CHANGES IN ROTATIONAL ANGULAR MOMENTUM DUE TO GRAVITATIONAL INTERACTIONS BETWEEN TWO FINITE BODIES ⋆

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Abstract. Interactions between an initially uniformly rotating body with a second degree and order gravity field and a sphere are analyzed. Explicit predictions of the change in rotational angular momentum of the non-spherical body are derived over one interaction (i.e. periapsis passage) between the bodies. The estimated changes are expressed in terms of trigonometric functions and generalized Hansen coefficients.

Key words: rotational dynamics, mutual gravitation, binary asteroids

1. Introduction

Interactions between a rotating second degree and order gravity field and a test particle can yield significant changes to the energy and angular momentum of the mutual orbit (Scheeres et al., 2000a). Previous work has developed estimates of the change in orbital energy and angular momentum following such an interaction (Scheeres, 1999). The current work generalizes this effect to the mutual orbit of a body with a second degree and order gravity field and a sphere with finite mass. Since no restricted approximation has been made the problem admits both the energy and angular momentum integrals, with these integrals involving contributions from both the translational and rotational dynamics of these bodies. Estimates of the change in rotational angular momentum of the non-spherical body are derived, applying a method used previously to estimate the change in mutual orbit of the bodies (Scheeres, 1999). The resulting estimates are a function of trigonometric terms and generalized Hansen coefficients. Applications of this result to the rotation of asteroids is given in Scheeres et al. (2000b).

2. Equations of Interaction

As shown in (Maciejewski, 1995) the equations of motion of two gravitationally interacting bodies can be expressed in terms of standard Lagrangian and Hamiltonian formulations. In the current application we allow one of the interacting bodies

⋆ Research was supported by NASA's Planetary Geology and Geophysics Program via grant NAG5-9017 to The University of Michigan.

Celestial Mechanics and Dynamical Astronomy **81:** 39–44, 2001.
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to be a sphere of finite mass and the other body to be a non-spherical body with finite mass and a second degree and order gravity field. Since no restricted approximations are made in the analysis, the classical integrals of energy and angular momentum are retained for the system. For the current analysis we only need the energy and angular momentum integrals of this two-body problem

$$E = \frac{1}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \frac{1}{2}\mathbf{W} \cdot \mathbf{I} \cdot \mathbf{W} - U(\mathbf{r}), \tag{1}$$

$$\mathbf{K} = \mathbf{H} + \frac{\mathcal{M}_c \mathcal{M}_s}{\mathcal{M}_c + \mathcal{M}_s}\mathbf{G}, \tag{2}$$

where E is the total energy of the system (minus the rotational energy of the sphere), $\mathbf{r}$ is the relative position vector between the body centers of mass, $\dot{\mathbf{r}}$ is the time derivative of the relative position vector (with respect to a non-rotating frame), $\mathbf{W}$ is the rotational velocity vector of the non-spherical body, $\mathbf{I}$ is its inertia tensor, $\mathcal{M}_s$ is the mass of the sphere, $\mathcal{M}_c$ is the mass of the non-spherical body, U is the mutual potential between these bodies, $\mathbf{K}$ is the total angular momentum of the system (minus the rotational angular momentum of the sphere), $\mathbf{H}$ is the rotational angular momentum vector of the non-spherical body in inertial space, and $\mathbf{G} = \mathbf{r} \times \dot{\mathbf{r}}$ is the specific angular momentum of the relative orbit of the bodies. Additional discussions of this specific system can be found in Scheeres et al. (2000b).

For the current system the mutual potential has the simplified form $U = -\mu/|\mathbf{r}| + U_{20} + U_{22}$, where

$$U_{20} = \frac{\mu C_{20}}{2|\mathbf{r}|^3}(1 - 3\sin^2\delta), \tag{3}$$

$$U_{22} = -\frac{3\mu C_{22}}{|\mathbf{r}|^3}\cos^2\delta\cos 2\lambda, \tag{4}$$

$$\mu = \mathcal{G}(\mathcal{M}_s + \mathcal{M}_c), \tag{5}$$

where $\mathcal{G}$ is the gravitational constant, and δ and λ are the latitude and longitude, respectively, of the sphere in the non-spherical body-fixed space. The parameters C_{20} and C_{22} are the second degree and order gravity coefficients, where we assume that their dimensions are given in units of length^2 and are defined with respect to a principal axis coordinate system.

In anticipation of future computations, the mutual perturbing potential $R = U_{20} + U_{22}$ can be rewritten as a function of the osculating elements of the relative orbit. In this formulation we assume that the non-spherical body is initially in uniform rotation about its maximum moment of inertia, its unperturbed rotational dynamics are then stated simply in terms of its rotational phase angle θ and its angular rotation rate $\dot{\theta} = W_z$, where W_z denotes the initial rotation rate about the non-spherical body's maximum moment of inertia. The perturbing potentials can be rewritten as

$$U_{20} = \frac{\mu C_{20}}{4r^3}[-1 + 3(\cos^2 i + \sin^2 i \cos 2(f + \omega))], \tag{6}$$

$$\begin{aligned} U_{22} = & -\frac{3\mu C_{22}}{r^3}\Big[\frac{1}{2}\sin^2 i \cos 2(\Omega - \theta) + \\ & + \cos^4(i/2)\cos 2(f + \omega + \Omega - \theta) + \\ & + \sin^4(i/2)\cos 2(f + \omega - \Omega + \theta)\Big], \end{aligned} \tag{7}$$

where $r = p/(1 + \mathrm{e}\cos f)$, and p, e, ω, Ω, and f are the usual osculating orbit elements of orbit parameter, eccentricity, argument of periapsis, longitude of the ascending node and true anomaly. Note that the body-fixed frame and the inertial frame are assumed to be aligned when $\theta = 0$.

3. Estimates of Change in the Rotational Angular Momentum

3.1. Computing the Estimates

An equation for the change in rotational angular momentum of the non-spherical body can be found from the total angular momentum integral **K**. Differentiating Equation (2) and rewriting yields the relation

$$\dot{\mathbf{H}} = -\frac{\mathcal{M}_s\mathcal{M}_c}{\mathcal{M}_s + \mathcal{M}_c}\dot{\mathbf{G}}, \tag{8}$$

where the **G** vector can be decomposed in osculating elements as

$$\mathbf{G} = G\begin{bmatrix} \sin i \sin\Omega \\ -\sin i \cos\Omega \\ \cos i \end{bmatrix}. \tag{9}$$

Starting from the canonical form of the Lagrange planetary equations (Brouwer and Clemence, 1961) the following differential equations can be derived:

$$\frac{\mathrm{d}(G\cos i)}{\mathrm{d}t} = -\frac{\partial R}{\partial\Omega}, \tag{10}$$

$$\frac{\mathrm{d}(G\sin i\sin\Omega)}{\mathrm{d}t} = -\frac{\sin\Omega}{\sin i}\left[\frac{\partial R}{\partial\omega} - \cos i\frac{\partial R}{\partial\Omega}\right] - \cos\Omega\frac{\partial R}{\partial i}, \tag{11}$$

$$\frac{\mathrm{d}(G\sin i\cos\Omega)}{\mathrm{d}t} = -\frac{\cos\Omega}{\sin i}\left[\frac{\partial R}{\partial\omega} - \cos i\frac{\partial R}{\partial\Omega}\right] + \sin\Omega\frac{\partial R}{\partial i}, \tag{12}$$

where R is the perturbing potential. Combining these results yields a set of differential equations for the rotational angular momentum of the non-spherical body in terms of the osculating elements of the mutual orbit.

To generate a first-order estimate of the change in rotational angular momentum of the non-spherical body Equation (8) can be integrated over a single interaction. In performing the quadrature we assume that the basic constants of the orbit and body rotation do not change, generating an estimate of the total change in the angular momentum of the non-spherical body. This is discussed in more detail in Scheeres (1999) in the context of Picard's method of successive approximation (Moulton, 1958). Under these assumptions, the total change in the non-spherical body's rotational angular momentum vector is

$$\Delta\mathbf{H} = \int_{-T/2}^{T/2} \dot{\mathbf{H}} \mathrm{d}t, \tag{13}$$

where T is the orbit period for an elliptic mutual orbit and is ∞ for a parabolic or hyperbolic mutual orbit. The only explicit functions of time under the integral is the true anomaly of the relative orbit and the rotational phase angle of the non-spherical body. We assume that the interaction is centered at periapsis (when $t = \theta = 0$) and thus the longitude of ascending node Ω and argument of periapsis ω in the resulting expressions are specified in the body-fixed frame.

3.2. EVALUATING THE INTERACTION FOR C_{20}

When the mutual orbit is elliptic or parabolic the quadrature in Equation (13) yields

$$\Delta\mathbf{H}_{20} = -\frac{3\pi}{2} C_{20} \frac{\mathcal{M}_c \mathcal{M}_s}{\mathcal{M}_c + \mathcal{M}_s} \sqrt{\frac{\mu}{p^3}} \sin(2i) \begin{bmatrix} \cos\Omega \\ \sin\Omega \\ 0 \end{bmatrix}. \tag{14}$$

For a mutual hyperbolic orbit between the bodies the total change in rotational angular momentum is

$$\begin{aligned} \Delta\mathbf{H}_{20} \;=\; & -C_{20}\sqrt{\mathrm{e}^2-1}\frac{\mathcal{M}_c \mathcal{M}_s}{\mathcal{M}_c + \mathcal{M}_s} \sqrt{\frac{\mu}{p^3}} \sin(i) \times \\ & \times \begin{bmatrix} 3\cos i \cos\Omega \left(1 + \frac{f_\infty}{\sqrt{\mathrm{e}^2-1}}\right) + \left(\frac{\mathrm{e}^2-1}{\mathrm{e}^2}\right) \times \\ \times \{\sin 2\omega \sin\Omega - \cos i \cos 2\omega \cos\Omega\} \\ 3\cos i \sin\Omega \left(1 + \frac{f_\infty}{\sqrt{\mathrm{e}^2-1}}\right) - \left(\frac{\mathrm{e}^2-1}{\mathrm{e}^2}\right) \times \\ \times \{\sin 2\omega \cos\Omega + \cos i \cos 2\omega \sin\Omega\} \\ 0 \end{bmatrix}, \end{aligned} \tag{15}$$

where $f_\infty = \arccos(-1/\mathrm{e})$ when $\mathrm{e} > 1$.

For interactions along an elliptic or parabolic orbit the total angular momentum of the rotation and orbit is conserved. For interaction during a hyperbolic flyby the projection of angular momentum along the z-axis is still conserved, but the total angular momentum magnitude is no longer conserved, and can have a change in its length. This implies that the mutual orbit can be given a change in inclination due to the interaction in this case.

3.3. EVALUATING THE INTERACTION FOR C_{22}

This case yields more complex results due to the interaction between the rotating gravity field and the mutual orbit. Performing the quadrature in Equation (13) yields

$$\begin{aligned}\Delta\mathbf{H}_{22} &= 6C_{22}\frac{\mathcal{M}_c\mathcal{M}_s}{\mathcal{M}_c+\mathcal{M}_s}\sqrt{\frac{\mu}{p^3}}\times\\ &\times\begin{bmatrix}\frac{1}{2}\sin i[\cos^2(i/2)\cos(2\omega+\Omega)I_2^1+\\ \quad+\sin^2(i/2)\cos(2\omega-\Omega)I_{-2}^1-\\ \quad-\cos i\cos\Omega I_0^1]\\ -\frac{1}{2}\sin i[\cos^2(i/2)\sin(2\omega+\Omega)I_2^1+\\ \quad+\sin^2(i/2)\sin(2\omega-\Omega)I_{-2}^1-\\ \quad-\cos i\sin\Omega I_0^1]\\ \cos^4(i/2)\sin 2(\omega+\Omega)I_2^1-\\ \quad-\sin^4(i/2)\sin 2(\omega-\Omega)I_{-2}^1+\\ \quad+\frac{1}{2}\sin^2 i\sin 2\Omega I_0^1\end{bmatrix},\end{aligned} \tag{16}$$

where

$$I_m^n = \int_{-f_\infty}^{f_\infty}\left(\frac{p}{r}\right)^n \mathrm{e}^{i(mf-2W_zt)}\mathrm{d}f, \tag{17}$$

and the time t is specified in the quadrature as an explicit function of true anomaly using the elliptic, parabolic, or hyperbolic forms of Kepler's equation. Note that $f_\infty=\pi$ if $\mathrm{e}\leqslant 1$, and $f_\infty = \arccos(-1/\mathrm{e})$ if $\mathrm{e} > 1$. A similar definition for the I_m^n integrals was given in Scheeres (1999), and the mathematical properties and computations of these integrals were discussed there. In particular that paper established that the contributions from I_2^1 and I_0^1 are much larger than the contributions from I_{-2}^1, implying that retrograde interactions have a markedly decreased strength. In Scheeres et al. (2000b) the above results are validated using simplified estimates and numerical integrations of the dynamics.

The integrals I_m^n can be directly related to the Hansen coefficients when the mutual orbit is elliptic ($\mathrm{e} < 1$)

$$I_m^n = 2(1-\mathrm{e}^2)^{n+1/2}\sum_{k=-\infty}^{\infty} X_k^{-(n+2),m}\frac{\sin\pi(k-2\sigma)}{k-2\sigma}, \tag{18}$$

$$\sigma = W_z\sqrt{\frac{a^3}{\mu}}, \tag{19}$$

and the Hansen coefficients are defined as (Brumberg, 1995)

$$X_s^{q,r} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{r}{a}\right)^q \mathrm{e}^{i(rf-sM)}\mathrm{d}M, \tag{20}$$

where M is the mean anomaly and s is generally assumed to be an integer. If we generalize the index s and allow it to be a real number, the integrals can be defined directly as

$$I_m^n = 2\pi(1-\mathrm{e}^2)^{n+1/2}X_{2\sigma}^{-(n+2),m}. \tag{21}$$

If the Hansen coefficients are further generalized for parabolic and hyperbolic orbits the relationships noted above can be extended to these cases as well. This specific generalization is not performed here.

4. Conclusions

A mathematical analysis of the effect of mutual gravitation on the rotational dynamics of a non-spherical body is presented. The model developed applies to the interaction of a non-spherical body with a sphere in a nominally conic orbit about the body. An estimate of the change in rotational angular momentum of the non-spherical body over one interaction is generated.

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