Elements of Heat Transfer Analysis

(A Brief Introduction to Heat Transfer)

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1. Introduction

A. General discussions

These lectures constitute a compact presentation of elementary heat transfer analysis, intended to accompany the study of thermodynamics in a first course on thermal sciences in mechanical engineering. Students with a good understanding of these pages and willing to use some deductive reasoning should be able to perform effective analyses of heat transfer for many engineering purposes. Occasional reference to more extensive property tables, charts and formulas in a dedicated textbook on heat transfer or handbook may be helpful for some problems [Incropera & DeWitt 1996, Mills 1992, Ozisik 1985][1][2]. Most of the methodologies illustrated here are algebraic. Calculus and differential equations are used sparingly, primarily to help students gain familiarity and benefit from the clarity and physical insight they bring to analysis, and to indicate what more advanced analysis might entail.

Heat transfer is concerned with the study of the transport of energy due to temperature differences. As engineers, we are more interested in the rate of heat transfer, denoted in these notes by $\dot{Q}$ [unit watts, or W, which is J/s], rather than in the amount of heat energy transferred[3]. In addition, we will also find it convenient to work with the *heat transfer rate per unit area* $\dot{\mathbf{q}}$, [W/m^2], which we will call the *heat flux*. Heat flux is clearly a vector, since it has a magnitude *and* a direction. For many purposes the direction is obvious by context, so we will find it adequate to work with the scalar version of heat flux, denoted by $\dot{q}$. Student should make an effort to distinguish these from the amount of heat transfer Q [J] and heat transfer per unit mass q [J/kg] used in the thermodynamic discussions of this course.

Heat transfer takes place in one of three modes: *conduction, convection* and *radiation*. We will briefly discuss each of these modes in this section. Practical problems of heat transfer frequently involve more than one component and more than one mode. The basic skills for dealing with multi-component, multi-mode heat transfer will be discussed in Section 2. More detailed discussions for basic calculations for each of the three modes will be discussed in the

[1] Square brackets[ ] with name(s) and year are keys to list of references. Otherwise [ ] are used indicate units and dimensions, as indicated in the next footnote.

[2] Another abbreviated introduction to heat transfer, with different emphasis, can also be found in Chapter 12 of Sonntag and Borgnakke, Introduction to Engineering Thermodyanics, Wiley, 2001, the current textbook for ME230.

[3] In some heat transfer books Q is used to denote the heat transfer rate [W] and q'' is used to denote heat flux [W/m^2].

next three sections, to be followed by other sections that will develop our skills further and introduce a few additional concepts.

B. Conduction Heat Transfer

Conduction is heat transfer due to temperature non-uniformities in a continuous substance, either stationary or in motion. Consider the plane slab shown in Fig. 1. Observations supplemented by common sense reasoning suggest that the heat transfer rate $\dot{Q}$ [W] must be proportional to the temperature difference $\Delta T \equiv |T_1 - T_1|$ [K], cross-sectional area A [m^2], and the inverse of the thickness B [m].

$$\dot{Q} = Ak\frac{\Delta T}{B} \tag{1}$$

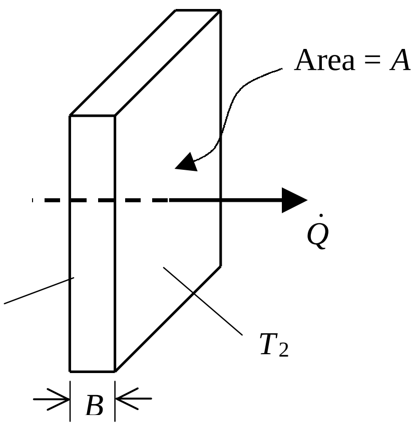

Fig. 1 Heat conduction through a plane slab

The coefficient of proportionality k is a material property, called the *thermal conductivity*. Unit consistency requires that the unit of k be [W/mK]. More careful analysis would indicate that a more general way to write this equation is

$$\dot{\mathbf{q}}_{\text{cond}} = -k\nabla T \tag{2}$$

where ∇T denotes the *temperature gradient*. This is a vector whose direction and magnitude are the direction and magnitude of the steepest slope of temperature increase. The negative sign is necessary since heat transfer only proceed from in the direction of decreasing temperature. The x-component of both sides of Eq. (2) is

$$\dot{q}_x = -k\frac{\partial T}{\partial x} \tag{3}$$

Similar equations can be obtained for the y- and z- components.

Eq. (2) is known as the Fourier's Law of Heat Conduction. The algebraic form, Eq. (1) can be used as an approximation for heat transfer through plane layers or thin non-planar layers in steady state if k is relatively uniform in the layer.

Since in general k is a material property[4], it is a function of T and p. In most cases, however, the pressure dependence is small and k depends primarily on temperature.

Typical thermal conductivities for several classes of material are shown in Table 1. It is advisable for students to be familiar with the relative magnitudes of thermal conductivities of different classes of material or representative values, to facilitate quick decisions in analyses or design. Other properties in in Table 1 will be discussed in later sections.

Table 1

Selected Thermal Properties

Material, at temperature(s)	k @ T's shown W/mK	ρc_p @ 300 K 10^6 J/m^3	$\alpha \equiv k/\rho c_p$ @ 300 K 10^{-6} m^2/s
Diamond, at 300, 400K	2300, 1540	1.78	949
Aluminum oxide, at 300, 600 K	46, 19	3.04	15.1
Metals			
Pure copper, at 300 K	401	2.44	225
Brass, a copper alloy, at 300, 600	110, 149	3.24	33.9
Aluminum alloy 2024-T6, at 300, 600 K	177, 186	2.42	77.0
Carbon steel, AISI 1010, at 300, 1000 K	60.5, 30.0	3.40	17.8
Stainless steel, AISI 302, at 300, 1000 K	15.1, 25.4	3.87	3.9
Common Non-metallic Materials			
Plate glass, at 300 K	1.4	1.88	0.747
Plastics – bakelite, at 300 K	1.4	1.90	0.735
Plastics – Teflon, at 300 K	0.35	--	--
Wood, common types, at 300 K	0.1-0.2	--	--
Glass fiber insulation, at 300 K	0.046	--	--
Urethane foam insulation, at 300 K	0.026	--	--
Liquids			
Water, at 300, 400 K	0.613, 0.688	4.18	0.147
Engine oil at 300 K	0.14	1.69	0.083
Gases			
Helium, at 300, 600 K	0.152, 0.252	0.000844	180
Air, at 300, 600 K	0.0263, 0.0469	0.001004	22.5

If better sources are not available, approximate conductivities at intermediate temperatures for solids and liquids can be obtained by linear interpolation. Interpolation, and modest extrapolation, for gas conductivities can best be done by assuming log$k$-versus-log$T$ as linear.

Note that k's for common engineering metals vary between 15 – 200 W/mK, with stainless steel near the bottom and aluminum alloys near the top. Pure metals have much higher conductivities than their own alloys, but are of no engineering significance.

Example 1 Basic Conduction

[4] The exceptions are very dilute gases or very thin (nanometer scale) layers, for which molecular-scale effects become important.

A hot water heater has an exposed surface area of 1.8 m^2. After it is insulated with 1 inch of fiberglass insulation, it is found that the inside surface of the insulation is at 120°F, and outside surface is at 90°F. Determine the heat loss through the insulation.

Solution:

ΔT = 120°F – 90°F = 30 F = 30*5/9=16.67 K

A = 1.8 m^2

B = 1 in = 1*0.0254=0.0254 m

k = 0.046 W/mK (best information conveniently available, from Table 1)

Hence

$\dot{Q} = Ak\,\Delta T/B$ = 1.8*0.046*16.67/0.0254 [m^2][W/mK][K/m] = 54.34 W

Students are strongly recommended to pay attention to the following in doing homework:

1. All quantities not in SI units are first converted to SI units to insure unit consistency.
2. Arithmetic expression is first written in symbolic form, as in the case of "= $Ak\,\Delta T$ " to assist grading and future checking, and to insure partial credit incase of numerical error. This is then followed by numerical substitutions in the same order so the grader can determine the value used for each parameter.
3. The numerical substitutions is accompanied by a unit consistency check in written form, as in [m^2][W/mK][K/m]. This can be great help in spotting errors. This may be omitted as you gain expertise and confidence, or when the problem is simple and obvious.

C. Convection and the Convective Heat Transfer Coefficient

The word *convection* refers to the conveyance of energy by moving material, usually a fluid. The detailed study of convection would include the study of the flow field and its effect on temperature distribution. It is a subject of interest not only to mechanical engineers but also to other professions, for example chemical and aeronautical engineers and atmospheric scientists.

In elementary heat transfer analyses, however, the study of *convection* primarily refers to the calculation of rate of heat transfer between a solid surface and a body of fluid, due to the combined effects of flow and conduction. The consequence of the combined effects of flow and conduction in the fluid is to concentrate the temperature variations in the fluid to a small layer adjacent to the surface, as illustrated by the shaded regions in Fig. **2**. This evokes the image that the fluid at large is well mixed, with nearly uniform temperature, and there exists a increasingly stagnant film near the wall which constitutes the primary resistance to heat transfer. This is the *thermal boundary layer*, also called *conduction boundary layer*.

For quantitative calculations, it is convenient to define the *convective heat transfer coefficient*[5] *h*, attributed to Issac Newton:

$$\dot{q} = h\Delta T; \qquad \text{where} \quad \Delta T \equiv \left|T_w - T_{fluid}\right| \tag{4}$$

As was the case in Eq. (1), ΔT is customarily employed as a positive number. The user should keep track of the direction of heat transfer by observing the sense of the temperature drop. The subscript w (wall) refers to the solid-fluid interface.

As might be expected, h varies with the flow configuration, the fluid velocity, the dimension of the immersed object or containing vessel, and the fluid properties. It should be recognized that this is not a fundamental equation, but is merely a definition of h for convenient computation of the convective heat flux. The actual value of h must be provided separately by detailed analysis or empirical correlations based on accumulation of data. Furthermore, there is also an ambiguity in assigning T_{fluid}. By convention, for *external flows*, illustrated in Fig. **2**a and **2**c, T_{fluid} is taken to be the undisturbed fluid temperature away from the wall, usually denoted as T_∞. For *internal flows*, illustrated in Fig. **2**b, T_{fluid} varies with axial position and is taken to be the average temperature over the cross section with the flow rate as the weighting factor. This will be discussed in greater detail in a later section.

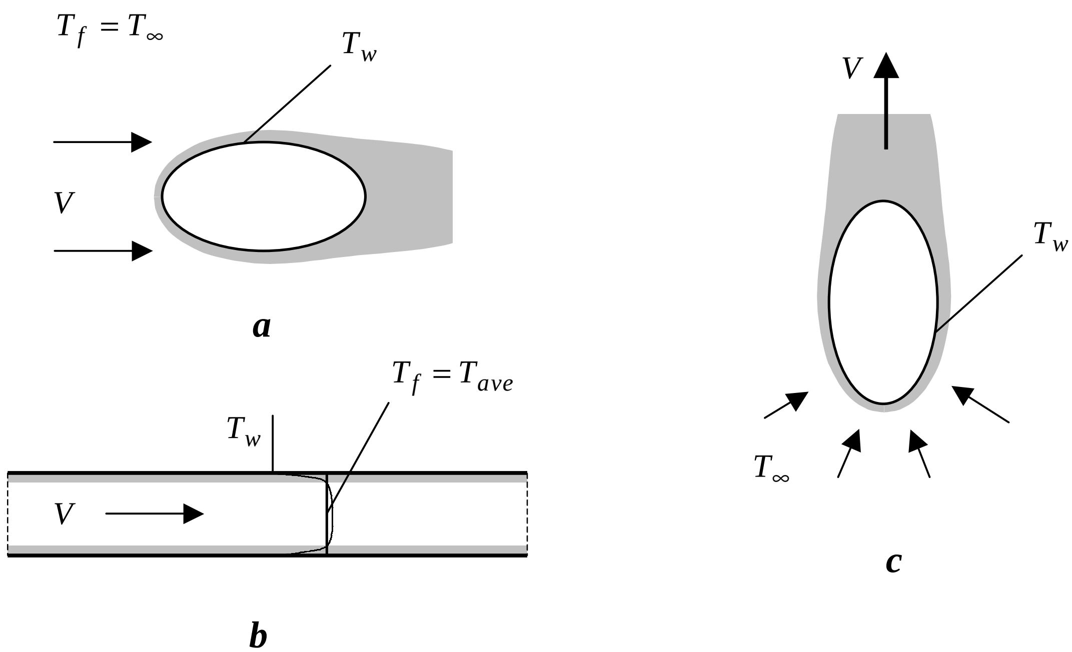

Fig. 2 Common configurations of convection. ***a. External flow Forced Convection, b. Internal flow Forced Convection, c. Free or Natural Convection***

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[5] The heat transfer coefficient is almost universally denoted by *h*, the same symbol used for specific enthalpy. Students should be alert to distinguish the two by context.

Forced convection refers to convection due to an imposed velocity, as shown in ***a*** and ***b*** of Fig. **2**, whereas *natural* or *free convection* refers to flows generated naturally by the buoyancy of heated air, as shown in ***c*** of Fig. **2**.

As an example for forced convection, consider a circular rod or pipe with surface temperature T_w, subjected to a cross wind of air at velocity *U* and temperature T_∞. This is an external flow, forced convection problem, for which ΔT is computed as $|T_w - T_\infty|$. For ranges of conditions shown by the inequalities, an approximate formula with about 5% accuracy is

$$h \cong 4.68 \frac{V^{0.65}}{D^{0.35}} \qquad \text{For air,} \quad 280\text{ K} < T_{mf} < 320\text{ K}, \quad 0.15 < VD < 6 \qquad (5)$$
$$(h \text{ in W/m}^2\text{K}, \; V \text{ in m/s}, \; D \text{ in m})$$

This formula has been derived from a more elaborate and more general formula (see Section 4) by using properties of air at 300 K. If the *mean film temperature* $T_{mf} \equiv (T_w + T_\infty)/2$ strays too far from 300 K, the assumed properties will no longer be valid, and results will not be accurate.

As an example for free convection, consider a heated horizontal pipe located in still air. In this case buoyancy would induce a local air flow just around the pipes, in the manner of Fig. **2**c. For ranges of conditions shown, an approximate formula with about 5% accuracy is

$$h = 1.515 \frac{(\Delta T)^{0.3}}{D^{0.1}} \qquad \text{For air,} \quad 280\text{ K} < T_{mf} < 320\text{ K}, \quad \Delta T < 60, \qquad (6)$$
$$0.001 < \Delta T\, D^3 < 100, \quad (h \text{ in W/m}^2\text{K}, \; \Delta T \text{ in K}, \; D \text{ in m})$$

Both of these formulas are approximations of the more elaborate and more general formulas to be discussed in Section 4. Note that these specialized formulas constructed for restricted applications must be used with the specified set of units, and must be converted carefully when parameters in other units are involved. In contrast, fundamental physical laws and basic definitions, such as Eqs. (2), (3) and (4), are general and can be used with any set of self-consistent units.

The exponents in these formulas are empirical and will change with conditions. Within the specified range of validity, the exponents give a concise indication of the dependence of *h* on parameters *U* and *D*. For example, it can be shown from Eq. (5) that for this forced convection configuration, a one percent increase of *U* will lead to a 0.65% increase in *h*, and each percent increase of *D* will lead to a 0.35% *decrease* in *h*. Similarly, Eq. (6) tells us that *h* for free convection is approximately proportional to 1/3-power of the temperature difference, and inversely proportional to the one-tenth power of *D*. This type of understanding of the trends is very valuable in engineering design and analysis.

Representative values of *h* are given in Table 2.

The forgoing discussion clearly shows that the convection heat transfer coefficient represents a form of conduction modified by fluid motion. Hence we cannot simultaneously consider the conduction and convection through the same body of fluid, or double counting would result.

Table 2. Typical Values of Heat Transfer Coefficients

(in W/m^2K) (From various sources)

Free Convection (Some T –dependence)	Gases	5 – 20
	Liquids	50 — 500
Forced Convection (Weak T –dependence)	Gases	25 –250
	Liquids	50 – 20,000
Boiling and Condensation (Strong-Weak T –dependence)		2,500 – 100,000
h for radiation near room temperature h for radiation around 600 K		5--10 ~50

Example 2. Forced Convection

An air heater consists of electric heating elements, 0.5” in diameter and 3 ft long, each dissipating 200 W. A fan blows air, at 25°C across the heaters at 5000 ft/min. Determine the heat transfer coefficient and the surface temperature of the heater.

Solution: From Eq. (4), and the definition of ΔT, the total heat transfer rate is

$$\dot{Q} = Ah\left(T_w - T_\infty\right)$$

which can be solved to yield

$$T_w = T_\infty + \dot{Q} / Ah$$

V = 5000*12*0.0254/60=25.4 m/s

D = 0.5*0.0254 = 0.0127 m

h = 4.68*25.4^0.65/0.0127^0.035 = 44.6 W/m^2K

A = 3*12*0.0254*pi()*0.0127=0.03648 m^2 (End surfaces neglected)

$T_w = T_\infty + \dot{Q} / Ah$ = 25+200/(44.6*0.03648) [W]/[m^2][W/m^2K] = 127.8°C

A retrospective check shows that the computed mean film temperature would be about 76°C, considerably above the allowed limit. This means that the results are likely to be quite inaccurate, with errors greater than 5%, which corresponds to a error in ΔT greater than 5 K. If more accurate results are desired, the more elaborate formulas of Section 4, or a dedicated text book on heat transfer, should be consulted.

Comments on Accuracy Although the result of the last example is not very accurate, it nevertheless can serve very useful engineering functions. For example, an estimated temperature of 128°C is probably sufficient to show that the surface is unlikely to start a fire if it is in contact with a piece of wood or paper. On the other hand, it is probably hot enough to present a burn hazard to humans and pets, and should be designed with safeguards to prevent direct contact.

Example 3 Free Convection

A horizontal heating element, 0.5" diameter and 3 feet long, is located in a room with still air at 10°C. The surface temperature of the heating element is maintained at 40°C. Determine the free convection heat transfer coefficient and the total heat transfer rate.

Solution

This rod has the same dimensions as the rod in Example 2.

From Eq. (6), with ΔT = 50 – 20 = 30 K,

$$h = 1.515\frac{(\Delta T)^{0.3}}{D^{0.1}} = 1.515*30\text{^}0.3/0.0127\text{^}0.1 = 6.504 \ \text{W/m}^2\text{K}$$

$$\dot{Q}_{fr-c} = Ah\Delta T = 0.03648*6.504*30 \ [\text{m}^2][\text{W/m}^2\text{K}][\text{K}] = 7.118 \text{ W}$$

D. Radiation Heat Transfer

It is apparent to most observers that very hot surfaces such as stoves and fireplaces, as well as the sun, can transmit heat without the help of an intervening material, and that the heat can be absorbed by objects which are in line-of-sight of such hot objects. This is thermal radiation, emitted by all objects not at absolute zero temperature. We now know that thermal radiation is a form of electromagnetic wave. All electromagnetic waves carry energy, and represent transmitted power. Typical thermal radiation differs from waves intended for signal transmission in that it is spread over a wide wavelength range (or equivalently, frequency range). The sun's radiation is thermal radiation with the bulk of its power spread over a wavelength range of 0.4 – 0.7 μm. Evolution has made our eyes adapt to this bountiful source of radiation. So we now call this range the *visible* light range. Objects at lower temperatures generally emit at longer wavelengths. Most earthbound radiation heat transfer occurs in the infrared range, with wavelengths above 1 μm. Thermal radiation has essentially the same properties as light, traveling in straight lines in uniform, transparent media and can be transmitted, absorbed, or reflected from surfaces.

Air at moderate temperatures is essentially transparent to infrared radiation, whereas most liquids and solids are not. Hence an important class of radiative heat transfer problems is concerned with heat exchange among surfaces, either in vacuum or immersed in air. The general principles and methodology of such analyses will be introduced in later sections.

To provide an example for the relative magnitude and mathematics of radiative heat transfer, we will cite without derivation one simple formula for an importance case of radiative transfer. This is the case when a small body *b* is completely enclosed by a larger enclosure *e*, as shown in Fig. 3*a*. We will later show that as long as *e* is at least two- or three-fold as large as *b*, essentially all of the radiation emitted from *b* will be absorbed by *e* after either single or multiple reflection in the cavity. On the other hand, most of the radiation emitted by *e*, in whichever direction, is likely to be reflected by other parts of *e* and in the process joined by additional emissions, so that when it finally reaches *b* it will nearly equal the ideal theoretical intensity corresponding to *e*'s surface temperature, independent of the surface properties. Under such conditions the net radiative heat transfer between *b* and *e* can be satisfactorily calculation by the approximate expression

$$\dot{Q}_{r,be} = A_b \varepsilon_b \sigma \left(T_b^4 - T_e^4\right) \tag{7}$$

In the above equation, σ is a universal physical constant called the Stephan-Boltzmann constant, equal to $5.67(10)^{-8}$ W/m^{2}K^4. ε is a dimensionless surface property called the *emissivity* [1]. Note that heat transfer is proportional to the difference of the 4th power of temperatures[6], but depends only on the surface area and emissivity ε of body *b*, and not on those of *e*. More detailed explanation for this apparent asymmetry in the formula will be discussed later.

The *emissivity* is the ratio of the emissive power of a surface compared to that of a perfect emitter, called a *black body*. For all but highly polished metals, the emissivity is typically between 0.7 and 1. Sample emissivities for selected surface are shown in Table **3**.

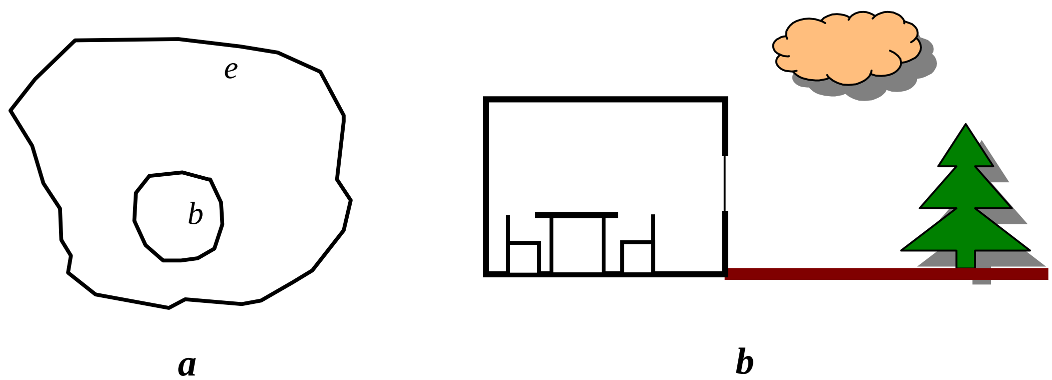

Fig. 3 ***a*. A small body completely enclosed in a larger body. *b*. A window pane with one side exchanging heat radiatively with interior walls, furniture and occupants, and another side exchanging heat with the exterior radiative environment, including ground, trees, clouds, mist and moisture in the atmosphere, and deep space.**

The above equation can be generalized to apply to the radiative exchange between any small surface exchanging heat radiatively with a *combination of bodies which enclose it optically*. Examples include the inside surface of the window pane shown in Fig. **3**b, exchanging heat radiatively with a combination of surfaces which completely occupy its view, including all the interior walls and the furniture. Note that common glass is opaque to infrared radiation. The

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[6]For essentially linear processes such as convection and conduction, it is immaterial whether temperature is Celsius or Kelvin. For a non-linear process such as radiation, it is essential to use the absolute temperature.

inside surface of the glass therefore cannot communicate radiatively with the outside. Thermal radiation inside the room does not pass through the window. Similarly, the exterior surface of the windowpane exchanges heat radiatively with the outside ground, the trees, cloud, droplet-laden atmosphere, and the deep space background. In either case T_e in the above equation is replaced by T_{mre}, a suitably defined *mean radiative environment temperature*:

$$\dot{Q}_{r,be} = A_b \varepsilon_b \sigma \left(T_b^4 - T_{mre}^4 \right) \tag{8}$$

For certain calculations it is more convenient to recast these results in terms the more familiar heat transfer coefficient. We may then define a *radiation heat transfer coefficient* in a manner similar to the convective heat transfer coefficient:

$$\dot{Q}_{r,be} \equiv A_b h_{rb} \left(T_b - T_{mre} \right) \tag{9}$$

Solving for *h* from these two equations, we have, after re-arrangement

$$h_{rb} = 4\varepsilon_b \sigma T_{ave}^3 \left[1 + \frac{1}{2} \left(\frac{\Delta T}{T_{ave}} \right)^2 \right] \qquad \cong 4\varepsilon_b \sigma T_{ave}^3 \quad \text{if} \quad \Delta T \le 0.3 T_{ave} \tag{10}$$

where $T_{ave} \equiv (T_b + T_{mre})/2$, $\Delta T \equiv \left| T_b - T_{mre} \right|$. The approximate form is usually adequate since rarely is ΔT greater than $0.3\,T_{ave}$.

Typical values of the radiative heat transfer coefficient are also listed in Table 2, for comparison with convective coefficients.

Table 3

Sample Ranges of Emissivities and Absorptivities

Clean, highly polished metals	0.01-0.1
Common metal surfaces	0.1-0.4
Oxidized metal surfaces	0.3-0.7
Clean Al_2O_3 (white), at room temperature At 1000-2000°C	0.7 0.35 –0.5
Most surfaces at room temperature, including some apparently white surfaces	0.7 – 1.0

Example 4 Radiative Heat Transfer and Combined Radiation and Convection

For the problem of Example 3, if the mean radiative environment temperature is also 10°C, determine the radiative heat loss and the total heat loss due to radiation and convection. Assume the emissivity of heating element to be 0.9.

Solution

It is important to convert all T 's to Kelvin: $T_w = T_a$ =40+273.15=313.15 K,

Similarly, T_{mre} =283.15 K, T_{ave} =298.15 K.

$$\dot{Q}_{r,be} = A_b \varepsilon_b \sigma \left(T_b^4 - T_{mre}^4\right)$$

= 0.03648*0.9*5.67E-8*(313.15^4-283.15^4) [m^2][1][W/m^{2}K^4][K^4] = 5.934 W

We can next try to solve the problem using the radiative heat transfer coefficient, and see if we get essentially the same result. Since the inequality is satisfied, we may use the simple approximate equation for *h*:

$h_{rb} \cong 4\varepsilon_b \sigma T_{ave}^3$ = 4*0.9*5.67E-8*298.15^3 [1] [W/m^{2}K^4][K^3]= 5.410 W/m^{2}K

$\dot{Q}_{r,be} \equiv A_b h_{rb} \left(T_b - T_{mre}\right)$= 0.03648*5.410*30 [m^2][W/m^{2}K][K]= 5.921 W

The difference between this and the previous result of 5.934 W is small.

Air is transparent. Hence convection and radiation can proceed independently and simultaneously. Hence the total heat transfer rate is the sum of the two, or

$\dot{Q}_{total} = \dot{Q}_{free-c} + \dot{Q}_{rad}$ = 7.118 + 5.934 = 13.052 W

In this problem, the ambient conditions for both convection and radiation are both equal to 20°C. Therefore we can also combine the two heat transfer coefficients and use a total heat transfer coefficient equal to 6.504+5.410 = 11.914 W/m^{2}K. Using this value

$\dot{Q}_{total} = A\left(h_{fr-c} + h_{rad}\right)\Delta T =$ 0.03648*11.914*30 [m^2][W/m^{2}K][K]= 13.04 W

Example 5 Combined Radiation and Convection -- Inverse Problem

If the heating element of Examples 3 and 4 is known to dissipate 20 W by convection and radiation, what is is the surface temperature?

Comments:

Combining radiation and convective heat transfer, we have

$$\dot{Q}_{total} = A\varepsilon_w \sigma \left(T_w^4 - T_{mre}^4\right) + Ah\left(T_w - T_\infty\right)$$

To solve for the unknown T_w involves a messy, highly non-linear algebraic problem. The difficulty is not alleviated by using the radiative heat transfer coefficient formulation, since without a known surface temperature, the radiative heat transfer coefficient cannot be determined. Hence such a problem must be solved using one of the iteration methods. This example will be completed in Section ***2.F***.

2. Formulation and Solution Methodologies

A. Conservation Laws

Heat transfer analysis usually requires two types of equations. The relationships discussed above, relating the conductive, convective and radiative heat fluxes to various forms of temperature non-uniformities, are called *constitutive relationships*. The second type of equations are the *conservation laws*. The primary conservation law used in heat transfer is the conservation of energy, though sometimes it is supplemented by the conservation of mass and conservation of momentum. Beyond the simplest of problems, at least some of the variables of interest, including temperature, heat flux, etc., are unknown. The conservation of energy is then used to tie them together in algebraic and/or differential equations, which are then solved to permit calculation of the desired quantities. Examples will be given in sub-section ***C*** below and in later sections.

In time-dependent heat transfer problems and in convection problems, changes in internal energy and enthalpy are often expressed in a per-volume basis as $\rho c_v dT$ and $\rho c_p dT$ respectively. The specific heats c_v and c_p always appear in combination with ρ, never alone. ρc_p can thus be viewed as a single property Sample values are shown in Table 1. It is useful to note that for solids and liquids, their values cluster around $2.5(10)^6$ J/m^{3}K, and is almost always within the range 1.7—$4.2(10)^6$ J/m^{3}K. This means crude but reliable estimates can often be made without detailed knowledge of the two separate properties ρ and c_p. The ability to do this can be of considerable value in analysis and design when precise property values are not available.

B. Unit and Dimensions

Units play important roles in heat transfer analysis. This is particularly true in the present transition period between traditional American units and the new standard, SI. While SI units are self-consistent, so that no conversion is necessary between work unit [N-m] and the unit of thermal energy [J], this is not true for the traditional American units. Careless use of inconsistent units is a frequent cause of errors.

Mathematical expressions based on physical laws must also have unit consistency – each term in the expression must be reducible to the same units. This is a valuable tool for checking the correctness of derivations and calculations for correctness. Students should develop the habit of checking units constantly as he/she is performing the analysis. Once the habit is developed, it can be effortless and is not time-consuming, but can often save a lot of the time wasted on erroneous computations.

Self-consistent units are derived from just a few basic quantities, called *dimensions*. For example, the SI unit of force, N, is really [kg-m/s^2], derived through the equation $F = ma$. Hence the dimension for force is [ML/T^2], where M, L and T denote the dimensions of mass, length and time. The checking of dimensional consistency is clearly a requisite for unit consistency. A simple and effective way to insure dimensional consistency is to render equations *dimensionless* by making all variables and parameters in the equation dimensionless. Examples of such an approach can be found in Section 4.

Another common source of error is the careless treatment of prefixes *k*, *M*, *m* (kilo, mega, milli), etc. While kg is part of the basic, self-consistent SI system, other units, such as kJ, mm, are not. For example, to determine the velocity corresponding to a given kinetic energy, using kJ in the equation $\text{K.E.} = mV^2/2$ would lead to erroneous answers. If energy is given in kJ, it is necessary to convert it to J before using this equation to calculate the velocity.

C. Resistances, Conductances and Combination Rules

To highlight the value of the concepts to be discussed below, we shall begin by considering a simple composite wall consisting of two layers of different materials bonded together, an inner layer of thickness B_a with conductivity k_a, and a outer layer of thickness B_b with conductivity k_b. The inside and outside temperatures are T_1, T_3 respectively. What is the heat transfer rate per unit area? The solution to this problem is not straightforward, because the temperature T_2 at the interface between the two layers is not known. The concept of resistance will greatly simplify the analysis of this type of problems.

The conduction and convection expressions, and the approximate expression for radiation discussed in Section 1 all show the heat transfer rates to be proportional to the temperature difference. This is analogous to the flow of electric current, which is proportional to the voltage difference across a component. The analogy permits us the define the *heat transfer resistance*, denoted by *R*, as the ratio of the temperature difference and the heat transfer rate, with unit [K/W]:

$$R \equiv \frac{\Delta T}{\dot{Q}} \quad \text{[K/W]}, \qquad \text{or} \qquad \dot{Q} \equiv \frac{\Delta T}{R} \tag{11}$$

For example, the conduction resistance for the plane slab, the convective resistance for an area *A* with coefficient *h*, and radiative resistance for a small object in a large enclosure, according to Eqs. (1)(4) and (10) respectively, are

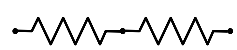
$$R_{k,slab} = \frac{B}{Ak}; \qquad R_v = \frac{1}{Ah}; \qquad R_{r,a} = \frac{1}{Ah_r} = \frac{1}{4A\varepsilon_a \sigma T_{ave}^3} \tag{12}$$

where the subscripts *k, v, r* denote conduction, convection, and radiation respectively.

The great benefit of borrowing the resistance concept from electrical circuitry is that the well-established combination rules for electrical resistances also apply here for heat transfer resistances. These rules are for the calculation of the equivalent resultant resistance for combinations of the individual resistances, according to whether they are in series or parallel:

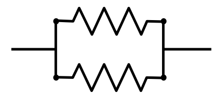
For Series Resistances: Equivalent resistance equals to sum of individual resistances

$$R_e = \sum_{i=1}^{n} R_i \quad \text{for series resistances} \tag{13}$$



For Parallel Resistances: Inverse of equivalent resistance equals sum of inverses of individual resistances

$$\frac{1}{R_e} = \sum_{i=1}^{n} \frac{1}{R_i} \quad \text{for parallel resistances} \tag{14a}$$

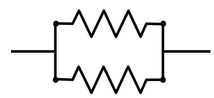


An alternative statement for Eq. (14) is to make use of the concept of ***conductance***, which is just the inverse of the resistance

$$G \equiv \frac{1}{R} \tag{15}$$

The combination rule for *parallel resistance* can now be stated as ***Equivalent Conductance is equal to the sum of conductances:***

$$R_e = \sum_{i=1}^{n} R_i \quad \text{for parallel resistances} \tag{14b}$$



Shown on the right of the equations are standard schematic representations for resistance networks, which we have also borrowed. This can be very valuable for constructing schematics to aid in visualizing the relationship between different heat transfer components in an overall system.

Derivation of the combination rules

The origin of the combination rules is the conservation of energy.

The combination rule for parallel resistances simply states that the total heat transfer is equal to the sum of heat transfer through all resistances. This is because for each resistance, $\dot{Q} = \Delta T / R$. Since ΔT is the same for all parallel resistances, the total heat transfer is simply the product of ΔT and the sum of all $(1/R)$'s.

To derive the combination rule for series resistances, we consider the composite wall of two slabs, described in the lead paragraph of this section. The problem consists of two components (slabs), *a* and *b*, in series, with known overall $\Delta T = T_1 - T_3$. The temperature of the median interface T_2 is not known. To solve this problem, we observe that in steady state, conservation of energy requires both slabs to have the same heat transfer rate $\dot{Q}$. Referring to the definition of resistance, we can write, for layers *a* and *b*:

$$\Delta T_a : \qquad T_1 - T_2 = R_a \dot{Q}$$

$$\Delta T_b : \qquad T_2 - T_3 = R_b \dot{Q}$$

We have two equations for the two unknowns $\dot{Q}$ and T_2. The algebra of eliminating the T_2 is particularly easy -- we simply add both sides of the equations. The result is a formula for $\dot{Q}$ in terms of the overall ΔT and $(R_1 + R_2)$, which is clearly the equivalent resistance:

$$T_1 - T_3 = \left(R_a + R_b\right)\dot{Q}$$

This proves the combination rule for two resistances in series. The proof can be easily extended to any number of resistances in series.

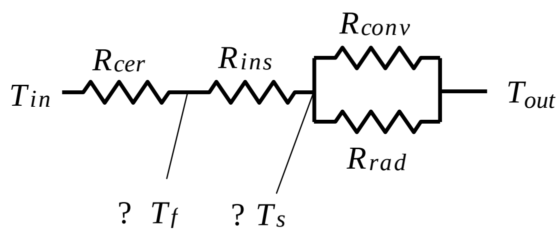
Application of the Combination Rules

The combination rules can be used systematically to reduce a complex system with multiple components to a simple system with only one equivalent resistance or conductance. This is illustrated by the following example.

Example 6 An industrial oven, with inside surface temperature at 120°C, consists of a 0.1 m thick ceramic wall of conductivity *k* = 1.6 W/mK, covered by a layer of insulation 0.12 m thick with *k* = 0.05 W/mK. The outside wall is exposed to air at 25°C with a convective heat transfer coefficient of 30 W/m^{2}K and to a mean radiative environment at 25°C, with a radiative heat transfer coefficient of 7 W/m^{2}K. (a) Determine the heat loss per m^2 of the wall. (b) Determine the interface temperature T_f and the surface temperature T_s .

Solution:

The circuit representation for this problem is shown



We shall first combine the two parallel components -- the convective and radiative heat transfer coefficients. We may use either Eq. (14a) or (14b), though latter is much simpler and simply states that radiation and convection in air can proceed independently and are additive. Hence the equivalent resistance R_{vr} for combined convection and radiation is

$$R_{vr} = \frac{1}{A\left(h_{conv} + h_{rad}\right)} = \frac{1}{30+7} = 1/(30+7) = 0.02703 \quad \text{K/W, for 1 m}^2$$

This reduced the problem to three resistances in series. The total resistance, for 1 m^2 of area, is hence

$$R_{tot} = R_{cer} + R_{ins} + R_{vr} = \frac{B_{cer}}{Ak_{cer}} + \frac{B_{ins}}{Ak_{ins}} + R_{vr}$$

$$= 0.1/1.6 + 0.12/0.05 + 0.02703 = 0.0625 + 2.4 + 0.02703 = 2.490 \text{ K/W}$$

(a) For heat transfer rate

$$\dot{Q} = \frac{\Delta T_{tot}}{R_{tot}} = (120 - 25)/2.490 = 38.15 \text{ W} \quad \text{per m}^2.$$

(b) To determine the temperatures, we make use of the relationship between temperature difference and resistance. First, for the ceramic wall

$$\dot{Q} = \frac{\Delta T_{cer}}{R_{cer}} = \frac{T_{in} - T_f}{R_{cer}}. \quad \text{Hence}$$

$T_f = T_{in} - \dot{Q}R_{cer}$ = 120-38.15*0.0625 = 120-2.384= 117.6°C.

Next, for the combined convection/radiation coefficients

$$\dot{Q} = \frac{\Delta T_{vr}}{R_{vr}} = \frac{T_s - T_{out}}{R_{vr}}. \quad \text{Hence}$$

$T_s = T_{out} + \dot{Q}R_{vr}$ = 25+38.15*0.02703 = 25+1.03 = 26.3°C.

Unit Area Resistance and the R-Value

In **Example 6,** we computed the heat transfer rate $\dot{Q}$ and resistance R for 1 unit area of the wall, but retained the units for the overall heat transfer and resistance, [W] and [K/W] respectively. If we desire, we can also express the results in terms of unit-area symbols and units. For heat transfer *per unit area*, this is the heat flux $\dot{q}$ [W/m^2]. Since resistance is *inversely proportional* to the area, the unit is [m^{2}K/W]. Note that we *can not* call this the resistance *per unit area* or *per* m^2 because of reciprocal relationship. We shall refer to it as the *unit area resistance*.

The *R-value* often cited in connection with building products in the US is the unit area resistance in [ft^{2}hr°F/Btu]. For example, consider a 3.5" thick layer of fiberglass insulation. With k = 0.046 W/mK, the unit area resistance value would be B/k = 3.5*.0254/0.04 = 2.22 m^{2}K/W. Converted to traditional American units, we have *R-Value* = 2.22*3.2808^2*1.8/3.4123 = 12.6 ft^{2}hr°F/Btu. This is close to, but not necessarily equal to the marked *R*-value of roll insulation being sold in building material stores because of variations of product and manufacturing processes.

D. Dominant and Non-dominant Components

Most real-life problems in engineering involve more than one component. In these problems it immensely important to have an appreciation for what is dominant and what is not. Knowing what is dominant is valuable in design iterations, as well as in allocating time and effort for analysis. Clearly effort expended in detailed, accurate analysis of a dominant component pays far greater dividends than effort expended on an unimportant component.

For most people the ability and habit to assess quickly what is dominant does not come naturally, but is developed after repeated practice.

Consider the problem of Example **6**. For this problem, the overall resistance, for 1 m^2 of area, was found to be 2.49 K/W. Of this, the insulation accounts for 2.4 K/W, or about 96% of total. This is clearly the dominant resistance. In contrast, the ceramic wall accounts for about 2.5%, and the combination of convection and radiation coefficients accounts for only 1.1%.

Suppose some one points out that the accuracy of the convection heat transfer coefficient is very low, only 30%. Is it worthwhile for us to re-evaluate these coefficients? Probably not. This is because a 30% error of 1.1% of the total resistance is only 0.33%, which is clearly too low to be of concern. If we wish to improve the accuracy of the overall computation, we must focus on the thickness of the insulation and the accuracy of the value of thermal conductivity.

Similar considerations may also suggest that we simply neglect the radiative contribution to heat transfer. Since it is much smaller than convection, and since even convection is not too important, the neglect of radiation would not introduce significant errors in this case.

Suppose we wish to reduce the total heat loss. Clearly increasing the thickness of the ceramic wall is not cost-effective, since it accounts for only 2.5% of the total resistance. A small increase of the thickness of the insulation would accomplish the goal much more economically.

Conversely, we may desire to change the design to increase the efficiency of heat transfer, in order to reduce the inside surface temperature for the same heat transfer rate. Clearly increasing the air cooling would not be effective, since convection only accounts for a negligible fraction of the resistance. Decreasing the thickness of insulation is the most effective measure for this purpose.

E. General Method for Linear Networks

Not all networks can be simplified by combination rules. For example, the simple network in Fig. **4.** is a non-combinable network. Fortunately, there is a general method for solving all linear resistance networks. We will first describe the method for the simple network shown in Fig. **4.** . Afterwards the general procedure will be described.

Let T_1 and T_2 and all the five resistances be known. We wish to determine the total heat transfer from node 1 to node 4, along with the temperatures at nodes 2 and 3.

Conservation of energy requires that in steady state, the total heat transfer *toward any node* must vanish. Applying this principle to nodes 2 and 3, we have

$$\frac{T_1 - T_2}{R_a} + \frac{T_3 - T_2}{R_c} + \frac{T_4 - T_2}{R_d} = 0 \tag{16}$$

$$\frac{T_1 - T_3}{R_b} + \frac{T_2 - T_3}{R_c} + \frac{T_4 - T_3}{R_e} = 0 \tag{17}$$

This constitute a set of two linear algebraic equations, with two unknowns T_1 and T_2. These of course can be solved in a straightforward manner, by any of the many methods for linear algebraic equations. Once T_1 and T_2 are known, the heat transfer can be obtained.

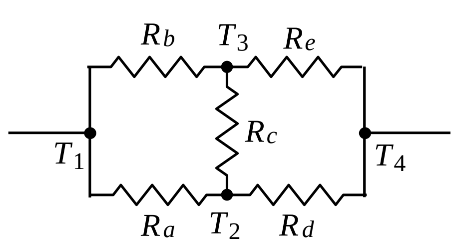


Fig. 4. Example of a non-combinable resistance network

This procedure can be extended to any network with any number of unknown nodal temperatures. Invoking the conservation of energy will usually lead to one equation for each node with an unknown temperature. Solution of these equations by Gaussian elimination or other matrix inversion schemes is straightforward. Iteration methods may also be used. Once the unknown temperatures have been found, the computation of heat transfer rates follows directly.

F. Temperature-dependent Coefficients, Non-linearities

There is a potential catch-22 situation that occur frequently in heat transfer analysis. Most resistances and conductances are temperature-dependent. The temperature-dependence of the coefficients for free convection and radiation is quite obvious. Even conduction and forced convection are at least somewhat temperature-dependent, because the conductivity and other physical properties are temperature dependent. When we begin to solve a heat transfer problem, the temperature distribution is frequently not known. How can we proceed?

The difficulty is a symptom of the non-linearity of the problem. With temperature-dependent resistances and conductances, neither the resistance-combination method nor the simultaneous equation method described in sub-sections ***C*** and ***E*** applies.

The answer to this difficulty is *iteration*. Rather than attempting to solve the problem directly, many iteration methods are far more efficient in this type of situations. For example, the Newton-Raphson method would be quite effective.

For most heat transfer problem, however, a special form of the *successive substitution* method is both efficient and intuitively easy to understand. For simple problems, the method can be easily implemented with a programmable computer. For either simple or complex problems, it can also be easily implemented with a spreadsheet program such as Excel.

To start the iteration process, reasonable guesses (these are called “first iterates”) are first assumed for each of the unknown temperatures required for the evaluation of the coefficients. These will permit the resistances to be determined. With known tentative resistances, the circuit can be constructed and solved, as if it were a linear problem. The temperatures from this solution are often improvements over the first iterates, and can be used to as second iterates to evaluate improved coefficients and resistances. The processes can be repeated until the temperatures no longer change.

The viability of the method depends on the fact that frequently, the coefficients are only weakly dependent on temperature. Thus a given relative error in temperature would lead to a smaller relative error in the coefficients. When these coefficients are used to compute new temperatures, the relative error would be smaller than those originally assumed.

As in all iteration methods, occasionally conditions are such that succeeding iterates do not represent improved values, but may either oscillate or continue to change without settling down to a converged value. In this case the iteration is said to diverge and another iteration method must be found. Fortunately, the successive substitution method described here will converge for most heat transfer problems.

We will illustrate this method by completing the solution of Example **5**.

Example 5 (continued from Section ***1.D***)

This is the same problem as Example 4, except that the surface temperature is unknown and that the heating element is dissipating 20 W.

With $T_{mre} = T_\infty$

$$\dot{Q} = A(T_s - T_\infty)(h_{fc} + h_r)$$

D = 0.0127 m, A = 0.03648 m^2 (Previously computed. End surfaces neglected)

For first iterate, assume surface temperature to be 313.15 K (40°C). The free convection and radiation heat transfer coefficients would then be 6.504 W/m^{2}K and 5.410 W/m^{2}K respectively.

$$R = \frac{1}{A(h_{fc} + h_r)} = 1/(0.03648*(6.504+5.410)) = 2.301 \text{ K/W}$$

$$T_s = T_\infty + \dot{Q}_{total} R = 283.15+20*2.301=329.17 \text{ K}$$

With this surface temperature, we can re-evaluate the free convection and radiation heat transfer coefficients

$$h_{fr-c} = 1.515 \frac{(\Delta T)^{0.3}}{D^{0.1}} = 1.515*(329.17-283.15)\text{^}0.3/0.0127\text{^}0.1 = 7.395 \text{ W/m}^2\text{K}$$

$$T_{ave} = (T_s + T_\infty)/2 = (329.17+283.15)/2 = 306.16 \text{ K}$$

$$h_{ra} \cong 4\varepsilon_a \sigma T_{ave}^3 = 4*0.9*5.67\text{E-}8*306.16\text{^}3 = 5.858 \text{ W/m}^2\text{K}$$

These can the used to calculate R, T_s in turn, starting a new cycle of iteration. In six iterations[7], the converged value for the surface temperature is found to be 325.3 K.

Iteration number	T_s Assumed	h_{fr-c}	T_{ave}	h_r (Exact)	R	T_s Computed
1	313.15	6.504	298.15	5.437	2.296	329.06
2	329.06	7.389	306.11	5.921	2.060	**324.34***
3	324.34	7.153	303.75	5.773	2.121	325.57
4	325.57	7.216	304.36	5.811	2.104	325.24
5	325.24	7.199	304.19	5.801	2.109	325.32
6	325.32	7.204	304.24	5.803	2.108	325.30
7	325.30	7.202	304.23	5.803	2.108	325.31

[7] It is interesting to note that should the given power of the heating element be 400 W or higher instead of 20 W, the iteration process described here would diverge. This is because at the high temperatures corresponding to these high power levels, radiation dominates. The radiation heat transfer coefficient at thse high temperatures is very strongly temperature-dependent.

8	325.31	7.203	304.23	5.803	2.108	325.31
9	325.31	7.203	304.23	5.803	2.108	325.31

*Iteration can stop at this point with less than 0.3% error.

3. Steady Conduction through Non-planer Geometries

A. The Differential Equation for Heat Conduction for Constant Density Media

(This subsection is for your enrichment only. Thorough understanding not required.)

Consider a solid or a stationary fluid with non-uniform but steady temperature $T(x,y,z)$. We will see that $T(x,y,z)$ obeys a differential equation that can be obtained from the conservation of energy. The differential equation can then be solved to determine the heat transfer behavior of complex body shapes.

We will consier the conservation of energy for a very small volume element $dxdydz$ center at location (x,y,z). Since q_x denotes the heat flux in the x-direction, $(dx)(\partial q_x/\partial x)$ denotes the difference between the heat flux on the right surface and that on the left surface. Hence the *net rate of outward* heat transfer due to q_x for this element is

$$(dydz)dx\frac{\partial q_x}{\partial x}$$

where the product in the parenthesis represents the cross-sectional area. Similarly, the *net rate of outward* heat transfer due to q_y and q_z for this element are

$$(dzdx)dy\frac{\partial q_y}{\partial y},\ (dxdy)dz\frac{\partial q_z}{\partial z}$$

The sum of these three terms is the total rate of outward heat transfer for the element $dxdydz$. Conservation of energy states that, in the absence of volume change (which leads to work), the total rate of outward heat transfer must be equal to the rate of decrease of internal energy, equal to the mass $(dxdydz)\rho$ times the rate of decrease of specific internal energy $-(\partial u/\partial t)$. Putting these together, we have

$$-(dxdydz)\frac{\partial u}{\partial t}=(dydz)dx\frac{\partial q_x}{\partial x}+(dzdx)dy\frac{\partial q_y}{\partial y}+(dxdy)dz\frac{\partial q_z}{\partial z}$$

Recognizing that $(\partial u/\partial t)=c(\partial T/\partial t)$, and making use of Eq. (3) and its counterparts in y- and z- directions, the differential equation for heat conduction becomes

$$\rho c\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}k\frac{\partial T}{\partial x}+\frac{\partial}{\partial y}k\frac{\partial T}{\partial y}+\frac{\partial}{\partial z}k\frac{\partial T}{\partial z} \tag{18}$$

Frequently simpler forms of this equation are satisfactory. For example, for constant thermal conductivity, k can be moved in front of the differentiation sign and with ρc to form $\alpha = k/\rho c$:

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{19}$$

In cylindrical coordinates with axi-symmetry (i.e., T independent of θ), the same equation takes the form

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) \tag{20}$$

For 2-D axi-symmetric problems in polar coordinates, the z-derivative term is omitted.

For *steady state* problems, the time derivatives vanish, the reduced equations just consist of contents of the parentheses set equal to zero.

B. Steady Conduction through a Cylindrical Shell

We now consider a cylindrical shell of length L, and inner and outer radii R_i and R_o, at temperatures T_i and T_o respectively. Although this problem can be solved from the more general partial differential equation discussed in the last section, the simple geometry permits a straightforward formulation in ordinary differential equations, which will be illustrated here. The cross section of the shell is shown in Fig. **5**. The total heat transfer rate Q at any radius r is given by Eq. (3) to be

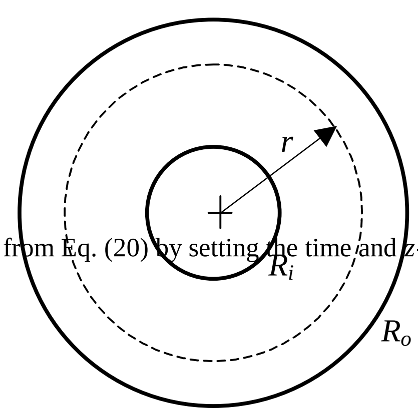
$$Q(r) = -L(2\pi r)\frac{dT}{dr} \tag{21}$$

In this case we expect the heat flux q_r to decrease as r increases, because the same total heat transfer is spread over a larger cylindrical area. So the differential equation required is obtained by required Q to be constant and unchanging:

$$\frac{d}{dr}Q(r) = 0 \qquad \text{or} \qquad \frac{d}{dr} r \frac{dT}{dr} = 0 \quad \text{if} \quad k = \text{uniform}^{8} \tag{22}$$

The boundary conditions are

$$T(R_i) = T_i, \qquad T(R_o) = T_o \tag{23}$$



[8] This equation can also be obtained from Eq. (20) by setting the time and z-derivatives equal to zero.

Fig. 5. The Cross-section of a cylindrical shell

This system can be solved easily by two successive integrations and evaluating the constants with the help of boundary conditions. The results is

$$Q = \frac{2\pi L}{\ln \frac{R_o}{R_i}} k\Delta T \tag{24}$$

C. More Complex Shapes, Conduction Shape Factors

For more complex body shapes than the plane slab and the cylindrical shell, the principle of the method of analysis is the same. The only difference is that in most cases the conservation of energy leads to a partial differential equation[9]

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{25}$$

with boundary conditions given at all exterior surfaces. Methods for solving such equations are beyond the scope of this brief introduction. They can be found in mathematical texts, especially those on numerical methods.

For frequently encountered body shapes, however, such solutions have already been obtained by engineers and tabulated in reference sources. We will briefly describe the use of such a resource.

It can be shown from dimensional considerations that all solutions of steady heat condition problems with an imposed ΔT are in the following format:

$$Q = S\,k\Delta T \tag{26}$$

The factor S, called the ***"Conduction Shape factor"***, contains all the important information on the shape and size of the body. It has the dimension $[L]$, which is a consequence of the fact that heat transfer rate is proportional to the cross section area, of dimension $[L^2]$, and inversely proportional to the length of the heat conduction path, of dimension $[L]$. Tabulations of the conduction shape factor can be found in many heat transfer reference sources, including most

[9] This can be obtained from Eq. (19) by setting the parenthesis = 0 and omitting the z-derivative term.

text books. An abbreviated list of some of the most useful shape factors is shown in Table 4. for illustrative purposes and to provide for simple calculations.

D. Temperature-dependent Conductivity and the Average Conductivity

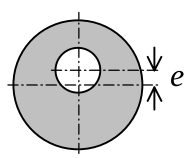
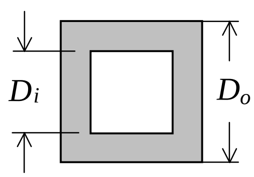
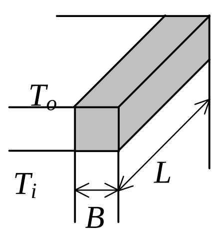
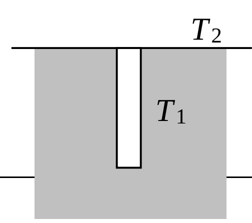
Thermal conductivities are strong functions of temperature. For problems spanning a small enough temperature range, this variation may be inconsequential. Assuming the conductivity to be constant and equal to any representative value within the range would lead to little error. On the other hand, if the variation of conductivity is large, one may ask if there is a technique to avoid the complications of solving non-linear equations? The answer is yes, at least for steady state conduction problems.

For steady conduction problems, it can be shown that the average thermal conductivity defined in the usual sense will yield the correct heat transfer calculations:

$$\bar{k} \equiv \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} k\, dT \tag{27}$$

Table 4. A Short List of Selected Conduction Shape Factors *S*

$$Q \equiv S k \Delta T$$

Description	Schematic	Conditions	Shape Factor
Eccentric cylindrical shell of length L, diameters D_i & D_o, and eccentricity e.		$D_o > D_i$ and $L >> D_o$	$\dfrac{2\pi L}{\cosh^{-1}\left(\dfrac{D_o^2 + D_i^2 - 4e^2}{2 D_o D_i}\right)}$
Rectangular shell of length L, with square cross sections of sides D_i & D_o.		$D_o / D_i < 1.41$ $D_o / D_i > 1.41$	$\dfrac{2\pi L}{0.785 \ln(D_o / D_i)}$ $\dfrac{2\pi L}{0.93 \ln(D_o / D_i) - 0.0502}$
Additive correction for edge of intersecting walls, when walls have been computed using plane slab formula		$L / B > 0.2$	$0.54L$
Cylinder, of diam. D & length L, surface at T_1, buried in semi-infinite medium normal to plane		$L >> D$	$\dfrac{2\pi L}{\ln(4L / D)}$

surface, at T_2			

Example 7. Determine the heat transfer rate per meter for a 2.5 cm diam. pipe at 100°C, insulated with 1 cm of fiberglass insulation with the outer surface temperature equal to 30°C.

Solution: We will solve this problem with two methods and compare the results.

Method 1. We will use Eq. (24) for the cylindrical shell.

$$Q = \frac{2\pi L}{\ln \frac{R_o}{R_i}} k\Delta T = 2*\pi*L*k*(100-30)/\ln(4.5/2.5) = 2*\pi*1*0.046*70/\ln(4.5/2.5) = 34.4 \text{ W}$$

Method 2, We will use the entry for eccentric cylinders in the shape factor table, and setting the eccentricity equal to zero.

$$Q = Sk\Delta T = \frac{2\pi L}{\cosh^{-1}\left(\frac{D_o^2 + D_i^2 - 4e^2}{2D_o D_i}\right)} k\Delta T = \frac{2\pi L}{\cosh^{-1}\left(\frac{4.5^2 + 2.5^2}{2 * 4.5 * 2.5}\right)} k\Delta T$$

$$= \frac{2\pi L}{\cosh^{-1}(1.1777)} k\Delta T = \frac{2\pi L}{0.5878} k\Delta T = \frac{2\pi}{0.5878} 0.046 * 70 = 34.4 \text{ W}$$

4. Convection Heat Transfer Coefficients

A. Correlations of h in terms of Nusselt Numbers

We have previously introduced the heat transfer coefficient h for the computation of convective heat transfer:

$$\dot{q} = h\Delta T; \qquad \text{where} \quad \Delta T \equiv \left|T_w - T_{fluid}\right| \tag{28}$$

Only two sample formulas for determining h were given. We shall now discuss how to determine h for wide ranges of conditions.

In principle, h can be evaluated from solutions of the partial differential equations of heat transfer and fluid flow. This can be quite involved and is the concern of more advanced studies of convection. For simple and frequently encountered flow configurations, use can made of compilations of formulas and correlations already established by analysis and experiments. These can be found for in handbooks and textbooks of elementary heat transfer. These formulas are typically given in the form of a dimensionless parameter called Nusselt number, defined as hL/k, where L is a characteristic length, either length, height or diameter as is appropriate for the problem. For *forced convection* when the velocity is imposed, the Nusselt number is represented as a function of the Reynolds number Re ≡ UL/ν , a dimensionless velocity for viscous flow, and the Prandtl number Pr ≡ ν/α, a fluid property equal to the ratio of momentum and heat diffusivities. ν denotes the kinematic viscosity [m^2/s], equal to μ/ρ, the dynamic viscosity [N-s/m^2 or kg/s-m] divided by density [kg/m^3]. One physical interpretation of ν is the diffusivity for momentum. α is the heat diffusivity [m^2/s], $\equiv k/\rho c_p$. For free convection, when convection is due to buoyancy associated with the temperature distribution, the velocity is unknown. In this case the Nusselt number is typically given as a function of the Rayleigh number $\mathrm{Ra} \equiv g\beta\Delta T L^3/\nu\alpha$.

To avoid ambiguity in the choice of the length parameter, frequently Nu, Re and Ra are written with the proper length parameter as a subscript. For example, Nu_D, Re_D, Ra_D use the diameter D for the length parameter L in their definitions.

As a general rule, fluid properties can vary significantly between the surface and the fluid interior.

A short list of the most important correlations are given below as examples and for the convenience of students. More complete lists can be found in dedicated textbooks. These are often divided into groups or chapters for *internal forced convection, external forced convection, and free convection*. Sample fluid properties required for these calculations are given in Table **5**.

a. Turbulent Flow inside Circular Tubes

This is an example of *internal forced convection*. A frequently cited correlation is

$$\mathrm{Nu}_D = 0.023\,\mathrm{Re}_D^{0.8}\,\mathrm{Pr}^{1/3} \quad \text{for} \quad \mathrm{Re}_D > 10^4,\quad L/D > 10,\quad 0.7 < \mathrm{Pr} < 160 \tag{29}$$

In this equation, the Nusselt and Reynolds numbers are based on the internal pipe diameter, and the velocity in the Reynolds number is the average fluid velocity (subscript *af*) over the cross-sectional area of the pipe.

$$\mathrm{Nu}_D \equiv \frac{hD}{k}; \qquad \mathrm{Re}_D \equiv \frac{U_{ave}D}{\nu} \tag{30}$$

$$U_{af} \equiv \frac{1}{A_c}\int_{A_c} u\,dA = \frac{\text{Volumetric flow rate}}{A_c} = \frac{\text{Mass flow rate}}{\rho A_c} \tag{31}$$

The definition of the heat transfer coefficient, and the computation of heat transfer, also make use of the average fluid (subscript *af*) temperature:

$$q_w \equiv h\left(T_w - T_{af}\right) \tag{32}$$

$$T_{af} \equiv \frac{1}{U_{af}A_c}\int_{A_c} u\,T\,dA \tag{33}$$

For many problems the fluid properties may change significantly between the wall temperature and the fluid temperature. Some correlations give specific remedies to account for this change. A general procedure, suitable for modest variations of property or when better information is not available, is to evaluate the properties at the *film temperature*, taken to be the mid-point between the wall temperature and the fluid temperature:

$$T_{film} = \frac{1}{2}\left(T_w + T_{af}\right) \tag{34}$$

b. Flow across Circular Tubes normal to the Direction of flow

This is an example of external flow. Earlier correlations are based on power law correlations similar to Eq. (29), with different sets of coefficients and exponents for different Reynolds number ranges. More recently a comprehensive equation covering the entire range of Re of interest has been obtained by Churchill and Bernstein:

$$\overline{\mathrm{Nu}}_D = 0.3 + \frac{0.62\mathrm{Re}_D^{1/2}\,\mathrm{Pr}^{1/3}}{\left[1+\left(\frac{0.4}{\mathrm{Pr}}\right)^{2/3}\right]^{1/4}}\left[1+\left(\frac{\mathrm{Re}_D}{282{,}000}\right)^{5/8}\right]^{4/5}; \qquad \mathrm{Re}_D\mathrm{Pr} > 0.2 \tag{35}$$

In this case the flow and heat transfer is only affect by the *outside* diameter, which is used in the definition of Nu and Re. For external flow the average fluid temperature is not defined. Instead, the *undisturbed fluid temperature*, often denoted by T_∞, is used in the definition of h and the calculation of heat transfer:

$$\overline{q}_w \equiv \overline{h}\left(T_w - T_\infty\right) \tag{36}$$

The symbol "bar" over $\overline{\mathrm{Nu}}$, $\overline{q}$, and $\overline{h}$ indicate that these are averaged Nusselt number, heat transfer coefficient, and heat flux over the entire external surface of the body.

The definition of the film temperature is analogous to that used in internal flow:

$$T_{film} \equiv \frac{1}{2}\left(T_w + T_\infty\right) \tag{37}$$

c. Turbulent Flow past a Plate Parallel to the Flow

$$\overline{\mathrm{Nu}}_L = \left(0.037\mathrm{Re}_L^{0.8} - 871\right)\mathrm{Pr}^{1/3}; \qquad 5(10)^5 < \mathrm{Re}_L < (10)^8,\ 0.6 < \mathrm{Pr} < 60 \tag{38}$$

This is another external flow configuration. The heat transfer coefficient is based on T_∞ as in all other external flow cases. The Nusselt and Reynolds numbers, however, are based on the plate length L (in the direction of flow).

d. Free Convection around Vertical Plates

$$\overline{\mathrm{Nu}}_H = \left\{0.825 + \frac{0.62\mathrm{Ra}_H^{1/6}}{\left[1+\left(0.429/\mathrm{Pr}\right)^{9/16}\right]^{8/27}}\right\}^2 \tag{39}$$

As in external flow forced convection, the heat transfer coefficient is defined on the basis of $\Delta T \equiv T_w - T_\infty$. As indicated by subscript, both Nu and Ra are based on height H as the characteristic length. The Rayleigh number Ra is

$$\mathrm{Ra} \equiv \frac{g\beta\Delta T H^3}{\nu\alpha} \tag{40}$$

e. β is the coefficient of thermal expansion, here defined on the basis of density variations

$$\beta \equiv -\frac{1}{\rho}\frac{\partial\rho}{\partial T}\bigg|_p \qquad \text{or} \qquad -\frac{1}{\rho_w}\frac{\rho_\infty - \rho_w}{T_\infty - T_w} \tag{41}$$

For practical purposes either of these is virtually equal to the volume coefficient of thermal expansion defined on the basis of specific volume. All three are also about 3× the linear coefficient of thermal expansion defined on the basis of length.

For gases the perfect gas law yields

$$\beta = \frac{1}{T_{film}} \tag{42}$$

f. Free Convection around Horizontal Cylinders

$$\overline{\mathrm{Nu}}_D = \left\{0.6 + \frac{0.387\mathrm{Ra}_D^{1/6}}{\left[1 + \left(0.559/\mathrm{Pr}\right)^{9/16}\right]^{8/27}}\right\}^2 \qquad \mathrm{Ra}_D \le 10^{12} \tag{43}$$

g. Free Convection above Heated Horizontal Plate or below Cooled Horizontal Plate

$$\overline{\mathrm{Nu}}_L = 0.15\,\mathrm{Ra}_L^{1/3} \qquad \mathrm{Ra}_L > 10^7 \qquad L \text{ is plate width} \tag{44}$$

h. Free Convection across Enclosed Vertical Air Spaces

This equation is for the combined heat transfer coefficient for both vertical surfaces, i.e., $\mathrm{Nu} \equiv h_{ab}L/k,\ q_{ab} \equiv h_{ab}\left(T_a - T_b\right)$ where T_a, T_b are the two surface temperatures. L is gap width, and H is gap height.

$$\overline{\mathrm{Nu}}_L = 0.42\,\mathrm{Ra}_L^{1/4}\left(\frac{H}{L}\right)^{-0.3} \qquad 10^4 < \mathrm{Ra}_L < 10^7,\ 10 < \frac{H}{L} < 40,\ 1 < \mathrm{Pr} < 2(10)^4 \tag{45}$$

Table 5. Some Properties Pertinent to Convection

	Density ρ [kg/m^3]	Dynamic viscosity: μ [N-s/m^2]	Kinematic viscosity ν [m^2/s]	Coeff. of Thermal Expansion β [K^{-1}]	Prandtl Number Pr [1]
Water, 20°C	998.2	1.002 E–3	1.004E–6	2.12E-4	7.0
60°C	983.3	0.464E-3	0.472E-6	5.25	3.0

100°C	958.4	0.282 E–3	0.294E–6	7.50E-4	1.76
Air, 20°C	1.204	1.82E–5	1.51E–5	$1/T$(abs.)	0.7
100°C	0.9461	2.17 E–5	2.29 E–5	$1/T$(abs.)	0.7

5. Radiation Fundamentals and Simple Calculations

The introduction of radiation in Section 1 was meant to be phenomenological, so the student can have an appreciation of its properties and magnitudes relative to conduction and convection. Here we will review its scientific basis and try to deduce the method of analysis for some simple configurations in a more rational way. Toward the end of these notes we will re-visit radiation analysis a third time, introducing an algebraic method for analyzing radiation among more than two surfaces.

A. Black Bodies and Stephan-Boltzmann Law

Early scientific observations and later thermodynamic theory established that all matter warmer than absolute zero emit and absorb thermal radiation, and that the best emitter is also the best absorber. Since in the visible spectrum black surfaces are the best absorbers, the term ***black body*** is used to describe the perfect emitter and the perfect absorber. Observations and theory further established that black bodies, the ideal emitters, emit thermal radiation at a rate proportional to the forth power of the absolute temperature. This is the Stephan-Boltzmann law of radiation:

$$\dot{q}_{rb}^{+} = \dot{q}_{b}^{+}(T) = \sigma T^{4} \tag{46}$$

σ, the *Stephan-Boltzmann* constant, is a universal constant related to the basic physical constants, and is equal to $5.67(10)^{-8}$ W/m^{2}K^4.

For emitters which are not perfect, a surface property called *emissivity* is defined as the ratio of real emitted power to the black-body emitted power:

$$\dot{q}_{r}^{+} = \varepsilon \dot{q}_{b}^{+}(T) = \varepsilon \sigma T^{4} \tag{47}$$

The superscript + indicates that this is the outbound radiation from the surface.

B. Absorptivity and Equality to Emissivity

To calculate the net radiation heat transfer, we must also evaluate the inbound radiation and the fraction that is absorbed by the surface. Since the inbound radiation can include radiation either emitted or reflected by any surface that can be "seen" by the surface under consideration, the calculation of the net radiation exchange require detailed knowledge of the geometrical relationships of all surfaces which are visible to each other.

We will call the inbound radiation, the *incident radiation heat flux* $\dot{q}_{ri}$. The fraction that is absorbed by the surface is related to another surface property, the *absorptivity*[10] α:

$$q_r^- = \alpha q_{ri} \tag{48}$$

Accordingly the net radiative heat transfer from a surface is

$$q_r = q_r^+ - q_r^- = \varepsilon\sigma T^4 - \alpha q_{ri} \tag{49}$$

Note that for opaque surfaces, whatever is not absorbed is reflected. Hence the *reflectivity* is just $(1 - \alpha)$.

Under conditions of thermodynamic equilibrium, it can be shown that the absorptivity and emissivity must be equal, $\alpha = \varepsilon$. For practical heat transfer calculations, thermodynamic equilibrium is never completely achieved. The equality of absorptivity and emissivity has nevertheless been found to be an expedient and satisfactory assumption:

$$\alpha = \varepsilon \tag{50}$$

This assumption is independent of the *gray body assumption* to be described in a later section, and can be justifiable even when the gray body assumption is not valid. However, the latter does lead to the same equality.

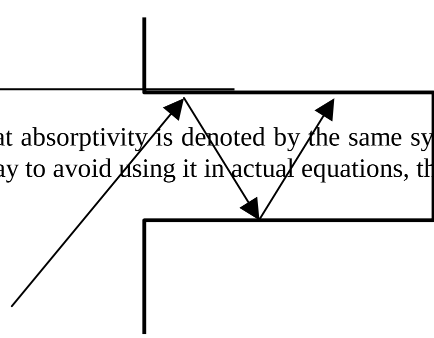
Since this assumption is almost always employed, the term *absorptivity* and its symbol α are rarely used. We tend to use *emissivity* and absorptivity synonymously and use ε for both. In the same vein, we also use $1 - \varepsilon$ to denote reflectivity. The student, of course, should be vigilant for conditions when this careless practice is not justified. For example, the absorptivity for solar radiation, mainly in the visible wavelengths, are likely to be quite different from the emissivity of surfaces at moderate temperatures, emitting primarily in infrared.

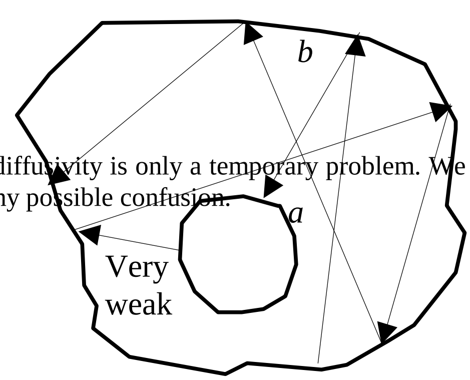
C. Multiple Reflection and Virtual Black Bodies

If a ray of radiation enters a deep cavity, it is likely to undergo a number of reflections before it can emerge again. Since for common material surfaces each encounter leads to the absorption of a large fraction (typically >50%) of the radiant power, most of the radiant power is absorbed before it can emerge.

The opening of the cavity thus acts as a *virtual black body*, and, if treated as a virtual surface, has an effective absorptivity/emissivity of nearly 1, and an effective reflectivity of nearly zero:

$$\varepsilon \approx \alpha \approx 1 \qquad 1 - \varepsilon \approx 0 \tag{51}$$





[10] The fact that absorptivity is denoted by the same symbol as heat diffusivity is only a temporary problem. We will soon find a way to avoid using it in actual equations, thus avoiding any possible confusion.

Fig. 6. Cavity and Virtual Black Bodies

A similar situation occurs when a large body encloses a smaller body. In this case any radiation emitted by the small body will be absorbed almost completely by the larger body before the ray of radiation strikes the small body again. Hence the large enclosing body also acts as a virtual black body.

D. Radiation Heat Transfer between Two Surfaces Close Together

A major complication in radiation calculations is the geometrical problem due to the fact that radiation simultaneously propagates to all directions. Thus accounting for the view angle is important, not only for one opposing surface, but for all surfaces within view, is necessary. Methods for accounting for such geometrical affects will be described in a later section. For two commonly encountered geometries, however, we may by-pass this complexity. The first of such simple geometries is for two surfaces so close together that very little radiation from these two surfaces ever strikes other surfaces.

Fig. 7. Pairs of surfaces so close together that effectively no other surface is in view

To consider the net radiative exchange, we first consider radiation emitted by *a*. From Section 1, we know this to be equal to $\varepsilon_a \sigma T_a^4$. When this radiation interacts with *b*, an amount equal to $\varepsilon_b \varepsilon_a \sigma T_a^4$ is absorbed, and an amount equal to $(1-\varepsilon_b)\varepsilon_a \sigma T_a^4$ is reflected back toward *a*. After reflection from *a*, an amount equal to $(1-\varepsilon_a)(1-\varepsilon_b)\varepsilon_a \sigma T_a^4$ is again directed toward *b*, and an amount equal to $(1-\varepsilon_a)(1-\varepsilon_b)\varepsilon_a \varepsilon_b \sigma T_a^4$ is absorbed. This reflection-reflection-absorption process is repeated again and again, each time the amount absorbed is reduced by a factor $(1-\varepsilon_a)(1-\varepsilon_b)$. The total amount absorbed, equal to the sum of each of such terms, can thus be represented by an infinite geometrical series, with the first term a_o equal to ratio equal to $\varepsilon_b \varepsilon_a \sigma T_a^4$ and the ratio *r* equal to $(1-\varepsilon_a)(1-\varepsilon_b)$. In math courses we have learned that the sum of a geometrical series is $a_o/(1-r)$. Thus the total radiation emitted by *a* and absorbed by *b* is

$$\dot{q}_{a\to b} = \frac{\varepsilon_a \varepsilon_b}{1-(1-\varepsilon_a)(1-\varepsilon_b)}\sigma T_a^4 \qquad = \frac{1}{\frac{1}{\varepsilon_a}+\frac{1}{\varepsilon_b}-1}\sigma T_a^4 \tag{52}$$

Similar expression can be obtained for the total radiation emitted by *b* and absorbed by *a*. Hence the net radiative exchange between the two is

$$\dot{q}_{r,ab} = \frac{1}{\frac{1}{\varepsilon_a}+\frac{1}{\varepsilon_b}-1}\sigma\left(T_a^4 - T_b^4\right) \tag{53}$$

E. Radiation Heat Transfer for a Small Body A Enclosed by Large Body B

This is discussed in connection with Fig. 6 and illustrated in Fig. 3 and on the right hand side of Fig. 6. Treating the enclosure as a black body and setting $\varepsilon_b = 0$ in Eq. (53), we have

$$\dot{q}_{r,ab} = \varepsilon_a \sigma\left(T_a^4 - T_b^4\right) \tag{54}$$

This was the result cited in Section 1 as an introduction to radiation.

F. Generic Forms of Equations for Radiative Exchange

These results suggest, as can be mathematically shown to be true later, that radiation heat exchange between any two surfaces can be written in the form

$$\dot{q}_{r,ij} = \mathsf{G}_{ij}\sigma\left(T_i^4 - T_j^4\right) \tag{55}$$

G_{ij} is a dimensionless coefficient depending on the surface properties and geometries of all the surfaces within view of surface *i* and *j*. These can be summed for all possible surfaces *j* which can exchange radiation with surface *i*, leading to an expression for total *radiative heat transfer* from surface *i* through exchanges with all surfaces it can see:

$$\dot{q}_{r,i} = \mathsf{G}_i\sigma\left(T_i^4 - T_{mre}^4\right) \tag{56}$$

Here G_i represent the sum of G_{ij} for all *j*:

$$\mathsf{G}_i \equiv \sum_{\text{all } j\text{'s}} \mathsf{G}_{ij} \tag{57}$$

The second temperature in Eq. (56), T_{mre} is the *mean radiative environment temperature*. If needed, a formula for computing it from the T_j's can be easily derived. This will not be done here.

A more general form of Eq. (10), not restricted to a small object completely enclosed by a larger environment, is

$$h_{ri} = 4G_i\sigma T_{ave}^3\left[1+\frac{1}{2}\left(\frac{\Delta T}{T_{ave}}\right)^2\right] \quad ; \quad T_{ave} \equiv \frac{1}{2}\left(T_i + T_{mre}\right), \quad \Delta T \equiv \left|T_i - T_{mre}\right| \quad (58)$$

For the two simple cases described in Sub-sections *C* and *D*, G_a are obviously $1/\left(1/\varepsilon_a + 1/\varepsilon_b - 1\right)$ and ε_a respectively. The radiant ambient temperature $T_{r,a}$ is just T_b.

It can be shown that G_i is always equal to or less than the emissivity of surface *i*:

$$G_i \le \varepsilon_i \quad (59)$$

Hence even if the configuration of the surfaces does not belong to either of the simple cases above, it is still possible to perform quick, rough estimates of radiation exchange without engaging in the complex calculation procedures to be described in a later section.

6. The Fin, or Extended Surface

Surface projections, when made of materials with sufficiently high conductivity, increase the heat transfer area and enhances heat transfer. Familiar examples include the fins around air-cooled motor cycle engines and fins around the hot water tubing in many baseboard heaters. They are also found in compact heat exchangers. The wires soldered between condenser coils of some refrigerators serve the same function, though they are sheets like the more conventional fins.

The end of the fin attached to the primary heat transfer surface is called the base of the fin, with temperature denoted by T_b . Toward the tip, temperature is closer to the ambient fluid temperature T_f than the base of the fin. Hence fins do not increase heat transfer in proportion to the area. This decrease in the effectiveness of the area is accounted for by fin efficiency, defined as follows.

$$Q_{fin} = \eta_{fin} h A_{fin}\left(T_b - T_f\right) \quad (60)$$

Formulas and graphs for determine the fin efficiency can be found in textbooks and handbooks.

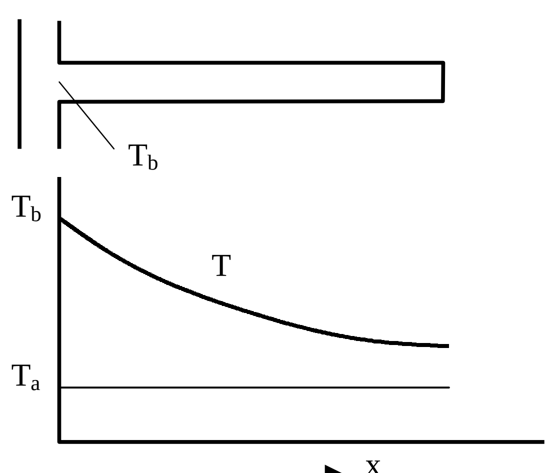


Fig. 8. A fin for increasing convection heat transfer

7. Heat Exchangers

A. Overall Heat Transfer Coefficient

This is just the overall unit conductance for composite walls. This is a quantity convenient for computations involving heat transfer through pipe walls with convective heat transfer coefficients on both sides. It is commonly denoted by the capital letter[11] *U*. Note that for most pipes, the outside area is larger than the inside area, hence a subscript o or i is often used to indicate on which surface *U* is defined. It can only be used with the area with the same subscript for the computation of heat transfer rates:

$$Q \equiv A_o U_o \Delta T \equiv A_i U_i \Delta T \tag{61}$$

$$U_o = \frac{1}{\frac{1}{h_o} + \frac{\Delta r}{k} + \frac{A_o}{A_i}\frac{1}{h_i}} \qquad [\mathrm{W / mK}] \tag{62}$$

$$U_i = \frac{1}{\frac{1}{h_i} + \frac{\Delta r}{k} + \frac{A_i}{A_o}\frac{1}{h_o}} \qquad [\mathrm{W / mK}] \tag{63}$$

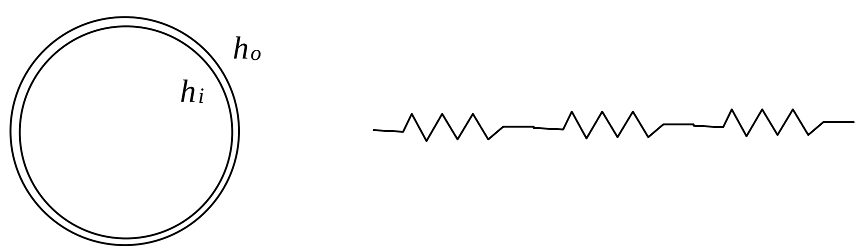


Fig. 9 Heat transfer through a pipe wall, with internal and external

[11] Not to be confused with the velcotu, also denoted by *U*.

heat transfer coefficients, and its network representation

These expressions have been derived using the series combination rule discussed above. Note that typically the wall thickness is thin relative to the radius, and the resistance through the metal wall is small relative to the total resistance. Hence we have approximated the cylindrical shell by a planar slab. If this approximation is not sufficiently accurate the proper logarithmic form of the resistance should be used.

8. Contact Resistance

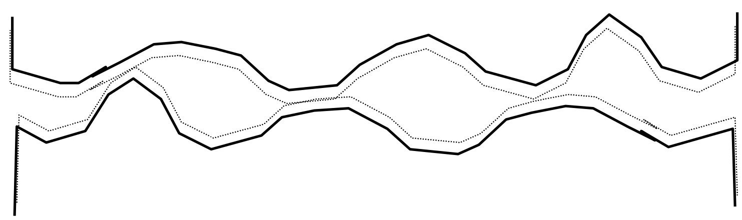


Fig 10 The formation of contact resistance between two rough and contaminated surfaces

Contact resistance is the resistance between two solid surfaces in contact. This resistance is due in a large part to the roughness of the surfaces, which prevents intimate contact between the two. Other contributors to the resistance include oxides and other impurities. The resistance is a strong function of the surface finish and prior history of exposure to contaminants. It is also a strong function of the pressure applied between the two bodies.

Contact resistances are important considerations in certain high heat flux applications, including cooling of power electronics and other electronic packaging considerations. Tabulations of contact resistances, sometimes expressed in conductance units, can be found in standard references. Because their values depend on surface condition, great care should be exercised in accepting these tabulations.

9. Transient Heat Transfer

A. Lumped Parameter Analysis

a. Formulation

Consider a relatively compact object without excessive protuberances, and with sufficiently high conductivity that the interior temperature can be assumed uniform. This may include simple blocks, rods and plates shown at the left of Fig. 11, as well as more complex objects shown at right. If this object, originally at temperature T_o, is subjected to cooling or heating in an environment of temperature T_∞ through a heat transfer coefficient *h*, what is the temperature as a function of time?

The increase or decrease of temperature with time reflects the change in stored energy. The key to this analysis is the assumption of uniform temperature, which simplifies the accounting of stored internal energy. We can express the latter as mcT or ρVcT . Here V denotes the total volume of the object. The rate of change of internal energy is then $\rho Vc(dT/dt)$. Conservation of energy takes the form

$$\rho c V \frac{dT}{dt} = -hA(T - T_\infty) \tag{64}$$

Here *A* denotes the surface area through which heat transfer to/from the ambient takes place. *h* can be viewed as the total conductance per unit area, including both convective and radiative contributions. It can be defined to account for resistance due to an insulating layer, if required. Thus both ρVcT and hA are *lumped parameters*, permitting a simple mathematical formulation of the problem.

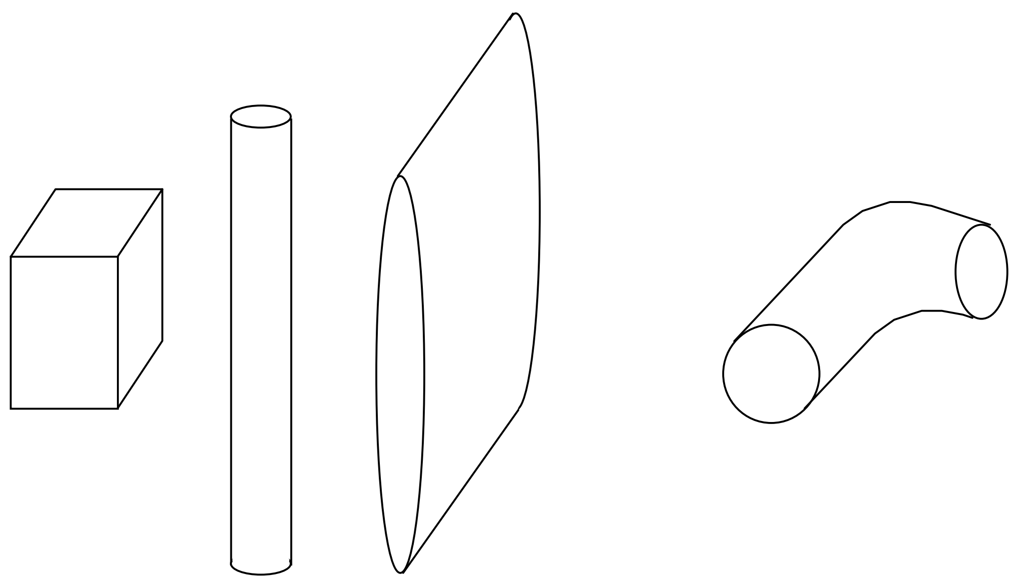


Fig. 11. Sample objects of compact shape suitable for lumped parameter analysis

This first order differential equation requires an initial value, which is

$$T(0) = T_o \tag{65}$$

b. Solution for Constant Coefficients

If ρ, c, h, are constants, this equation is linear, and can be solved easily by *separation of variables*, moving all terms and factors containing *T* to the left of the equation sign, and terms and factors containing *t* to the right. This makes the left hand side an exact differential for (T −

T_∞) and the right hand side an exact differential for t. The solution, after using the initial condition to determine the constant of integration, is

$$\boxed{\frac{T - T_\infty}{T_o - T_\infty} = e^{-t/\tau_c}} \tag{66}$$

where τ_c is a combination of all the parameters

$$\boxed{\tau_c \equiv \frac{\rho c V}{hA}} \tag{67}$$

τ_c is sometimes called the *time constant* of the system. It represents the time required to reduce the temperature difference by a factor e. It is thus a measure of the speed of response, or more precisely the sluggishness of response, of the system.

c. Solution for Non-constant Coefficients

The parameters ρ, c, h are rarely true constants, as thermal properties tend to vary with temperature. Free convection, depending on the temperature difference, and radiation, depending approximately on the third power of the absolute temperatures, can have strong temperature dependences. What simple recourse is available to us if we wish to improve upon the constant coefficient assumption?

The solution is quite simple indeed. Re-arranging Eq. (64) and expressing τ_c as $\tau_c(T)$, a function of temperature, we have

$$\frac{dT}{dt} = -\frac{1}{\tau_c(T)}(T - T_\infty) \tag{68}$$

Initially, as $T = T_o$, the rate of temperature change is

$$\left.\frac{dT}{dt}\right|_o = -\frac{1}{\tau_c(T_o)}(T_o - T_\infty) \tag{69}$$

We can expect that this rate will remain relatively constant for a small time interval δt, say a few percent of the value of the initial time constant $\tau_c(T_o)$. At the end of this interval, $t_1 = 0 + \delta t$, the new temperature can be calculated approximately as

$$T(t_1) \cong T_o + \delta t \left.\frac{dT}{dt}\right|_o = T_o - \delta t \frac{T_o - T_\infty}{\tau_c(T_o)} \tag{70}$$

At t_1, the new temperature will permit a new evaluation of $\tau_c(T)$. The process can be repeated many times to continue the evaluation of the desired function $T(t)$. Such repetitive computations can be trivially managed by common computer softwares, such as Microscoft *Excel*, among others. The degree of accuracy can be regulated by the user by simply changing the time interval δt.

d. Assessing the Accuracy of the Uniform Temperature Assumption – Biot Number

A simple analysis can lead to a valuable test for the accuracy of the uniform temperature assumption. As we will demonstrate, however, the test is not foolproof and should be used with keen observation of the physical configuration involved.

From Eq. (3) we expect that the magnitude of interior temperature variation is inversely proportional to the conductivity and directly proportional to the length of the conduction path. Also the largest departure from surface temperature would occur at the deepest interior. This would typically be the mid-point of the thickest section. From this point outward the majority of heat conduction would be in the direction of the shortest distance to the surface. It is this path of heat conduction that would govern the magnitude of the interior temperature variation. The latter along such a heat conduction path is sketched in Fig. 12.

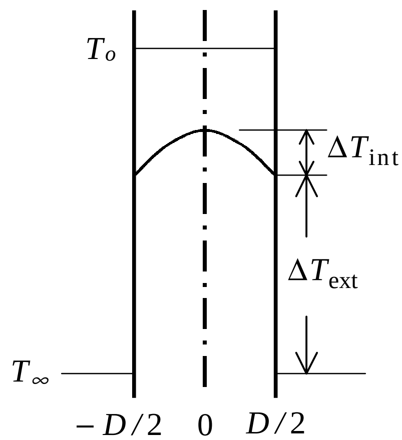


Fig. 12. Interior and exterior temperature variations in a cooling body

In this figure, D is the thickness of the *thickest section* in the *thinnest direction*, because this is the critical heat conduction path discussed above. For example, for a rod, D is the diameter. For the plate of non-uniform thickness shown, D is the center thickness.

At the surface, the conservation of energy requires that the heat flux by conduction must be equal to the product of h and the exterior temperature difference $h(T_w - T_\infty) \equiv h\Delta T_{\text{ext}}$:

$$k \left.\frac{dT}{dx}\right|_{x=0} = h\Delta T_{\text{ext}} \tag{71}$$

Even without knowing the exact shape of the curve, it is reasonable to expect that it can be approximated by a parabola. For a parabola, it can be shown that

$$\left.\frac{dT}{dx}\right|_{x=0} = 4\frac{\Delta T_{\text{int}}}{D} \tag{72}$$

Combining the two equations, and using an *approximately equal* sign $\cong$ because the parabola is only an approximation, we have

$$\frac{\Delta T_{\text{int}}}{\Delta T_{\text{ext}}} \cong \frac{1}{4}\frac{hD}{k} \tag{73}$$

The dimensionless parameter on the right is called the Biot Number

$$\mathrm{Bi}_D \equiv \frac{hD}{k} \tag{74}$$

Note that it resembles the Nusselt number, except that the conductivity k is that of the solid, and not that of the fluid.

ΔT_{int} is the maximum interior temperature difference, and is not the actual error. The error should be $\Delta T_{\text{int},m} \equiv T_m - T_w$, the difference between the mean interior temperature and the wall temperature. It can be shown that for a parabolic temperature distribution, depending on geometery,

$$\frac{\Delta T_{\text{int},m}}{\Delta T_{\text{int}}} \cong 0.4 - 0.67 \tag{75}$$

The larger coefficient, 0.67, is for flat plates. Accordingly, the relative error is

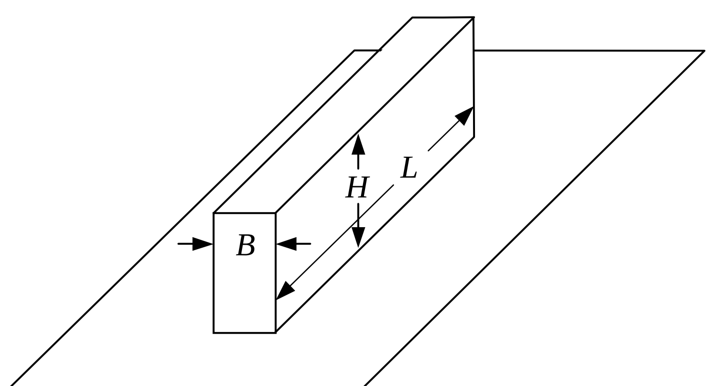
$$\boxed{e \le \frac{1}{6}\mathrm{Bi}_D} \tag{76}$$

This estimate should help the student decide whether lumped parameter is of sufficient accuracy. The accuracy required, of course, should be a value judgment of the user, considering the circumstances and application of his calculation. A 10% accuracy can be quite adequate for some calculations, and not adequate for another calculation. For example, if a problem has a Biot number of 0.3, the user may consider the expected 5% accuracy to be quite satisfactory, and decide against a more time-consuming numerical analysis.

This error estimate is a more intelligent way to assess the suitability of lumped parameter analysis than the rigid Bi < 0.1 condition stated in many textbooks.

Example 8 (This is a multi-issue example consisting of several steps of analysis, closer to a real engineering problem than the more familiar single-issue homework problems.)

A just-molded plastic part, 10 mm thick (B) × 20 mm high (H) × 60 mm long (L), is ejected from a mold at 100°C and rests on it narrow, long side (10 mm × 60 mm side) in a room with air and wall temperatures at 25°C. Determine the cooling time necessary for the part to reach 60°C, which is cool enough for packing. The properties are k = 0.8 W/mK, ρc = 1.9E6 J/m^{2}K, ε = 0.8.



Solution:

We begin with the crudest assessment of the situation, then gradually refining the calculations.

e. What physics? Quick assessments

Cooling by forced convection? free convection? Radiation? No mention of imposed air flow. From Table 2, free convection and radiation are probably comparable.

Is temperature nearly uniform or non-uniform inside the part? This is governed by the Biot number Bi = *hL/k*. *What length parameter L should be used in the Biot number*? Use *B*, because *B*, rather than *H*, *L*, represent the shortest heat conduction path, hence governs the magnitude of the interior ΔT.

For a quick assessment (to be refined later): From Table 2, assume h = 10 W/mK, h_r = 10 W/mK. Tentative Bi = 20*.01/0.8=0.25. Not small enough for lumped parameter analysis according to the *Bi* < 1 criteria in some books, but perhaps acceptable in view of the error estimate $\Delta T_{\text{int}} / \Delta T_{ext} \approx 1/4\text{Bi}$ derived by us in class. We will proceed for lack of an alternative method of calculation.

We now proceed to do the detailed calculation.

f. What temperature should be used to evaluate properties, ΔT and *h'*s?

During the cooling process, the part's (surface) temperature changes from 100°C to 60°C. Accordingly the mean fluid temperature, the fluid properties, the ΔT for free convection, and the two temperatures from radiative heat transfer coefficient will all be changing. The most accurate approach would be to continuously re-evaluate these coefficients as we follow the cooling process. A less accurate but much simpler approach is to use an average value of the surface temperature for the evaluation of the properties and heat transfer coefficients. For this purpose we choose a medium surface temperature 80°C, midway between 100°C and 60°C.

g. Free convection.

If we neglect the heat transfer through the bottom surface, there are 5 surfaces we need to be concerned with.

Key parameter is Rayleigh no. $\text{Ra} \equiv \dfrac{g\beta\Delta T L^3}{\alpha\nu}$

Vertical surfaces:

What length parameter to use for the Reyleigh and Nusselt numbers? Obviously **Height!**

$$\overline{\mathrm{Nu}}_H = \left\{0.825 + \frac{0.62\mathrm{Ra}_H^{1/6}}{\left[1+(0.429/\mathrm{Pr})^{9/16}\right]^{8/27}}\right\}^2$$

Temperature to evaluate properties: "Film temperature" = aver of wall temp and fluid temp. During the cooling process, wall temp gradually changes from 100°C to 60°C. *How about using a medium temp of* 80°C!

For a medium surface temperature 80°C,

ΔT = 80°C-25°C=55 K (Note we got K, the proper units, w/o converting first!)

T_{film} = (80+25)/2=52.5°C or 325.65 K (whichever is more convenient for the property tables)

Properties (evaluated at 325.65 K, by interpolation if needed: β = 1/325.65 K^{-1}, ν = 1.8E-5 m^2/s, Pr = 0.7, α = (1.8/0.7)*1E-5=2.6E-5 m^2/s, k = 0.028 (~) W/mK.

$$\mathrm{Ra} \equiv \frac{g\beta\Delta T H^3}{\alpha\nu}$$ =(9.81/325.65)*55*0.02^3/(2.6E-5*1.8E-5)=28322

$\overline{\mathrm{Nu}}_H$ =(0.825+0.62*28322^(1/6)/(1+(0.429/0.7)^(9/16))^(8/27))^2=13.84

h = $\overline{\mathrm{Nu}}_H$ *k/H = 13.84*0.028/0.02 = 19.38 W/m^2K *Higher than guessed value used in initial assessment!*

Horizontal Surfce on Top

This requires a different formula involving the Rayleigh number, for which and for the Nusselt number the key length parameter is the width B. The calculation is similar, and will not be done here. We will just use the same heat transfer coefficient for both the sides and the top, incurring possibly a small error.

h. Radiation:

Simultaneous with convection.

Use radiation heat transfer coefficient – more compatible with convection.

Small body, at medium surface temperature 80°C, or 353.15 K, inside large enclosure, at 25°C, or 298.15 K. *Radiation equations require the use of absolute temperatures*.

$$q_{rad,i,j} = \varepsilon_i\sigma\left(T_i^4 - T_j^4\right)$$

Hence $h_{r,i} = \varepsilon_1\sigma\left(T_i + T_{r,a}\right)\left(T_i^2 + T_{r,a}^2\right)$=0.8*5.67E-8*(298.25+353.15)*(298.25^2+353.15^2)

=6.31 W/m^2K.

i. Effective (total) heat transfer coefficient

$h_{eff} = h + h_r$ = 19.38 + 6.31 = 25.69 W/m^2K

j. Calculation of cooling time

The solution of the diff eq for lumped parameter analysis is

$$\frac{T - T_\infty}{T_o - T_\infty} = e^{-t/\tau}$$

where τ is the time constant:

$$\tau = \frac{\rho c V}{hA}$$

Volume V = 0.01*0.02*0.06=1.2E–5 m^3.

Surface area A =2*0.02*0.06+2*0.02*0.01+0.01*0.06=2.4E–3+0.4E–3+0.6R–3 = 3.4E–3 m^2

With ρc = 1.9E6 J/m^{3}K,

τ = 1.9E6*1.2E-5/(25.69*3.4E-3)=261 s

Take natural log of both sides of the solution

$$t_{60} = -\tau \ln\left(\frac{T - T_\infty}{T_o - T_\infty}\right) = -261*\ln((60-25)/(100-25)) = 198.9 \text{ s}$$

k. Estimating the accuracy

The inaccuracy of the lumped parameter formula was due to the assumption of uniform temperature inside the part. This temperature is used to represent different physical quantitaties on both sides of the differential equation, which was used to derive the cooling rate formula:

$$\rho c V \frac{dT}{dt} = -hA\left(T - T_\infty\right)$$

On the left hand side, T is used to calculate the stored internal energy. On the right hand side, T is used to compute the heat loss. If the temperature is not uniform, what should be used on the left hand side should be the average temperature. What should be used on the right hand side should be the surface temperature. Hence the error is the difference between the average interior temperature and the surface temperature. The *relative error*, or *relative accuracy*, is

$$\text{Relative error} = \frac{\Delta T_{ave-sur}}{\Delta T_{sur}} = \frac{T_{ave} - T_{sur}}{T_{sur} - T_\infty} == \frac{T_{ave} - T_{sur}}{\Delta T_{ext}}$$

Since T_{ave} lies approximately halfway between the center temperature and the surface temperature, relative error should be approximately 1/2 of $\Delta T_{\text{int}} / \Delta T_{ext}$. Depending on the shape of the body, this estimate is about ±20%.

In class, I have shown that $\Delta T_{\text{int}} / \Delta T_{ext} \approx \frac{1}{4}\text{Bi}$. Hence

$$\text{Relative error} \approx \frac{1}{6}\text{Bi}$$

With the new value of h_{eff} , 25.69 W/m^{2}K,

$$\text{Bi} = 25.69{*}.01/0.8=0.32$$

Hence the expected error is about 5%, which is acceptable for many purposes.

10. Thermal Radiation Exchange for Gray, Diffuse, Opaque Surfaces

This is the third time we visit the subject of radiation exchange. Previously we were only able to treat two surfaces exchanging radiation with each other. Since any object within view can play a role in radiation exchange, more frequently it is necessary to treat more than two surfaces exchanging heat radiatively at the same time. Fortunately there is a straightforward algebraic procedure which can treat such a system. The method appears messy, but it is really not difficult if there are only a few surfaces that need to be considered.

This is a very tersely written section, meant only for those who are serious about learning this methodology.

A. Reviewing Some Basic Relationships

For diffuse, gray, opaque surface, we assume

$$\varepsilon = \alpha \tag{77}$$

$$\rho = 1 - \alpha \tag{78}$$

We will also make use of algebraic relationships that follow from the definitions of irradiance and radiocity. Irradiance, denoted by the capital letter J is the total radiation intensity incident on a surface. Radiocity, denoted by G, is the total outgoing radiation intensity. It follows these definitions

$$J = \varepsilon E_b + (1 - \varepsilon)G \tag{79}$$

Another form of this equation can be obtained by solving for G

$$G = \frac{J - \varepsilon E_b}{1 - \varepsilon} \tag{80}$$

B. Partial Interception of Radiation as a Geometrical Problem

As shown in Fig. 12, radiation calculation requires knowledge on fraction of emitted radiation actually intercepted by another surface. A related complication is that other surfaces, not shown in Fig. 12, may intercept part of the radiation, and reflect it back to both surfaces. This makes radiation calculations potentially quite complex.

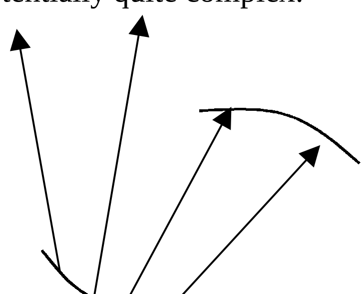


Fig. 13 Illustration of the Geometrical Problem of Partial Interception

Two Simple Cases without Geometrical Difficulties have already been discussed. These are (a) Two surfaces so close together that all the radiation from one would be intercepted by the other. (b) A small body completely enclosed by a bigger body. We will now learn the methodology to deal with more complex geometries.

C. View Factor

$$F_{ij} \equiv \frac{\text{Part of diffuse radiation from } i, \text{ intercepted by } j}{\text{Total diffuse radiation from } i} \tag{81}$$

View factor is a geometrical parameter, independent of radiation properties. Its use is only meaningful for diffuse surfaces.

(Note, for concave surfaces $F_{ii} \neq 0$)

Compilations: In ID3, Table 13.1 and graphs on pp 681-685. Table 6.1, pp 497-499, and Fig. C.3a through C.3c, pp865,866 of Mills. Extensive tables in Siegel & Howell, Thermal Radiation Heat Transfer, Hemisphere Pub.

Cross-string Rule for 2-D bodies: This is a very elegant theory due to Hottel. See Fig. 14. For string lengths AD, AC, etc., which are stretched taut around convex surfaces and blockages as shown. L_1, L_2 are full lengths of curves, without spanning across valleys.

$$L_1F_{12} = L_2F_{21} = \frac{1}{2}(AD + BC - AC - BD) = \frac{1}{2}(crossed\ \ strings - uncrossed\ \ strings) \tag{82}$$

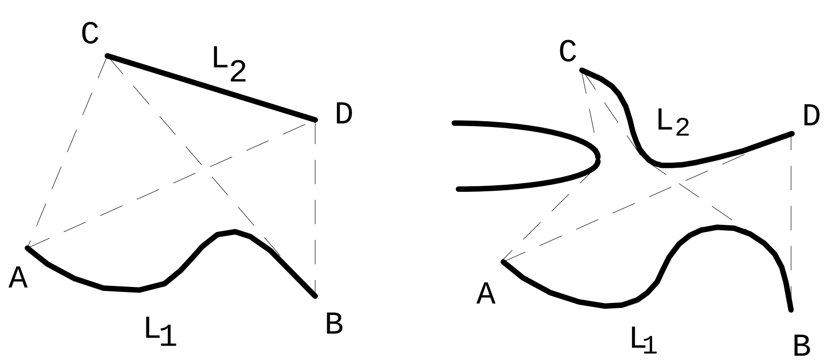


Fig. 15. The Cross-string Rule for View Factors of 2-D Bodies

View Factor Algebra (Useful for evaluating view factors from other known view factors)

Reciprocity: $$\boxed{A_i F_{ij} = A_j F_{ji}} \tag{83}$$

Summation #1. All surfaces seen by A_i: $$\boxed{\sum_{j=1}^{N} F_{ij} = 1} \tag{84}$$

Summation #2. If $A_k = \sum_{j=1}^{N_k} A_j$: $$\boxed{\sum_{j=1}^{N_k} F_{ij} = F_{ik}} \tag{85}$$

Variation: summation #2 + reciprocity: $$\boxed{A_k F_{ki} = \sum_{j=1}^{N_k} A_j F_{ji}} \quad \text{if} \quad A_k = \sum_{j=1}^{N_k} A_j \tag{86}$$

D. Radiation Heat Transfer Calculations among Diffuse Gray Surfaces

Method works only for diffuse gray surfaces, and only if all surfaces in view of any surface in the group is included. Some authors state the latter condition by requiring enclosures.

E. Among Surfaces of Known Temperatures

T_i known, hence $E_{bi} = \sigma T_i^4$ known, for every surface.

Heat transfer from surface i can evaluated in two ways (from definitions):

Method 1:

$$\dot{Q}_i = A_i\left(\varepsilon_i E_{bi} - \varepsilon_i G_i\right) = A_i \varepsilon_i E_{bi} - A_i \frac{\varepsilon_i}{1-\varepsilon_i} J_i + A_i \frac{\varepsilon_i^2}{1-\varepsilon_i} E_{bi} = \frac{A_i \varepsilon_i}{1-\varepsilon_i}\left(E_{bi} - J_i\right)$$

Method 2:

$$\dot{Q}_i = \sum_{j=1}^{N} J_i A_i F_{ij} - \sum_{j=1}^{N} J_j A_j F_{ji} = \sum_{j=1}^{N} J_i A_i F_{ij} - \sum_{j=1}^{N} J_j A_i F_{ij} = \sum_{j=1}^{N} A_i F_{ij}\left(J_i - J_j\right)$$

Equating the two:

$$\boxed{\frac{A_i \varepsilon_i}{1-\varepsilon_i}\left(E_{bi} - J_i\right) = \sum_{\text{All } j \neq i} A_i F_{ij}\left(J_i - J_j\right)} \tag{87}$$

Of these, E_b's, A's, F's are known, and J's are unknowns. There is one such equation for each surface, or N equations for N unknowns, hence J's can be easily solved. If the emissivities are not functions of temperature, this is just a set of linear algebraic equations.

F. Graphical Network Representation

Sometimes network representation is easier to visualize. E_b's and J's act as voltages at junctions. Samples are shown in Fig. 16. The coefficients $A\varepsilon/(1-\varepsilon)$ and A_iF_{ij} act as conductances (1/R) of the resistors shown. For simpler networks this representation may also lead to easy methods of solution using resistance combination rules. Note $A_iF_{ij} = A_jF_{ji}$.

Network representation does not add new information, but may help in visualizing the problem.

G. One or More Surfaces of Unknown Temperatures

The given temperature condition at a surface may be replaced by a given heat flux/heat transfer rate condition, or by a given heat transfer relationship depending on a given ambient temperature. Each such condition adds an algebraic relationship, for an additional unknown E_b. Hence the count of unknowns and equations are still equal.

Important special cases include radiation shields and re-radiating surfaces. These are insulated from the non-radiative environment so net heat flux = 0. This means E_b = J. So the unknowns are reduced.

H. Treatment of Some Non-gray, Non-diffuse Surfaces

If one (or a few) surface are non gray, the conservation of energy for that surface can be modified to account for the un-equal emissivity and absorptivity, if good estimates of these properties are available. This should not affect the number of unknowns and equations, hence the general formulation and solution schemes are basically unaffected. Similarly ad hoc treatment can be devised for one or a few non-diffuse surfaces. In addition, methodology for treating one or two specular reflecting surface can be found in some texts.

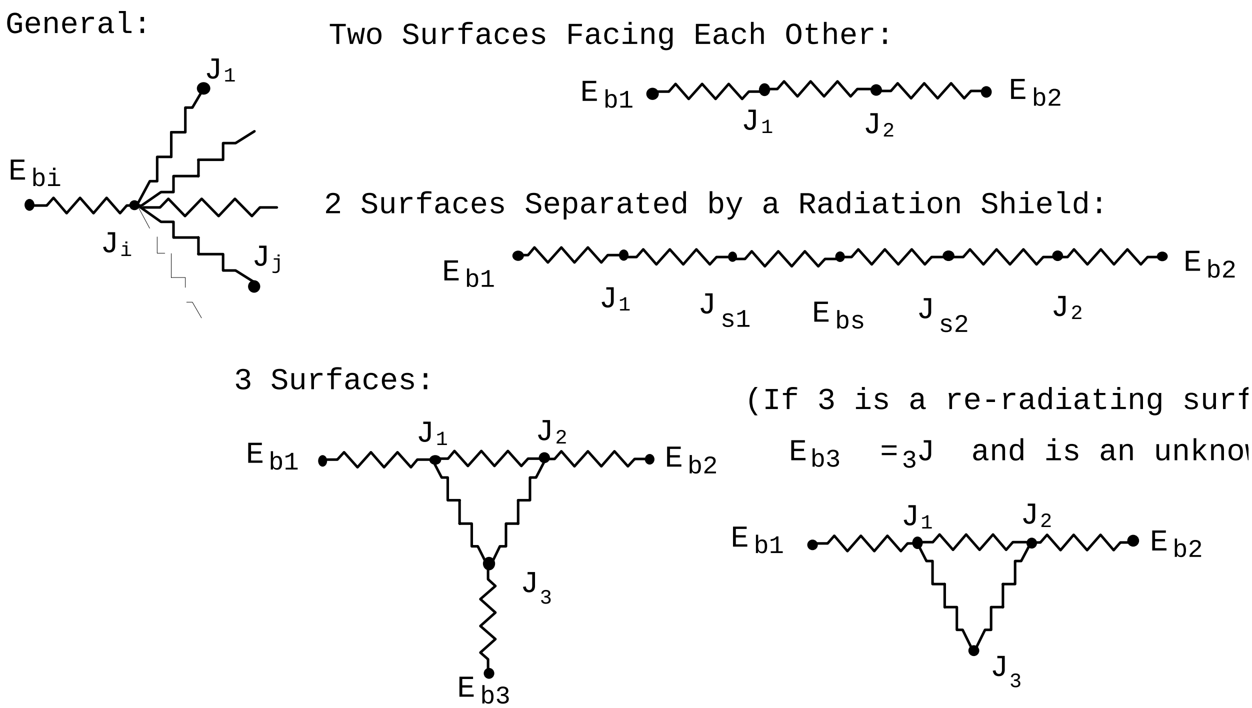


Fig. 17. Network Represention of Radiative Exchange

11. Evaluating the Radiation Heat Transfer Coefficient

For constant emissivities, solution of the linear equation is a linear combination of the given E_b's. Hence the heat exchange between any pair of surfaces can be put in the form

$$\dot{Q}_{ij} = A_i \mathsf{G}_{ij}\left(E_{bi} - E_{bj}\right) = A_i \mathsf{G}_{ij}\sigma\left(T_i^4 - T_j^4\right) \tag{88}$$

The factor G_{ij} in this equation is not the view factor, but involves all the view factors and all the properties of all the surfaces. We will not be concerned with the evaluation of here, though it can be easily done by following though the linear algebraic steps outlined in the last section.

As discussed earlier, when radiation is not dominant while in the presence of convection and conduction, it is sometimes convenient to linearize the radiation term by introducing radiation heat transfer coefficient:

$$\dot{Q}_{rad,ij} \equiv A_i h_{rad,ij}\left(T_i - T_j\right) \tag{89}$$

$$h_{rad,ij} \equiv \frac{A_i \mathsf{G}_{ij}\sigma\left(T_i^4 - T_j^4\right)}{A_i\left(T_i - T_j\right)} = \frac{\mathsf{G}_{ij}\sigma\left(T_i^4 - T_j^4\right)}{\left(T_i - T_j\right)} = \mathsf{G}_{ij}\sigma\left(T_i^3 + T_i^2 T_j + T_i T_j^2 + T_j^3\right) \tag{90}$$

Note that unlike the convective coefficient *h* of which there is only one for each surface, radiation exchange can take place between surface *i* with any number of mutually visible surfaces, *j* = 1, 2 etc. As discussed in Section 5, we may sum these contributions together and write an expression in terms of a *mean radiant environment temperature* T_{mre}. In that case the total radiative contribution to heat transfer for surface *i* can be expressed in a simpler expression:

$$\dot{Q}_{rad,i} \equiv A_i h_{rad,i}\left(T_i - T_{mre}\right) \tag{91}$$

where

$$h_{rad,i} = \mathsf{G}_i \sigma\left(T_i^3 + T_i^2 T_{mre} + T_i T_{mre}^2 + T_{mre}^3\right) \tag{92}$$

$$\mathsf{G}_i \equiv \sum_{\text{all } j\text{'s}} \mathsf{G}_{ij} \tag{93}$$

Algebraic rearrangement can lead to a simpler form of Eq. (92), as well as an approximate expression:

$$h_{rad,i} \cong 4\mathsf{G}_i \sigma T_{ave}^3\left[1 + \frac{1}{2}\left(\frac{\Delta T}{T_{ave}}\right)^2\right] \qquad \cong 4\varepsilon_{\mathrm{a}}\sigma T_{ave}^3 \quad \text{if} \quad \Delta T \le 0.3 T_{mre} \tag{94}$$

where

$$T_{ave} \equiv (T_a + T_{mre})/2\,,\ \Delta T \equiv \left|T_a - T_{mre}\right|. \tag{95}$$

Typical values of radiative heat transfer coefficients are shown in Table 2.

12. Comments on the Solution of Heat Transfer Network Problems

Many practical steady heat transfer problems can be represented by a network of resistances or conductances. For problems involving conduction and surface convection coefficients, the net work leads to simultaneous algebraic equations in terms of T’s. For problems involving radiation among gray diffuse surfaces only, the equations are in terms of E_b’s, J’s, or T^4’s. In both cases the equations are nominally linear. In principle, their solution can be straightforward, by such standard techniques as Gauss elimination. In reality, the coefficients, containing temperature-dependent physical properties and sometimes buoyancy effects, always cause the equations to be weakly non-linear. The non-linearity becomes potentially more severe in the so called mixed mode problems, when radiation must be analyzed together with conduction and convection. The following techniques may significantly simplify the mathematical labor involved.

A. Iteration by Successive Updating of Coefficients

If the temperature dependencies are introduced into the formulation, resulting in a set of non-linear algebraic equations, direct solution of these equations can be very laborious for simle problems, and virtually impossible for complex problems. Instead, a number of iteration methods can be quite simple and effective.

Successive updating of coefficients is a special case of a large class of elementary iteration methods called successive substitution. It is especially suited to weakly non-linear problems due to slightly non-constant coefficients. To begin the iteration, an initial set of approximate temperatures is assumed, permitting the evaluation of properties and heat transfer coefficients. With these coefficients then assumed constant and independent of temperature, the linear algebraic equations are solved to yield a second iteration of temperatures. These improved temperatures are then used to evaluate the improved coefficients. This process is repeated to obtain a third iteration of the temperatures, then a third set of improved coefficients. Typically this process will converge within a few cycles or on the order of 10 cycles . The process is quite easy to set up using modern software, such as Excel and Matlab.

B. Taking Advantage of Resistance Combination Rules

Whenever possible, the network should be simplified as much as possible by the use of combination rules for series and parallel resistances. This usually reduces the number equations which must be solved simultaneously. For many simple problems, the process may eliminate simultaneous equations completely.

C. Point-by-point Successive Substitution

An easy-to-program and effective method for solving a large number of simultaneous algebraic equations obtained from point-by-point conservation of energy is point-by-point successive substitution. The basic idea of the method is as follows. After assigning initial temperatures as iterates, each of the temperatures is calculated and updated in succession, using the equation governing local conservation of energy, while all other temperatures needed in the equation are treated as known and equal to their respective updated values. The process is repeated until convergence is achieved[12].

Specifically, for the conduction/convection system, the equation for updating T_i is obtained from Eq. (44):

$$T_i = \left(\sum_{\text{All } j \neq i} G_{ij} T_j \right) \div \left(\sum_{\text{All } j \neq i} G_{ij} \right) \tag{96}$$

Thus, after T_i has been computed from Eq. (96), the updated value is stored in the memory, replacing the old value. So when T_{i+1} is being calculated it can benefit from the improved value of T_i. The implementation of this method for finite difference or finite volume methods is called the Gauss-Seidel iteration.

For the radiation problem, the unknowns which correspond to temperatures are the J's. The iteration equation is obtained from Eq. (61).

---

[12] Convergence can best be verified by evaluating the residual. The alternative method of checking the change per iteration is not reliable, since it may mistake a slow convergence rate for actual convergence.

$$J_i = \frac{\frac{\varepsilon_i}{1-\varepsilon_i} E_{bi} + \sum_{\text{All } j \neq i} A_i F_{ij} J_i}{\frac{\varepsilon_i}{1-\varepsilon_i} + \sum_{\text{All } j \neq i} A_i F_{ij}} \tag{97}$$

Point-by-point successive substitution almost always converges when applied to heat transfer problems in this manner. The method frequently converges even when the individual algebraic equation is non-linear, as long as the individual equation can be solved by a convenient method, including iterative method. This robustness makes this a valuable method for non-linear problems when both radiation and convection/conduction are important. However, like all successive substitution methods, convergence is not always assured. The user should be vigilant for signs of non-convergence.

References

Many excellent undergraduate textbooks for heat transfer exist. New texts also appear regularly. Those listed only serve as samples of what is available.

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