partial \theta}, \quad V = -\frac{\partial \Psi}{\partial r}, \quad u' = \frac{1}{r} \frac{\partial \psi'}{\partial \theta}, \quad v' = -\frac{\partial \psi'}{\partial r}. \]  

(3.9)

Substituting the axisymmetric basic stream function (3.9) into equation (3.8) yields

\[ \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} - \frac{1}{r^2} \frac{\partial \Psi}{\partial r} = 0. \]  

(3.10)

To satisfy the boundary conditions of concentric cylinders of the basic state solution is

\[ \Psi = \frac{\gamma^2}{1 - \gamma^2} \ln \left( \frac{r}{\gamma} \right) + \frac{r^2 - \gamma^2}{2\gamma^2 - 2}. \]  

(3.11)

Next we develop the kinematic boundary conditions for the disturbed shaft problem. The bearing center is located on the origin of a global cylindrical coordinate system \((r, \theta)\) while a moving local coordinate system \((r_0, \theta_0)\) is attached to the shaft center at \( r = r' \) and \( \theta = \theta' \) as sketched in Figure 1. The relation between the global and local coordinates system is described by

\[ \begin{align*}
  r \cos \theta &= r_0 \cos \theta_0 + r' \cos \theta' \\
  r \sin \theta &= r_0 \sin \theta_0 + r' \sin \theta'.
\end{align*} \]  

(3.12a,b)

The bearing surface is \( r = \gamma \) in the global coordinate system and the shaft surface is \( r_0 = 1 \) on the local one. The radial position of the translating shaft surface in the global
system is
\[
r = r_s(\theta) = 1 + r' \cos(\theta - \theta') - \frac{(r')^2}{2} \sin^2(\theta - \theta') + O(r'^2). \tag{3.13}
\]

We use both Cartesian and cylindrical coordinate systems to specify the center of the translating shaft under disturbances at \((x', y')\) or \((r', \theta')\). Again, \(P\) implies a perturbation quantity. The no-slip boundary conditions are quite straightforward:
\[
\psi_\theta(r = \gamma) = -\psi_r(r = \gamma) = 0 \tag{3.14}
\]
where \(\gamma\) is the radius ratio of the bearing to the inner shaft. The no-slip boundary conditions require the fluid velocity to be equal to the vector summation of the spinning and translating velocities of the shaft. One can express the translating velocity of the shaft center in Cartesian coordinate system \(x\) and \(y\) directions as \((\dot{x}', \dot{y}')\) although the governing equations are expressed in cylindrical coordinates. From the geometric relation, the exact inner boundary conditions on the disturbed shaft surface \(r_0 = \Gamma\) are [be consistent with previous eqn ... how to do pd’s]
\[
\left. \frac{\partial \psi}{\partial \theta} \right|_{r_0=1} = \dot{x}' \cos \theta + \dot{y}' \sin \theta \tag{3.15a}
\]
\[
\left. \frac{\partial \psi}{\partial r} \right|_{r_0=1} = 1 - \dot{x}' \sin \theta + \dot{y}' \cos \theta. \tag{3.15b}
\]

The boundary conditions (3.15a,b) are transformed from the local system to the global system by using chain rules, linearized by neglecting higher-order terms, and approximated from the undisturbed position by using Taylor series expansion. This leads to
\[
\psi_\theta + \psi_{r\theta}(x' \cos \theta + y' \sin \theta) + x' \sin \theta - y' \cos \theta - \dot{x}' \cos \theta - \dot{y}' \sin \theta = 0 \tag{3.16a}
\]
\[
\psi_r + \psi_{rr}(x' \cos \theta + y' \sin \theta) + 1 - \dot{x}' \sin \theta + \dot{y}' \cos \theta = 0 \tag{3.16b}
\]
on the undisturbed shaft surface \(r = 1\). Again, the stream function \(\psi\) is decomposed into the summation of a basic state stream function \(\Psi \Gamma\) given in (3.11) and a disturbance \(\psi'\). Substituting this decomposition into the inner boundary conditions (3.16a,b) and
neglecting the higher-order terms results in
\[
\psi_x' = -\frac{\gamma^2 + 1}{\gamma^2 - 1}(x' \cos \theta + y' \sin \theta) + \dot{x}' \sin \theta - \dot{y}' \cos \theta
\]
\[
\psi_y' = -(x' \sin \theta - y' \cos \theta) + \dot{x}' \cos \theta + \dot{y}' \sin \theta
\] (3.17a)

on the undisturbed shaft position.

As in lubrication theory we define a complex variable \( z' = x' + iy' \). Multiplying \( z' \) by \( e^{-i\theta} = \cos \theta - i \sin \theta \) leads to a new complex function \( Z' \)
\[
Z' = z' e^{-i\theta} = (x' \cos \theta + y' \sin \theta) - i(x' \sin \theta - y' \cos \theta).
\]

We assume the perturbation stream function \( \psi' \) is periodic in \( \theta \) with unit wave number. A normal mode expansion in time is suggested for both \( \psi' \) and \( Z' \) such that
\[
\psi'\right\} = \Re \left\{ \hat{\psi}(r) e^{\sigma t - i \theta} \right\}
\]
\[
Z' = \hat{\zeta} e^{-i\theta} e^{\sigma t},
\] (3.19)

where \( \Re \{ \} \) indicates the real part. Consequently the inner boundary conditions (3.17a,b) are expressed as the real parts of
\[
\hat{\psi}(1) - (1 + i\sigma) \hat{\zeta} = 0, \quad \frac{d}{dr} \hat{\psi}(1) + \left( \frac{\gamma^2 + 1}{\gamma^2 - 1} - i\sigma \right) \hat{\zeta} = 0.
\] (3.20a)

The outer boundary conditions become
\[
\hat{\psi} = -\hat{\psi}_w = 0 \quad \text{on} \quad r = \gamma.
\] (3.21)

We assume the vorticity \( \zeta \) is composed of a mean flow vorticity \( \bar{\zeta} = -\nabla^2 \Psi \) and a disturbance vorticity \( \zeta' = -\nabla^2 \psi' \) that is induced by oil whirl. The mean flow stream function \( \Psi \) results in a constant mean flow vorticity
\[
\bar{\zeta} = \frac{2}{1 - \gamma^2}.
\] (3.22)

If the stream function \( \psi \) is replaced by \( \Psi + \psi' \) and the higher-order terms are ignored in the vorticity equation (3.8) the linearized disturbance equation becomes
\[
\Re \left( \frac{\partial \zeta'}{\partial t} - \frac{1}{r} \frac{d}{dr} \left( \frac{1}{r} \frac{d \zeta'}{d\theta} \right) \right) = \frac{\partial^2 \zeta'}{\partial r^2} + \frac{1}{r} \frac{\partial \zeta'}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \zeta'}{\partial \theta^2}
\] (3.23)

Using a normal mode expansion for the disturbance vorticity function \( \zeta'(r, \theta, t) = \hat{\zeta}(r)e^{\sigma t - i\theta} \) the vorticity equation (3.23) can be simplified to
\[
\frac{\partial^2 \hat{\zeta}}{\partial r^2} + \frac{1}{r} \frac{\partial \hat{\zeta}}{\partial r} + \left[ -\Re \left( \sigma + i \frac{1 - (\gamma/r)^2}{\gamma^2 - 1} \right) - \frac{1}{r^2} \right] \hat{\zeta} = 0.
\] (3.24)

We define two complex variables \( \nu = \sqrt{1 - i \Re \gamma^2/(\gamma^2 - 1)} \) and \( \xi = \sqrt{-\Re \left[ \sigma + i (\gamma^2 - 1) \right]} \). When \( \Re \neq 0 \) \( \nu \) will be complex and the general solutions of \( \hat{\zeta}(r) \) are Bessel functions \( J_\nu(\xi \ r) \) and \( J_{-\nu}(\xi \ r) \). An appropriate general solution of the perturbation stream function \( \psi' \) can be obtained by integrating the Poisson equation of the disturbance vorticity
\[
\frac{d^2 \hat{\psi}}{dr^2} + \frac{1}{r} \frac{d \hat{\psi}}{dr} - \frac{1}{r^2} \hat{\psi} = C J_\nu(\xi \ r) + D J_{-\nu}(\xi \ r)
\] (3.25)
to give
\[
\dot{\psi}(r) = \frac{A}{r} + Br + \frac{r}{2\xi} \int_0^{2\xi} [CJ_\nu(\eta) + DJ_\nu(\eta)] d\eta - \frac{1}{2r\xi^3} \int_0^{2\xi} \eta^3 [CJ_\nu(\eta) + DJ_\nu(\eta)] d\eta
\]
(3.26)
where constants $A$, $B$, and $D$ are determined by the kinematic boundary conditions (3.20-3.21). When $Re \to 0$ the argument of the Bessel function $\xi$ is always small. Using a two-term ascending series approximation for small argument given in Abramowitz and Stegun for $J_\nu(\eta)$ and $J_\nu(\eta)$ leads to
\[
\dot{\psi}(r) = \frac{A}{r} + Br + C \left(\frac{\xi r}{2}\right)^\nu \frac{4(\nu + 5) r^3 - r^4 \xi^2}{4\Gamma(\nu) \nu(\nu + 1)(\nu + 3)(\nu + 5)}
\]
\[-D \frac{2}{4\Gamma(-\nu)} \frac{4(\nu - 5) r^2 + r^4 \xi^2}{4\Gamma(-\nu) \nu(\nu - 1)(\nu - 3)(\nu - 5)}.
\]
(3.27)
Viscous and pressure forces induced by the rotating shaft contribute to the subsequent shaft motion. The force exerted on the shaft surface is calculated by integrating stresses along the shaft surface. The dynamic boundary condition is nondimensionalized by choosing appropriate scales listed in Table 2? and $\rho R_0^2$ is selected as the mass scale in this section. The nondimensionalized dynamic equation is
\[
Re M \left(\frac{\tau}{\rho}\right) = \oint \left(\nabla \mathbf{V} + \nabla \mathbf{V}^T\right) - p \mathbf{\delta} \right] \cdot \mathbf{n} dA,
\]
(3.28)
where $M$ is the dimensionless mass of shaft per unit length and $\mathbf{\delta}$ denotes an identity tensor. The traction vector $\mathbf{t}$ is [Is this needed? What about $u$ and $v_\theta$]
\[
\mathbf{t} = \left[\begin{array}{c}
 \frac{2}{r} \frac{\partial u}{\partial r} - \frac{\partial p}{\partial r} \\
 \frac{1}{r} \frac{\partial u}{\partial \theta} \sin \phi + \frac{1}{r} \frac{\partial u}{\partial \theta} \cos \phi
\end{array}\right] \cdot \mathbf{n} dA.
\]
(3.29)
The normal vector on surface $\mathbf{n}$ is expressed as
\[
\mathbf{n} = \cos(\theta + \phi) \mathbf{i} + \sin(\theta + \phi) \mathbf{j} = \cos \phi \mathbf{e}_r + \sin \phi \mathbf{e}_\theta.
\]
(3.30)
A Taylor series expansion is applied to the integrand so that the integration is performed on the undisturbed rather than the disturbed surface. The perturbation pressure $p'$ is evaluated by integrating the Navier-Stokes equations with respect to $r$ or $\theta$, i.e.,
\[
\psi' = -i \left\{ r \hat{\psi}_{rr} + \hat{\psi}_r - \left[ \frac{2}{r} + Re \left( \sigma r - i \left( \frac{\gamma^2}{r^2} - 1 \right) \right) \right] \hat{\psi}_r + \left( \frac{2}{r^3} + i \frac{2 Re}{\gamma^2 - 1} \right) \hat{\psi} \right\} e^{e \phi - e \theta}.
\]
(3.31)
The $x$-component of the induced force on the disturbed shaft surface $r_x(\theta)$ is
\[
Re M \hat{x}' = \oint \left\{ -p \cos(\theta + \phi) + 2 u_r \cos(\theta - \phi) - \left[ r \left( \frac{u}{r} \right)_r + \frac{u_r}{r} \right] \sin(\theta - \phi) \right\} dA
\]
\[= i \pi e^{e t} \left( \hat{\psi}_{rr} - \left[ 3 + Re (\sigma - i) \right] \hat{\psi}_r + \left( 3 + i \frac{2 Re}{\gamma^2 - 1} \right) \hat{\psi} \right) - Re t \hat{x}' + O(r^2).
\]
Similarly the $y$-component of total pressure and viscous force is expressed as
\[
Re M \hat{y}' = \int_0^{2\pi} \left\{ -p \sin(\theta + \phi) + 2 u_r \sin(\theta - \phi) + \left[ r \left( \frac{u}{r} \right)_r + \frac{u_r}{r} \right] \cos(\theta - \phi) \right\} r_x(\theta) d\theta
\]
Dynamic Stability of Pressurized Journal Bearings

Table 1. Critical mass and whirling frequency for $Re = 1$ and given gap aspect ratios

<table>
<thead>
<tr>
<th>Clearance Ratio</th>
<th>0.01</th>
<th>0.05</th>
<th>0.10</th>
<th>0.50</th>
<th>1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Critical Mass</td>
<td>133.2</td>
<td>32.9</td>
<td>20.6</td>
<td>13.0</td>
<td>15.5</td>
</tr>
<tr>
<td>Frequency Ratio</td>
<td>0.495</td>
<td>0.476</td>
<td>0.453</td>
<td>0.308</td>
<td>0.200</td>
</tr>
</tbody>
</table>

$$ e^{ct} \left\{ \ddot{z} - \left[ 3 + Re (\sigma - i) \right] \dot{z} + \left( 3 + i \frac{2Re}{\gamma^2 - 1} \right) \dot{z} \right\} - Re \pi y' + O(y^2).$$

Adding the $x$-component to $i$ times the $y$-component and neglecting the higher-order terms gives

$$ Re (\sigma^2 M + \pi) \ddot{z} = i \pi \left\{ \ddot{z} - \left[ 3 + Re (\sigma - i) \right] \dot{z} + \left( 3 + i \frac{2Re}{\gamma^2 - 1} \right) \dot{z} \right\}. \quad (3.32)$$

According to the inner boundary condition (3.20), the complex displacement is $\ddot{z} = z = \frac{1}{1 + \frac{Re}{Re}}$. We acquire eigenvalues at a specified gap aspect ratio, Reynolds number, and unit length shaft mass. The eigenvectors associated with eigenvalues are used to assemble the perturbation solution after appropriate normalization.

The shaft is stable when the orbit radius decreases, corresponding to all eigenvalues having negative real parts. The stability of rotating shaft is controlled by the shaft mass for a given geometry and Reynolds number. A secant iteration is used to isolate the [define earlier] critical mass that leads to neutral stability in the most dangerous eigenmode at a specified geometry and Reynolds number. For example, at $\delta = 1$ and $Re = 11$, the shaft is stable if its mass per unit length is equal or less than 15.5. In this case, the shaft whirls at one-fifth of its rotating frequency. Table 1 lists the critical unit length shaft mass and whirling frequency ratios at given gap aspect ratios $\delta$. Reynolds number has almost no influence in determining the critical mass and whirling frequency ratio providing that the Reynolds number is small. Consequently, the formulation derived in the next section is sufficient to determine critical mass and orbiting frequency ratio for low Reynolds number flow. Nonetheless, (3.26) needs more terms in the ascending series approximating $J_{\sigma}(\xi)$ when the argument $\xi$ becomes large. This corresponds to moderate Reynolds number or small gap aspect ratio where the asymptotic series may be required (although not attempted here). For example, we tried to find the critical mass for $\delta = 0.001$ but the secant iteration diverges. For $\delta \geq 0.01$, the iteration converges and agrees well with those calculated in the next section.

The centrifugal force is the major stabilizing mechanism. The journal is unstable if it is neglected. Detailed derivation and information are available in Han?

3.2. Small Reynolds Number Expansion

The mass per unit length scale $\mu/\omega$ used by Schumack? results in a linear relation between the shaft critical mass and Reynolds number. Since we wish to expand in $Re$ in this section, $Re \mu/\omega = \rho R^2$ is suggested as a better mass scale. Using $\rho R^2$ as the mass scale leads to a $Re$ appearing on the left-hand side of the dynamic boundary condition. This result implies that a Reynolds number expansion about $Re = 0$ might be appropriate to study the shaft stability as $Re$ approaching zero.

The two-dimensional linearized disturbance vorticity equation (3.23) is chosen as the
governing equation. The basic state stream function $\Psi$ is given in (3.11) and the perturbation vorticity is denoted by $\zeta'$. We use a normal mode expansion in $t$ and $\theta$ to the perturbation stream function $\psi'$ and vorticity $\zeta'$, i.e.,

$$
\psi' = (\psi_0 + Re \, \psi_1 + \cdots) = \left(\dot{\psi}_0 + Re \, \dot{\psi}_1 + \cdots\right) e^{\sigma t - i\theta} 
$$

(3.33)

and the perturbation vorticity is denoted by $\zeta'$. We use a normal mode expansion in $t$ and $\theta$ to the perturbation stream function $\psi'$ and vorticity $\zeta'$, i.e.,

$$
\zeta' = (\zeta_0 + Re \, \zeta_1 + \cdots) = \left(\dot{\zeta}_0 + Re \, \dot{\zeta}_1 + \cdots\right) e^{\sigma t - i\theta}.
$$

(3.34)

The complex growth rate $\sigma$ approximated by $\sigma = \sigma_0 + Re \, \sigma_1 + \cdots$ and the complex displacement approximated by $z = z_0 + Re \, z_1 + \cdots$ are substituted into the vorticity equation (3.23). The coefficients of various order of $Re$ are required to satisfy the governing equations and boundary conditions of related power. The coefficient of the lowest-order vorticity equation is

$$
\frac{d^2 \dot{\zeta}_0}{dr^2} + \frac{1}{r} \frac{d \dot{\zeta}_0}{dr} - \frac{1}{r^2} \dot{\zeta}_0 = 0.
$$

(3.35)

The homogeneous vorticity equation is equidimensional and the general solution is

$$
\zeta_0 = \left( a \, r + \frac{b}{r} \right) e^{\sigma t - i\theta}.
$$

(3.36)

The definition of vorticity $\Gamma \zeta = -\nabla^2 \psi$ and (3.36) result in a Poisson equation for stream function $\psi_0$

$$
\frac{d^2 \dot{\psi}_0}{dr^2} + \frac{1}{r} \frac{d \dot{\psi}_0}{dr} - \frac{1}{r^2} \dot{\psi}_0 = -a \, r - \frac{b}{r}.
$$

(3.37)

The general solution of the Poisson equation is

$$
\dot{\psi}_0 = -\frac{a}{8} \, r^3 - \frac{b}{2} \, r \ln r + \left( \frac{b}{4} + c \right) \, r + \frac{d}{r}
$$

(3.38)

where $a, b, c, \Gamma$ and $d$ are constants to be determined by boundary conditions. Applying the linearized kinematic boundary conditions (3.20-3.21) the lowest-order kinematic boundary conditions is phrased as

$$
\dot{\psi}_0(1) - (1 + i \sigma_0) \, \dot{z}_0 = 0
$$

(3.39a)

$$
\frac{d \dot{\psi}_0(1)}{dr} + \left( \frac{\gamma^2 + 1}{\gamma^2 - 1} - i \sigma_0 \right) \, \dot{z}_0 = 0
$$

(3.39b)

$$
\dot{\psi}_0(\gamma) = 0 \quad \text{and} \quad \frac{d \dot{\psi}_0(\gamma)}{dr} = 0.
$$

(3.39c)

The lowest-order dynamic boundary condition is derived from equation (3.32)

$$
\frac{d^2 \dot{\psi}_0(1)}{dr^2} - 3 \frac{d \dot{\psi}_0(1)}{dr} + 3 \, \dot{\psi}_0(1) = 0.
$$

(3.40)

The condition that the determinant vanishes results in the lowest-order eigenvalue

$$
\sigma_0 = \frac{i}{\gamma^2 + 1}.
$$

(3.41)

Note that the ratio of shaft orbiting and spinning frequency is only determined by the geometry i.e. the radius ratio of journal to the sleeve. For $\gamma = 2\Gamma$ a wide gap journal bearing the frequency ratio is 0.2. As the gap ratio decreases i.e. in the lubrication limit $\gamma \rightarrow 1$ the whirling frequency approaches one-half of the shaft rotational frequency.
These lowest-order eigenvalues agree with our spectral element eigenvalue results developed in Section 4 and the half-frequency whirl observed in Hori for plain cylindrical journal bearings. Table 3 lists the lowest-order of the eigenvalues which is the ratio of orbiting to rotating frequency of selected radius ratios.

To get a unique solution for the under-determined system we impose an additional condition to normalize the eigenfunction

$$\hat{\psi}(1) = 1. \tag{3.42}$$

We solve the system for a given radius ratio $\gamma$ to obtain an eigenvector and assemble the eigensolution. Note that the shaft mass per unit length has no influence in determining the eigensolution at the lowest-order.

To find the effect of fluid inertia and shaft mass we proceed to the next order. Balancing the first-order coefficient of Reynolds number in the vorticity equation leads to

$$\sigma_0 \hat{\zeta}_0 + \frac{i}{r} \frac{d\Psi}{dr} \hat{\zeta}_0 = \frac{d^2 \hat{\zeta}_1}{dr^2} + \frac{1}{r^2} \hat{\zeta}_1 - \frac{1}{r} \hat{\zeta}_1, \tag{3.43}$$

which has the general solution

$$\hat{\zeta}_1(r) = c r + d \frac{r}{r} + a \frac{i \gamma^2 r (r^2 - 2 \gamma^2 \ln r - 2 \ln r + \gamma^2 + 1)}{4(\gamma^4 - 1)}$$

$$+ b \frac{i \gamma^2 (4 r^2 \ln r + 1 + \gamma^2 - 2 r^2 + 2 \ln r + 2 \gamma^2 \ln r)}{4 r (\gamma^4 - 1)}. \tag{3.44}$$

Solving the Poisson equation for the first-order stream function $\hat{\psi}_1(r)$ gives

$$\hat{\psi}_1(r) = c r + \frac{f}{r} - a \frac{i \gamma^2 (1 - 12 \gamma^2 \ln r - 12 \ln r + 15 + \gamma^2 + 2 r^2)}{192 \gamma^4 - 192}$$

$$- b \frac{i \gamma^2 r [4 \gamma^2 + 4 \gamma \ln r + 4 (\ln r)^2 + 4 \gamma^2 \ln r]}{32 \gamma^4 - 32} - g \frac{r^2}{8} + h \frac{1 - 2 \ln r}{4}, \tag{3.45}$$

where constants $a$ and $b$ are components of eigenvector determined by the lowest-order mode and $c$ and $d$ are unknowns in the first-order correction.

The first-order kinematic boundary conditions are

$$\hat{\psi}_1(1) - (1 + i \sigma_0) \hat{z}_1 - i \sigma_1 \hat{z}_0 = 0 \tag{3.46a}$$

$$\frac{d\hat{\psi}_1}{dr}(1) + \left( \frac{\gamma^2 + 1}{\gamma^2 - 1} - i \sigma_0 \right) \hat{z}_1 - i \sigma_1 \hat{z}_0 = 0 \tag{3.46b}$$

$$\hat{\psi}_1(0) = 0 \quad \text{and} \quad \frac{d\hat{\psi}_1}{dr}(0) = 0. \tag{3.46c}$$

The first-order dynamic boundary condition at $r = 1$ is expressed as

$$(M \sigma_0^2 + \pi) \hat{z}_0 = i \pi \left[ \frac{d^3 \hat{\psi}_1}{dr^3} - 3 \frac{d\hat{\psi}_1}{dr} - \frac{d\hat{\psi}_0}{dr} (\sigma_0 - i) + 3 \hat{\psi}_1 + \frac{2 i}{\gamma^2 - 1} \hat{\psi}_0 \right]. \tag{3.47}$$

We have five equations and six unknowns. To form a determinate system another equation in the form of the correction to the normalization condition (3.42) is used

$$\hat{\psi}_1(1) = 0. \tag{3.48}$$

The secant method is applied to search iteratively for the shaft critical mass that leads to a negligible real part in the most dangerous eigenvalue. The lowest-order influences
Table 2. Comparisons of critical mass and frequency ratio between Sections 3.1 and 3.2 for chosen clearance ratio

<table>
<thead>
<tr>
<th>Clearance Ratio δ</th>
<th>Critical Mass $M_c$</th>
<th>Frequency Ratio $\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Section 3.1</td>
<td>Section 3.2</td>
<td></td>
</tr>
<tr>
<td>0.010</td>
<td>133.2</td>
<td>0.495</td>
</tr>
<tr>
<td>0.100</td>
<td>20.6</td>
<td>0.453</td>
</tr>
<tr>
<td>1.000</td>
<td>15.5</td>
<td>0.200</td>
</tr>
</tbody>
</table>

the imaginary part determining the shaft whirling to rotation frequency ratio and the first-order affects the real part determining the growth rate of orbit radius.

Comparisons of the critical mass and the frequency ratio of whirling to rotation between Sections 3.1 and 3.2 are listed in Table 2 for selected clearance ratio $\delta = \gamma - 1$. These results are valid providing the Reynolds number is small. The computation using the small Reynolds number expansion is more efficient than using the Bessel function formulation in Section 3.1 and does not have any difficulty for small clearance ratio. Comparisons with the spectral element eigenvalue algorithm are very good and shown in Section 4.

4. A SPECTRAL ELEMENT METHOD FOR JOURNAL BEARINGS DYNAMIC STABILITY ANALYSIS

Most research regarding the stability of shaft movement inside a sleeve are conducted under the assumptions made by lubrication theory. However, lubrication theory lacks the ability to handle important factors that might stabilize shaft movement like fluid inertia and cavitation. When the gap clearances between cylinders are not small, one-dimensional lubrication theory becomes suspect. A spectral domain decomposition method pioneered by Schumack is implemented to both directly simulate the movement of a rotating journal inside a bearing and execute a linearized analysis of shaft dynamic stability. Unlike lubrication theory which ignores the effects of fluid inertia and finite thickness effects, almost every possible factor that might affect the dynamic stability of translating shaft are included in our studies except for cavitation and non-Newtonian effects which are not our current research interest. We concentrate on numerically solving the eigenvalue problem emerged from linearizing the momentum and force equations subject to appropriate boundary conditions by using spectral element methods in this section. A brief description of the direct time marching simulation of shaft movement is in Schumack.

We assume an infinitely long circular shaft is rotating and translating inside an incompressible fluid that is constrained by an infinitely long stationary sleeve. The gap clearance $c = R_o - R_i$ is selected as the length scale. The journal rotating speed is chosen as the velocity scale such that the shaft is rotating with unit angular velocity. A torque is supplied to the shaft so that the conservation of angular momentum is preserved. The aspect ratio of gap clearance to shaft radius is defined as

$$\delta = \frac{c}{R_i} = \frac{R_o}{R_i} - 1 = \gamma - 1.$$
We use a convective time scale \( \frac{c}{R_0 \omega} = \frac{\delta}{\omega} \) and a diffusive pressure scale \( \frac{\mu R_0 \omega c}{\mu} = \frac{\mu \omega}{\delta} \).

Reynolds number is defined as \( Re = \frac{\rho R_0 \omega c}{\mu} \). The frequency ratio of shaft whirl to shaft rotation \( \omega \) is

\[
\Omega = \frac{I(\sigma) R_0 \omega}{c} = \frac{I(\sigma)}{\delta},
\]

where \( I() \) indicates the imaginary part of a complex number. The shaft unit mass is scaled by \( c R_i = \frac{\mu \delta}{\omega} \) and the force is scaled by \( \mu R_i c \).

### 4.1. Eigenvalue Problem Formulation

We first study the problem where the shaft has an equilibrium position at the bearing center. Providing that the shaft deviates from its equilibrium center by a small displacement the dynamic stability is explored by linear stability analysis. Velocities and pressure are decomposed into basic state solution and perturbation components

\[
v = V + v', \quad p = P + p',
\]

where capital letters denote basic state and prime quantities represent disturbances. We calculate the basic steady state solution first. Then the perturbation velocity \( v' \) and perturbation pressure \( p' \) are approximated in terms of \( r, s, t \) as [What are \( h_i \)?] [give refs to Schumack and other earlier spectral element papers]

\[
\begin{align*}
    v'(r, s, t) &= e^{\sigma t} \sum_{n,m} u_{nm} h_n(r) h_m(s) \\
    p'(r, s, t) &= e^{\sigma t} \sum_{n,m} p_{nm} L_n(r) L_m(s),
\end{align*}
\]

where \( \sigma \) denotes the complex growth rate. In this section unless otherwise noted \( r \) is a mapped isoparametric coordinate and is not to be confused with the radial coordinate in the previous section. The fluid domain is described isoparametrically as

\[
\begin{align*}
x(r, s) &= \sum_{n,m} x_{nm} h_n(r) h_m(s) \\
y(r, s) &= \sum_{n,m} y_{nm} h_n(r) h_m(s).
\end{align*}
\]

We can linearize the momentum equations by substituting (4.1) into Navier-Stokes equations and cancelling the basic state equation and neglecting the higher-order terms. This yields

\[
Re (\sigma v' + V \cdot \nabla v' + v' \cdot \nabla V) = -\nabla p' + \nabla^2 v'.
\]

The variational form of inner product between momentum equations and the vector test function \( \mathbf{T} = T(r, s) \mathbf{e}_r + T(r, s) \mathbf{e}_s \), where \( T(r, s) = h(r) h(s) \Gamma \) is

\[
\int_{\Omega} \left( \sigma v' + V \cdot \nabla v' + v' \cdot \nabla V \right) \cdot \mathbf{T} - p \nabla \cdot \mathbf{T} + \nabla v' \cdot \nabla T + \nabla v' \cdot \nabla T \, d\Omega = 0.
\]
We evaluate the integration of this equation on a term by term basis. The variational
viscous term \( \int_\Omega \nabla u' \cdot \nabla T + \nabla v' \cdot \nabla T \, d\Omega \) is converted to
\[
\int_{-1}^{1} \int_{-1}^{1} \frac{1}{|J|} \left\{ \left[ \frac{\partial u'}{\partial r} \frac{\partial y}{\partial s} - \frac{\partial y}{\partial y} \frac{\partial u'}{\partial r} \right] \left[ \frac{\partial T}{\partial r} \frac{\partial y}{\partial s} - \frac{\partial y}{\partial x} \frac{\partial T}{\partial r} \right] + \left[ \frac{\partial u'}{\partial r} \frac{\partial x}{\partial s} - \frac{\partial x}{\partial y} \frac{\partial u'}{\partial r} \right] \left[ \frac{\partial T}{\partial s} \frac{\partial x}{\partial s} - \frac{\partial x}{\partial r} \frac{\partial T}{\partial s} \right] \right\} \, dr \, ds
\]
which is discretized using Gauss-Lobatto quadrature to \( A_{ij\,kl}(u_{kl} + v'_{kl}) \). The Jacobian \( J \) is defined as
\[
J = \frac{\partial x}{\partial r} \frac{\partial y}{\partial s} - \frac{\partial x}{\partial s} \frac{\partial y}{\partial r}.
\]  

(4.6)

The discretization of the unsteady term \( \sigma \int_\Omega \mathbf{v}' \cdot \mathbf{T} \, d\Omega \) is expressed as
\[
\sigma \int_{-1}^{1} \int_{-1}^{1} \mathbf{v}' \cdot \mathbf{T} |J| \, dr \, ds
\]
which is simplified to \( \sigma B_{ij\,kl}(u_{kl} + v'_{kl}) \). The pressure term \( -\int_\Omega p' \nabla \cdot \mathbf{T} \, d\Omega \) is converted to
\[
- \int_{-1}^{1} \int_{-1}^{1} p' \left[ \frac{\partial T}{\partial r} \frac{\partial y}{\partial s} - \frac{\partial y}{\partial x} \frac{\partial T}{\partial r} + \frac{\partial T}{\partial s} \frac{\partial x}{\partial s} - \frac{\partial x}{\partial r} \frac{\partial T}{\partial s} \right] \frac{|J|}{J} \, dr \, ds,
\]
which is discretized and simplified to \( -(C_{ij\,kl} + C_{j\,kl}) p'_{kl} \).

The fluid inertia terms \( \int_\Omega (\mathbf{V} \cdot \nabla \mathbf{v}' + \mathbf{v}' \cdot \nabla \mathbf{V}) \cdot \mathbf{T} \, d\Omega = 0 \) is converted to
\[
\int_{-1}^{1} \int_{-1}^{1} \left\{ \left[ \frac{\partial \mathbf{V}}{\partial r} \frac{\partial y}{\partial s} + \frac{\partial \mathbf{V}}{\partial s} \frac{\partial y}{\partial r} \right] \left[ \frac{\partial \mathbf{T}}{\partial r} \frac{\partial y}{\partial s} + \frac{\partial \mathbf{T}}{\partial s} \frac{\partial y}{\partial r} \right] + \left[ \frac{\partial \mathbf{V}}{\partial r} \frac{\partial x}{\partial s} + \frac{\partial \mathbf{V}}{\partial s} \frac{\partial x}{\partial r} \right] \left[ \frac{\partial \mathbf{T}}{\partial s} \frac{\partial x}{\partial s} + \frac{\partial \mathbf{T}}{\partial s} \frac{\partial x}{\partial r} \right] \right\} \frac{|J|}{J} \, dr \, ds.
\]
The continuity equation for incompressible flow is multiplied with a scalar test function \( q(r,s) = L(r) L(s) \) and integrated to get \( \int_\Omega \nabla \cdot \mathbf{v}^\prime \, d\Omega = 0 \). This is converted to
\[
\int_{-1}^{1} \int_{-1}^{1} q \left[ \frac{\partial \mathbf{v}^\prime}{\partial r} \frac{\partial y}{\partial s} - \frac{\partial \mathbf{v}^\prime}{\partial s} \frac{\partial y}{\partial r} + \frac{\partial \mathbf{v}^\prime}{\partial x} \frac{\partial x}{\partial r} - \frac{\partial \mathbf{v}^\prime}{\partial r} \frac{\partial x}{\partial s} \right] \left[ \frac{J}{J_0} \right] \, dr \, ds
\]
and discretized to \( D_{ijkl} u_{k1}^\prime + D_{ijkl} v_{l1}^\prime = \).

The discretized Navier-Stokes equations and continuity equation are now expressed as
\[
\begin{align*}
\text{Re} \, \sigma \, B_{ijkl} u_{k1}^\prime + \text{Re} \, E_{ijkl}^+ u_{k1}^\prime + A_{ijkl} u_{k1}^\prime - C_{ijkl} p_{k1}^\prime &= 0 \quad (4.7a) \\
\text{Re} \, \sigma \, B_{ijkl} v_{l1}^\prime + \text{Re} \, E_{ijkl}^- v_{l1}^\prime + A_{ijkl} v_{l1}^\prime - C_{ijkl} p_{k1}^\prime &= 0 \quad (4.7b) \\
D_{ijkl} u_{k1}^\prime + D_{ijkl} v_{l1}^\prime &= 0, \quad (4.7c)
\end{align*}
\]
To specify a unique pressure solution the additional pressure constraint is applied:
\[
\int_\Omega p^\prime \, d\Omega = 0. \quad (4.8)
\]
The basic state solution is calculated and used to construct \( E_{ijkl}^+ \) and \( E_{ijkl}^- \). The spectral coefficients \( u_{k1}^\prime, v_{l1}^\prime, \) and \( p_{k1}^\prime \) are unknowns that must be determined by the discretized equations as well as the boundary conditions.

Since the shaft translates inside the sleeve, four new scalar unknowns, i.e. the displacement of the deviating shaft \((x^\prime, y^\prime)\) and the translating velocity \((\dot{x}^\prime, \dot{y}^\prime)\) are introduced. Therefore, four more equations are needed to generate a determined system. Two of the equations are from the assumptions of normal mode expansion, i.e.
\[
\begin{align*}
\dot{x}^\prime &= \frac{dx^\prime}{dt} = \Re (\sigma \, x^\prime) \quad (4.9a) \\
\dot{y}^\prime &= \frac{dy^\prime}{dt} = \Re (\sigma \, y^\prime). \quad (4.9b)
\end{align*}
\]
The other two equations are from Newton’s second law of motion, i.e.
\[
\begin{align*}
F_x^\prime &= M \frac{d\dot{x}^\prime}{dt} \quad (4.10a) \\
F_y^\prime &= M \frac{d\dot{y}^\prime}{dt}. \quad (4.10b)
\end{align*}
\]
the real part of the most dangerous eigenvalue changes sign. and unstable if it is positive. The critical shaft mass shaft whirl to shaft spin frequency. The shaft is stable if the real part of determines the growth rate of shaft orbit radius and the imaginary part is the rate of shaft whirl to shaft spin frequency. Given the journal bearing geometry, Reynolds number, shaft mass per unit length, we can compute the complex eigenvalue \( \sigma \). The real part of the most dangerous eigenvalue changes sign.
Dynamic Stability of Pressurized Journal Bearings

Table 3. Comparisons between perturbation methods and spectral element eigenvalue algorithms for $Re = 0$ and selected clearance ratio in concentric cylinders

<table>
<thead>
<tr>
<th>Clearance Ratio</th>
<th>Critical Mass Perturbation</th>
<th>Critical Mass Eigenvalue</th>
<th>Frequency Ratio Perturbation</th>
<th>Frequency Ratio Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>1264.0</td>
<td>1261.47</td>
<td>0.4995</td>
<td>0.49955</td>
</tr>
<tr>
<td>0.005</td>
<td>258.89</td>
<td>258.865</td>
<td>0.4975</td>
<td>0.49754</td>
</tr>
<tr>
<td>0.010</td>
<td>133.25</td>
<td>133.239</td>
<td>0.4950</td>
<td>0.49506</td>
</tr>
<tr>
<td>0.100</td>
<td>20.636</td>
<td>20.6354</td>
<td>0.4525</td>
<td>0.45251</td>
</tr>
<tr>
<td>0.500</td>
<td>13.040</td>
<td>13.0401</td>
<td>0.3077</td>
<td>0.30771</td>
</tr>
<tr>
<td>1.000</td>
<td>15.519</td>
<td>15.5183</td>
<td>0.2000</td>
<td>0.20002</td>
</tr>
</tbody>
</table>

4.2. Hydrodynamic Stability of Concentric Cylinders

The Schumack algorithm has optimal performance with square $N \times N$ truncations for single elements if the corner singularities do not exist in the flow field. In our numerical experiments with various degrees of freedom for a specified accuracy, we observe that larger $N$ is required when the aspect ratio of the gap clearance to shaft radius departs from 0.1. [Do you mean 1.0?] When the fluid inertia is considered (no matter how small the magnitude of the Reynolds number is), the whirling shaft becomes conditionally stable due to centrifugal pressure. A comparison between the perturbation method developed in Section 3.2 and the spectral element eigenvalue algorithm developed in this section is presented in Table 3. The results of Table 3 agree with the perturbation analysis to the number of digits shown.

Figure 3 illustrates the influence of the clearance ratio $\delta$ on shaft critical mass for $Re = 1$. Journals with mass per unit length below the curve have negative growth rates while those above the curve are unstable to oil-whirl instability. A minimal critical shaft mass is observed around $\delta \approx 0.45$ implying the system is most unstable when the bearing radius is about 50% larger than that of journal. When the gap becomes much smaller or larger than the shaft radius, the shaft appears to achieve unconditional stability. The whirling to spinning frequency ratio $\Omega$ versus the gap ratio $\delta$ at $Re = 1$ is also shown. The shaft orbiting frequency is nearly half of its spinning frequency for a very small gap aspect ratio $\delta$ and approaches zero for a large gap ratio. These curves seem independent of $Re$ when $Re \leq 1$. Above this value, the critical mass decreases and the frequency ratio increases as Reynolds number increases. For low Reynolds number flow, the geometry appears to be the primary factor in determining the orbiting frequency rather than fluid inertia.

4.3. Hydrodynamic Stability of Eccentric Cylinders

The equilibrium position of the journal is determined by the load and the shaft spinning velocity. Hydrodynamic stability of loaded eccentric cylinders for both wide and small gaps are investigated in this section. As expected, higher resolution is required for
eccentric cylinders than for the concentric case. Most computations are based on a 19×19 Gaussian-Lobatto grid similar to Figure 2.

The influence of Reynolds number on the critical mass and the frequency ratio of an eccentric spinning journal is negligible in the small gap case. The clearance ratio is $\delta = 0.01$ and the eccentricity is $\epsilon = 0.1$. In contrast, the eccentricity ratio has a strong effect on the shaft critical mass. Figure 4 shows the shaft critical mass increases rapidly as the eccentricity ratio $\epsilon$ increases for both gap aspect ratios at $\delta = 0.01$ and $\delta = 1$. This indicates that increasing eccentricity (and hence load) improves the stability of spinning journal for any specified gap aspect ratio. This observation agrees with the comments
Neale [1] and Pan [2] that lightly loaded journals are more likely to whirl. Higher resolution is required as the eccentricity increases. For example, a $17 \times 17$ Gauss-Lobatto quadrature grid is sufficient for $\epsilon = 0.001$ but a $21 \times 21$ mesh is required for $\epsilon = 0.3$ at $\delta = 1$ to obtain $M_c$ to an accuracy of $10^{-4}$. The frequency ratio is almost constant for low eccentricity. However, the ratio decreases significantly after the eccentricity exceeds 0.1.

The effect of the clearance ratio $\delta$ on the shaft critical mass are illustrated in Figures 5 for different eccentricity ratio $\epsilon$ at $Re = 1$. A $20 \times 20$ mesh is required in our computation due to the eccentricity. For either concentric slightly eccentric or moderately eccentric cylinders, the lowest shaft critical mass exists at $\delta \approx 0.45$ where the whirling journal suffers the worst instability. The critical mass for the slightly eccentric case $\epsilon = 0.01$ is almost identical to that for concentric cylinders (and the plotted lines are almost coincident). As the eccentricity increases from $\epsilon = 0$ to $\epsilon = 0.1$, the critical mass increases (apparently by a constant factor of 1.5). The frequency ratio for concentric and eccentric cylinders are demonstrated for $Re = 1$ and various gap aspect ratio. The frequency ratio has a maximum of 0.5 when the clearance ratio $\delta \rightarrow 0$. Low eccentricity $\epsilon = 0.05$ has a nearly identical frequency ratio as the concentric cylinders. Only as the eccentricity increases close to unity does the frequency ratio decrease slightly.

4.4. *Hydrodynamic Stability of Elliptical Bearings*

Elliptical bearings are used to reduce the hydrodynamic instability. We study elliptical bearings in this section to demonstrate the capability of our spectral element eigenvalue algorithm for a complicated geometry. Two spectral elements are essential to explore concentric elliptical bearings due to the corner singularity. We investigate the concentric case only because using two elements instead of one significantly increase memory requirements. On each element, a $17 \times 17$ Gaussian-Lobatto quadrature grid similar to Figure 2 is used. A four-element grid might be required for small clearance ratio elliptical bearings.

The effect of bearing ellipticity $\xi$ on the shaft critical mass is illustrated in Figure 6. The shaft critical mass increases as the ellipticity increases. This indicates the hydrody-
Dynamic stability of the journal improves as the bearing becomes more elliptic. The dual high pressures in the convergent regions help stabilize a whirling shaft by balancing the forces in the angular direction. The orbiting frequency ratio decreases from 0.21 which is the frequency ratio of concentric cylinders of $\delta = 1$ to 0.14 at ellipticity $\xi = 1$.

The whirling journal cannot be stabilized when the fluid inertia is not considered, i.e. $Re = 0$ in a non-cavitating flow except when the shaft mass per unit length is negligible. In this special case the shaft is neutrally stable. The fluid inertia leads to the centrifugal pressure which is an important stabilizing mechanism. The magnitude of the fluid inertia is not critical as long as it exists. The influence of Reynolds number on the critical mass and the frequency ratio is negligible for small Reynolds number. Reynolds number is not a factor in decreasing the critical mass or increasing the frequency ratio until it exceeds $Re = 1$. As the Reynolds number exceeds unity the shaft critical mass decreases. This indicates the whirling shaft becomes less stable if we increase the angular velocity after it reaches a threshold speed or if we increase the clearance ratio. The fluid inertia actually destabilizes the shaft for moderate Reynolds number.

5. Results and Comparisons

Comparing results between the perturbation methods and the lubrication theory of a concentric journal bearing case we find the rotating shaft is always unstable providing that the fluid inertia effect is ignored i.e. $Re \equiv 0$. However taking the limit as $Re \to 0$ allows a finite critical mass of the shaft to be obtained. The whirling shaft is neutrally stable if its mass per unit length and the fluid inertia are both negligible. This conclusion agrees with Myers and Poritsky. They applied lubrication theory without including fluid inertia to study vertical concentric cylinders. The eigenvalues we calculate using the perturbation methods also agree with our eigenvalue spectral element algorithm without the fluid inertia effect.

When fluid inertia is considered in a no-load (concentric) journal bearing case the rotating shaft is stable if the shaft mass per unit length is less than the critical value $M_c$. The influence of ellipticity ratio on critical mass and frequency ratio at $\delta = 1$ and $Re = 1$ (elliptical bearings)
even as $Re \to 0$. Excellent agreement of stability boundary for shaft critical mass at different clearance ratios exists between the Bessel function formulations of Section 3.1, the small Reynolds number expansions of Section 3.2, and the spectral element eigenvalue algorithm of Section 4 as demonstrated in Figure 7. In addition, the ratio of shaft whirl to rotation frequencies between these methods agree well as illustrated. Numerical comparisons of shaft critical mass and frequency ratio between the perturbation methods and the spectral element eigenvalue algorithm are listed in Table 3 for small Reynolds number and chosen clearance ratio. It is widely observed in practice that half-frequency resonant whirl usually happens when the shaft is lightly loaded or operated at very high speed. Our calculations show that the shaft is whirling at nearly half of the shaft spinning velocity when the gap aspect ratio is very close to zero.

Brennen investigated the dynamic stability of journal in the annulus flow using perturbation methods to solve the full equations analytically. Even though he failed to calculate any accurate numerical solution nor conclude whether the rotating shaft is stable or not, the asymptotic force coefficients are obtained in his analysis. The validity of his force coefficients is still under evaluation. Brennen suggested a dominant viscous drag could damp the shaft whirl motion in a large gap bearing. A sub-synchronous whirl frequency is possible under his investigation of general gap journal bearing. However, in his formula to evaluate the force exerted on the shaft surface, he appears to consider normal and shear stresses only without pressure. Unbalanced pressure is considered as a primary factor that causes oil whirl since the force in angular direction is usually non-zero. Lubrication theory evaluates force on the shaft surface by integrating only pressure along the surface since pressure usually is much larger than both normal and shear stresses.

For loaded or eccentric cylinders, a comparison between the eigenvalue spectral element results and Collins et al. is made by Schumack. Collins et al. used a perturbation study of the linear stability in the context of lubrication theory to evaluate the shaft stability with the fluid inertia effect. They conclude that the journal is conditionally stable if the fluid inertia is considered in the momentum equations. The neutral stability...
curve and operating curve compare well with Schumack's eigenvalue result. We exercise
the eigenvalue spectral element code and find that increasing the shaft eccentricity can
significantly stabilize an orbiting journal as shown in Figures 4. Hollis et al. and Lund et al.
showed that the shaft is unconditionally stable when the shaft eccentricity is greater
than 0.8 qualitatively agreeing with our observation. We did not attempt to calculate
oil whirl for $\xi > 0.3$ since the required storage is too large for these high resolution cases.

A time-marching spectral element algorithm developed by Schumack shows that concentric cylinders are stable providing the shaft mass per unit length is lower than a
critical value. Nonetheless the critical mass obtained from the time-marching code is
about 30% to 50% higher than what we obtained from the eigenvalue code and perturbation
analyses. For example if $Re = 1$ and gap aspect ratio $\delta = 1$ the critical mass is
15.5 from the eigenvalue code and 19.5 from the time-marching code. The ratio of shaft
whirl to shaft spin frequencies is 0.2 from the eigenvalue code and 0.3 from the time-
marching code. We are still improving the time-marching code. But to date no changes
have been made to the algorithm that we now feel predicts incorrect stability boundaries.
An investigation of the hydrodynamic stability of eccentric cylinders by Gwynllyw et al.
using a time-marching spectral element algorithm predicts the journal is unconditionally
unstable for a non-cavitating and constant viscosity lubricant in contrast to our results.

Non-circular bearings are used in industry to reduce shaft instability although their
load capacities are considerably lower than plain circular bearings. Due to the comp-
plicated geometry of elliptical bearings have not often been analyzed in the past except
by lubrication theory. Pinkus pioneered a simple finite difference computation of the
Reynolds equation and ended up with qualitatively acceptable results for the oil flow
coefficient and power loss factor compared to experiments. Our spectral element eigen-
value code shows that increased ellipticity of elliptical bearings can stabilize the whirling
shaft for the concentric case as shown in Figure 6. We also tested the dependence of
shaft critical mass on Reynolds number for concentric cylinders, eccentric cylinders and
elliptical bearings. We find the critical mass is almost independent of Reynolds number
providing it is small but as mentioned previously is required to be non-zero.

6. Concluding Remarks

According to our studies the fluid inertia plays an important role in stabilizing the
whirling shaft. Both perturbation analysis and numerical computation agree that the
shaft becomes unconditionally unstable if the fluid inertia is set identically equal to zero.
The centrifugal pressure which is the primary stabilizing mechanism for a non-cavitating
journal bearing disappears from the dynamic boundary condition when the fluid inertia
is not considered. Lubrication theory which ignores fluid inertia may experience some
difficulties and acquire irrational conclusions.

Increasing the ellipticity of the bearing will increase the shaft critical mass and stabilize
oil whirl. The range of pressure in the flow field can be significantly reduced in the
elliptical or three-lobe bearings; dramatically improving the instability of a translating
rotor. The clearance ratio $\delta$ has important effects on both shaft critical mass and whirling
frequency. Once the gap becomes small or large compared to the shaft radius it improves
the dynamic stability of rotating shaft. The eccentricity of journal bearing is determined
by the static load. As the eccentricity increases the critical mass per unit length increases
rapidly; stabilizing the whirling shaft.

The frequency ratio of journal whirl to rotation frequency is primarily determined by
the geometry of journal bearings rather than the fluid inertia. The critical mass (for unit length of shaft) is constant for Reynolds number in all of the geometries we explored in Section 4 providing the fluid inertia is small but finite. Higher resolution is required as the gap aspect ratio approaches extremes or the eccentricity approaches unity.

This work was sponsored by National Science Foundation contracts ECS-90012263 and OCE-9119459 and Department of Energy contract KC-070101. Most of the computations were performed on workstations of the Computer Aided Engineering Network at University of Michigan.

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