Taylor vortices between elliptical cylinders

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Critical parameters (Reynolds number and wave number) signaling the onset of Taylor vortices are calculated for the flow between "elliptical" cylinders. The spinning inner cylinder is circular; the stationary outer cylinder is composed of two circular arcs and is similar to an ellipse. It is shown that increasing ellipticity destabilizes the flow and increasing eccentricity stabilizes the flow. The spectral element method is used to calculate the base flow and to solve the linear stability problem. They showed that increasing eccentricity stabilizes the flow. Subsequent papers by DiPrima and Stuart and Eagles et al. investigated the nonlinear characteristics of Taylor vortices between eccentric circular cylinders. Oikawa et al. numerically investigated the linear stability of the flow between eccentric circular cylinders without the restrictions of small eccentricity or clearance. Oikawa's results show good agreement with the DiPrima and Stuart analysis for small eccentricity and clearance. They also agree well with experiments by Vohr and Karasudani.

There are no analytical or numerical studies of the stability of flows between noncircular cylinders. Lewis solved for the base flow between a rotating circular inner cylinder and fixed square outer cylinder using finite differences. Snyder, Mullin, and Terada and Hattori experimentally studied flow stability for a variety of inner/out outer noncircular cylinder arrangements and characterized the bifurcation to Taylor vortices for the various geometries. The two nondimensional parameters describing the concentric elliptical geometry are the ellipticity ratio, \( e^* = e/d \), and the minimum gap radius ratio, \( \eta = R/(R_i + d) \), where \( e \) is the distance between the inner cylinder center and the center of the circle describing either lobe of the outer cylinder, \( d \) is the minimum gap width, and \( R_i \) is the inner cylinder radius (see Fig. 1).

We use the spectral element method to solve the base flow and linear stability problems. For the two-dimensional base flow, we solve the incompressible Navier–Stokes equations in a Cartesian coordinate system:

\[
\text{Re} \left( \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} (\mathbf{u} \nabla \mathbf{u} + \nabla \mathbf{u}^T) \right) = \nabla^2 \mathbf{u} - \text{Re} \nabla p, \tag{1}
\]

\[
\nabla \cdot \mathbf{u} = 0. \tag{2}
\]
The Reynolds number is $Re = \alpha R_d / \nu$, where $\nu$ is the kinematic viscosity. The skew-symmetric form for the nonlinear terms is used to minimize aliasing errors. The unknowns within an element are approximated in a series of orthogonal polynomials:

$$u = \sum_{m=0}^{N} \sum_{n=0}^{N} u_{mn} h_m(r) h_n(s)$$

and

$$p = \sum_{m=0}^{N-2} \sum_{n=0}^{N-2} p_{mn} L_m(r) L_n(s),$$

where $h_m(r)$ is the Lagrangian interpolant of order $N$ constructed from Legendre polynomials, $L_m(r)$ is the Legendre polynomial of order $m$, $u_{mn}$ is the velocity at the $mn$th grid point, $p_{mn}$ is the $mn$th spectral coefficient for pressure, and $r$ and $s$ are the computational coordinates related to the Cartesian coordinates by $Gx = x$, $Gy = y$, $Gz = z$.

For the stability analysis, we assume small perturbations of the form

$$\begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix} = \begin{bmatrix} U \\ V \\ 0 \\ P \end{bmatrix} + e^{\sigma t} \begin{bmatrix} \hat{u} \cos kx \\ \hat{v} \cos kx \\ \hat{w} \sin kx \\ \hat{p} \cos kx \end{bmatrix},$$

where $k$ is the real wave number along the $z$ axis and $\sigma$ is (in general) a complex growth rate. Substituting (6) into the nonconservative form of the three-dimensional Navier-Stokes equations and linearizing yields

$$\text{Re}(\sigma \hat{u} + U \cdot \nabla \hat{u} + \hat{u} \cdot \nabla U) = -\text{Re} \nabla \hat{p} + \nabla^2 \hat{u} - k^2 \hat{u},$$

$$\text{Re}(\sigma \hat{v} + U \cdot \nabla \hat{v}) = \text{Re} k \hat{p} + \nabla^2 \hat{v} - k^2 \hat{v},$$

$$\nabla \hat{u} + k \hat{u} = 0.$$
As with eccentric circular cylinders, increasing eccentricity stabilizes the flow and results in smaller vortices. For given eccentricity, ellipticity, and $\eta$, the flow is more stable for angles that result in smaller clearances between the two cylinders.

We found no Hopf bifurcation for the elliptical geometry results presented here. However, since Hopf bifurcations do occur for eccentric cylinders with small gap and high eccentricity, we suspect that at large enough values of $\eta$ and $e^*$ a Hopf bifurcation will also occur for elliptical geometries. Computer resource limitations prevented us from trying the necessary higher truncations. Perhaps a mapping that stretches the radial coordinate in the narrowest parts of the flow field would allow solutions with reasonable truncations.

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