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Solutions of the Optimal Feedback Control Problem using Hamiltonian Dynamics and Generating Functions

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***Abstract*— We show that the optimal cost function that satisfies the Hamilton-Jacobi-Bellman (HJB) equation is a generating function for a class of canonical transformations for the Hamiltonian dynamical system defined by the necessary conditions for optimality. This result allows us to circumvent the final time singularity in the HJB equation for a finite time problem, and allows us to analytically construct a nonlinear optimal feedback control and cost function that satisfies the HJB equation for a large class of dynamical systems. It also establishes that the optimal cost function can be computed from a large class of solutions to the Hamilton-Jacobi (HJ) equation, many of which do not have singular boundary conditions at the terminal state.**

I. INTRODUCTION

For a general nonlinear system with arbitrary performance criteria, optimal state feedback control laws can be derived from the solution to the HJB equation. The HJB equation does not have closed-form solutions in general, thus much research has been performed to find practical approaches for obtaining sub-optimal feedback controls.

Based on the time duration of interest, optimal control formulations can be classified as infinite horizon regulator or finite-horizon targeting problems[1]. In the former case, the time variable does not explicitly appear in the optimal feedback control. Due to this simplification, many algorithms have been proposed which provide numerical approximations of the optimal feedback regulator problem (*cf.* Durbeck[1], Lukes[2], Huang et al[3], Beard et al[4], Cloutier[5], Tsiotras et al[6], Curtis et al[7], etc.). In Beeler et al[8] five different methodologies were analyzed and compared.

In the finite horizon targeting problem, the optimal feedback control is an explicit function of both spatial and time variables in general. Due to this difficulty, relatively few results exist in the literature, a representative sample can be found in Burghart[9], Garrard et al[10], Saridis et al[11], Fax et al [12]. All the methodologies discussed in these papers are applicable to both finite and infinite horizon problems, however only Fax et al[12] treats examples of *finite horizon* problems and considers the case where the terminal condition is specified at a point (the hard constraint problem). Even though the scope of their method is limited due to their underlying assumptions, they did not provide the solution to the hard constraint problem they considered. All this implies that the finite horizon problem is more difficult to solve than the infinite horizon problem due to the explicit dependence of the HJB equation on time. Furthermore, when the final condition is completely specified and there is no control constraint, the optimal control law becomes singular at the final condition, which adds to the problem difficulty.

Independent of these results, there have been studies which have taken advantage of the theory of canonical transformations for Hamiltonian systems in an optimal control context[13], [14], [15]. These studies were very limited, however, and only considered the control of Hamiltonian systems during periods of null control and did not take full advantage of the Hamiltonian nature of the necessary conditions.

This paper presents a more general approach to finding optimal feedback controls, combining the fact that the necessary conditions for optimality define a Hamiltonian system with the abstract properties of this class of dynamical systems. We first formulate the optimal feedback control problem and introduce the sufficient and necessary conditions for optimality (section 2). After a short introduction to Hamiltonian dynamical systems and canonical transformations, the core discussion of how to employ these results to obtain optimal feedback control follows (section 3). Then we propose a systematic numerical implementation based on series approximation, and explain how it circumvents the final condition singularity for the finite horizon problem (section 4). Application of this approach to the optimal control of nonlinear orbital rendezvous follows (section 5).

II. PROBLEM STATEMENT

Consider the minimization of a general performance index for an arbitrary initial point (x,t)[2]

$$J(x,t;x_f,t_f) = \phi(x(t_f),t_f) + \int_t^{t_f} L(x(\tau),u(\tau),\tau)d\tau$$

subject to the following system with final time constraints

$$\dot{x}(t) = F(x(t),u(t),t) \quad , \quad \psi(x(t_f),t_f) = 0.$$

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[1]Alternatively, the infinite horizon problem can be viewed as a special case of finite horizon problem by forcing the final time to be infinite.

[2]In this document, x represents the initial state as well as the time history of the state, as does u, and t stands for the initial time as well as the general time variable for an interval of interest.

Here we assume that $x \in \Re^n$, $t \in \Re$, $u \in \Re^m$, and $\psi \in \Re^{p \leq n}$. Our objective is to evaluate the optimal trajectory and find the optimal cost and the associated optimal feedback control law for a given initial point $(x,t) \in \Re^{n+1}$.

We define the *hard constraint problem* as the problem described above with terminal boundary conditions specified at a fixed point in $\Re^n$, i.e.,

$$\phi(x(t_f),t_f) \equiv 0 \quad , \quad \psi(x(t_f),t_f) = x(t_f) - x_f$$

where the pair (x_f,t_f) is given. In this case the cost function as well as the associated optimal feedback control is singular at the given terminal boundary point[16]. This singularity restricts conventional dynamic programming approaches using the HJB equation. Although the Ricatti transformation technique provides a feedback control law for linear, quadratic, hard constraint problems[16], it is not applicable to systems with non-linear dynamics or with non-quadratic performance criteria.

With these observations in mind, we focus our analysis on the hard constraint problem, which is presumed to be one of the most challenging types of problems in classical optimal control[3].

Sufficient Conditions for Optimality

Let us first define the Hamiltonian as

$$H(x,\lambda,u,t) = L(x,u,t) + \lambda^T F(x,u,t)$$

According to the classical derivation of the sufficiency conditions from dynamic programming[16][17], if

1) In the domain considered for (x,t), the Hamiltonian has a unique minimizer with respect to u such that

$$u = arg\min_{\bar{u}} H\left(x, \frac{\partial J}{\partial x}, \bar{u}, t\right)$$

2) $J(x,t)$ is sufficiently smooth (or analytic) and satisfies the HJB equation with the given boundary condition

$$-\frac{\partial J}{\partial t}(x,t) = \min_u H\left(x, \frac{\partial J}{\partial x}, u, t\right) \quad , \quad J(x_f,t_f) = 0$$

then J is the optimal cost and u is the corresponding optimal control law.

In general, the HJB equation is a nonlinear first order partial differential equation for the spatial variables x and the time variable t.

Necessary Conditions for Optimality

Now re-consider the Hamiltonian defined above:

$$H(x,\lambda,u,t) = L(x,u,t) + \lambda^T F(x,u,t) \tag{1}$$

[3]This does not imply that the applicability of our approach is confined to the hard constraint problem. Later we will discuss, briefly, how to manipulate other types of boundary conditions.

The standard derivation from the variational calculus and Pontryagin's principle provides the well-known 1st order necessary conditions[16]:

$$\begin{aligned} \dot{x} &= H_\lambda(x,\lambda,u,t) && (2)\\ \dot{\lambda} &= -H_x(x,\lambda,u,t) && (3)\\ u &= arg\min_{\bar{u}} H(x,\lambda,\bar{u},t) && (4) \end{aligned}$$

where λ is the costate. Substituting (4) into (1)- (3) yields

$$\begin{aligned} H(x,\lambda,t) &= L(x,t) + \lambda^T F(x,t)\\ \dot{x} &= H_\lambda(x,\lambda,t)\\ \dot{\lambda} &= -H_x(x,\lambda,t) \end{aligned}$$

which is a Hamiltonian canonical system for states and costates. Since we have specified the initial and terminal states and need to find the initial or terminal costates to satisfy the ODE's, the optimal control problem is reduced to the two point boundary value problem (TPBVP).

Our alternative approach to solving the optimal control problem evaluates the cost function and the corresponding optimal feedback control using properties of Hamiltonian dynamical systems. The core procedural scheme we use employs the theory of canonical transformations to solve the TPBVP, which is elucidated in the next section.

III. Boundary Value Problems in Hamiltonian Systems

This section briefly introduces the application of canonical transformation theory to solve boundary value problems in Hamiltonian systems. A more detailed description of this theory can be found in Guibout and Scheeres[18]. For a general review of Hamiltonian Dynamical systems see Greenwood [19].

Hamiltonian Systems and Canonical Transformations

Suppose we have a system whose equations of motion can be represented by Hamilton's canonical form

$$\begin{bmatrix} \dot{q}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial H(q(t),p(t),t)}{\partial p} \\ -\frac{\partial H(q(t),p(t),t)}{\partial q} \end{bmatrix}$$

where

- $H = H(q(t),p(t),t)$ is the Hamiltonian of the system,
- $q(t) = [q_1(t) \quad q_2(t) \quad \cdots \quad q_n(t)]^T$ is the generalized coordinate vector,
- $p(t) = [p_1(t) \quad p_2(t) \quad \cdots \quad p_n(t)]^T$ is the generalized momentum vector conjugate to $q(t)$.

In our application we restrict ourselves to canonical transformations with time as independent parameter, i.e., solutions to the given dynamical system.

Consider a transformation between (q,p,t) and (Q,P,T) defined by

$$\begin{aligned} Q(T) &= Q(q(t),p(t),t,T) && (5)\\ P(T) &= P(q(t),p(t),t,T). && (6) \end{aligned}$$

If the transformation is canonical, there exists a new Hamiltonian $K = K(Q(T),P(T),T)$ such that the equations of motion have the form:

$$\begin{bmatrix} \dot{Q} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} \frac{\partial K(Q,P,T)}{\partial P} \\ -\frac{\partial K(Q,P,T)}{\partial Q} \end{bmatrix}$$

We note that for this class of transformations $K \equiv H(Q(T),P(T),T)$, however we retain the K notation for convenience in the following. In order to relate K and H recall Hamilton's principle

$$\delta I = \delta \int_{t_0}^{t_f} L dt = 0 \qquad (7)$$

where the Lagrangian L is defined as $L(q,\dot{q},t) = p^T \dot{q} - H(q,p,t)$.

Then, by application of Hamilton's principle (7) for the old and new coordinates, we find that the Lagrangians differ by at most the total time derivative of an arbitrary function F:

$$p^T \dot{q} - H(q,p,t) = P^T \dot{Q} - K(Q,P,T) + \frac{dF}{dt} \qquad (8)$$

The function F is called a *generating function* and depends on the old and new states and times (i.e., $4n+2$ variables). Using the $2n$ relations from (5) and (6), we see that F is reduced to a function of $2n+2$ variables. Assume that F is dependent upon n old states and n new states. Then the generating function can have one of the classical forms:

$$F_1(q,Q,t,T), \quad F_2(q,P,t,T), \quad F_3(p,Q,t,T), \quad F_4(p,P,t,T)$$

If, for instance, q and Q are independent variables, then F_1 should be used.

When treating the transformations as solutions we generally choose the states (Q,P) to be the initial conditions at time T, which are constants of motion and hence have a constant Hamiltonian K (which is equivalent to setting $K \equiv 0$). Treating t as the independent variable and expanding the total time derivative of the generating function, we can find the standard results for F_1 and F_2:[19]

$$\begin{aligned} p &= \frac{\partial F_1(q,Q,t,T)}{\partial q} && (9) \\ P &= -\frac{\partial F_1(q,Q,t,T)}{\partial Q} && (10) \\ 0 &= H(q,p,t) + \frac{\partial F_1(q,Q,t,T)}{\partial t} && (11) \end{aligned}$$

$$\begin{aligned} p &= \frac{\partial F_2(q,P,t,T)}{\partial q} && (12) \\ Q &= \frac{\partial F_2(q,P,t,T)}{\partial P} && (13) \\ 0 &= H(q,p,t) + \frac{\partial F_2(q,P,t,T)}{\partial t}. && (14) \end{aligned}$$

Similar equations can be found for $F_3(p,Q,t,T)$ and $F_4(p,P,t,T)$. The generating functions satisfy a partial differential equation found by substituting for p in (11) and (14):

$$\frac{\partial F_1(q,Q,t,T)}{\partial t} + H\left(q, \frac{\partial F_1(q,Q,t,T)}{\partial q}, t\right) = 0$$
$$\frac{\partial F_2(q,P,t,T)}{\partial t} + H\left(q, \frac{\partial F_2(q,P,t,T)}{\partial q}, t\right) = 0$$

These equations are usually referred to as the Hamilton-Jacobi (HJ) equation, and again similar equations exist for F_3 and F_4. For the applications we make, it is also important to derive the HJ equation for F_1 treating the state (q,p,t) to be a set of fixed final time conditions (again, equivalent to setting $H(q,p,t) \equiv 0$) and the state (Q,P,T) to be the moving variable with $T \leq t$. Derivation of this equation yields[20]:

$$-\frac{\partial F_1(q,Q,t,T)}{\partial T} + H\left(Q, -\frac{\partial F_1(q,Q,t,T)}{\partial Q}, T\right) = 0 \qquad (15)$$

A crucial property of the generating functions related to a given transformation is that they are linked to each other via Legendre transformations, which can be represented by the following identities:

$$\begin{aligned} F_2(q,P,t,T) &= F_1(q,Q,t,T) + P^T Q \\ F_3(p,Q,t,T) &= F_1(q,Q,t,T) - p^T q \\ F_4(p,P,t,T) &= F_2(q,P,t,T) - p^T q \end{aligned} \qquad (16)$$

Given an analytical solution to any generating function, it is then possible to evaluate the analytical form of any other generating function as long as some uniqueness conditions are satisfied[18]. This is an important property as some generating functions are easier to compute than others.

Solving Boundary Value Problems with Generating Functions

The choice of the appropriate generating function depends on the type of boundary condition of the problem to be solved. For the hard constraint problem as specified here, the F_1 generating function is the appropriate choice. Indeed, if we can find $F_1(x,x_0,t,t_0)$, we can directly evaluate the initial and final costates from (9)- (10):[4]

$$\lambda_0 = \left.\frac{\partial F_1}{\partial x_0}\right|_{t=t_f, x=x_f}, \quad \lambda_f = -\left.\frac{\partial F_1}{\partial x}\right|_{t=t_f, x=x_f}$$

If the terminal boundary condition is not specified, then the transversality condition

$$\lambda(t_f) = \frac{\partial \phi(x(t_f),t_f)}{\partial x(t_f)}$$

provides the terminal condition. In this case $F_3(\lambda, x_0, t, t_0)$ is the appropriate choice. Else, if the terminal condition is

[4]The 1st order necessary condition for optimality can be mapped into the notation of a Hamiltonian system by making the identifications: $x \equiv q$ and $\lambda \equiv p$.

given by a hyper plane $\psi(x(t_f),t_f)=0$ in $\Re^{p\leq n}$, we will have mixed terminal conditions for both states and costates in general. In this case, a more generalized kind of generating function can be implemented, which would mix all 4 kinds of variables (initial and terminal states and costates).

Once the generating function has been found, the unknown boundary conditions are simply evaluated by the algebraic manipulation of (9)-(10) and (12)-(13) without solving a differential equation. For our example, once we find the F_1 generating function and obtain the initial costates λ_0 from the corresponding necessary conditions, we can evaluate the optimal trajectory by simple forward integration. Note that we do not need to find F_1 directly from the HJ equation, and can instead find it from a different generating function (which may be easier to solve for) via a Legendre transformation. This result is at the heart of our analytical solution to the problem described in the next section.

If we fix the final state and time of our system (x_f,t_f) and keep the initial state and time free, then F_1 also generates the optimal feedback control law:

$$u = arg\min_{\bar{u}} H\left(x,-\frac{\partial F_1(x_f,x,t_f,t)}{\partial x},\bar{u},t\right) \tag{17}$$

Equivalence of F_1 and the Optimal Cost Function

To show the equivalence of the F_1 function and the cost function of the HJB equation, we will show that F_1 satisfies the sufficiency conditions. First define the function

$$V(x,t) = -F_1(x_f,x,t_f,t). \tag{18}$$

Thus, we fix the final state and time and keep the initial state and time as free variables. Then V satisfies the modified HJ equation (15) and the optimal control law (17) by definition:

$$\frac{\partial V(x,t)}{\partial t} + H\left(x,\frac{\partial V(x,t)}{\partial x},t\right) = 0$$

$$u = arg\min_{\bar{u}} H\left(x,\frac{\partial V(x,t)}{\partial x},\bar{u},t\right)$$

Finally, in order to show equivalence we must evaluate the terminal boundary condition for F_1 at $(t=t_f, x=x_f)$. For our dynamical system, at the terminal condition we must find the identity transformation:

$$x = x_f \quad \lambda = \lambda_f$$

The functions F_1 and F_4 cannot generate identity transformations, although F_2 and F_3 can[19]. For F_2 we see that the function $F_2 = x_f^T\lambda$ generates the identity transformation

$$x = \frac{\partial F_2}{\partial \lambda} = x_f \quad , \quad \lambda_f = \frac{\partial F_2}{\partial x_f} = \lambda$$

Thus at the terminal conditions we know the form that F_2 must have. Then, from the Legendre transformation (16), we can solve for F_1 at the terminal condition

$$F_1(x_f,x_f,t_f,t_f) = \left[F_2(x_f,\lambda,t_f,t) - \lambda^T x\right]\Big|_{t=t_f} \equiv 0$$

Thus $V(x_f,t_f) = -F_1(x_f,x_f,t_f,t_f) \equiv 0$ (and we see that F_1 is singular as it loses independence of its arguments). Therefore the function $V(x,t)$ satisfies the sufficient conditions for optimal control. Furthermore, we see that the F_1 generating function satisfies both the necessary and sufficient conditions for optimality, and hence that the optimal cost function is:

$$J(x,t) = -F_1(x_f,x,t_f,t). \tag{19}$$

IV. Evaluation of the Optimal Cost Function via Generating Functions

We have shown that the optimal cost function is fundamentally linked to the generating function F_1 and that F_1 is related to other generating functions by the Legendre transformation. In fact, the Legendre transformation plays a crucial role in the evaluation of F_1, as described below.

Solving the HJ equation requires, at the least, that the value of the generating function be specified at some epoch. Now recall that for our canonical transformation the old and new coordinates are equal when $t=T$, and thus the generating function solution to the HJ equation must define an identity transformation at $t=T$. F_1 cannot generate such a transformation since the initial and final positions are equal and not independent at $t=T$[5]. On the other hand, F_2 is well defined at $t=T$ and generates the identity transformation, as noted previously. Therefore, given the Hamiltonian of a system we can solve the HJ equation for F_2 from the initial time and then evaluate F_1 through the Legendre transformation at a later time. In [18] this fact is used to derive a specific solution algorithm for a class of problems where:

1) the system $\dot{x} = F(x(t),u(t),t)$ is analytic and has a zero equilibrium, i.e., $F(x=0,u=0,t)=0$
2) the integrand of the cost function $L(x(t),u(t),t)$ is analytic

In this approach we expand the Hamiltonian H and the generating functions F_1 and F_2 as Taylor series in their spatial arguments about the zero condition, these expansions can be made to arbitrarily high order. Then, using the HJ equation, we find a series of ordinary differential equations for the coefficients of the series expansions of F_1 and F_2. We solve these equations for F_2, using our initial boundary conditions to generate initial conditions for the ODEs. Then, at a later epoch where F_1 is defined, we use the Legendre transformation and inversion of series to compute the coefficients of F_1, and use these to initialize the ODEs that define this generating function. Then by integrating forwards and backwards in time we construct time varying solutions for F_1 that are non-linear, analytic functions of its spatial arguments x and x_0. See [18] for a complete description of this solution method.

[5] In fact, this is an equivalent statement that the optimal cost function is singular at the terminal time for hard constraint problem

V. Application to Nonlinear Optimal Rendezvous Maneuvers in a Central Gravity Field

We apply this specific algorithm to the following problem: consider minimizing

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^T(t)u(t)dt$$

subject to the system dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 2x_4 - (1+x_1)\left(\frac{1}{r^3} - 1\right) + u_1 \\ -2x_3 - x_2\left(\frac{1}{r^3} - 1\right) + u_2 \end{bmatrix}$$

where $r = \sqrt{(x_1+1)^2 + x_2^2}$. This system represents the planar motion of a particle in a central gravity field, expressed in a rotating coordinate frame. The origin of this frame corresponds to a circular orbit, the coordinates (x_1, x_2, x_3, x_4) represent radial displacement, tangential displacement, radial velocity, and tangential velocity deviations from the circular orbit, and (u_1, u_2) represent the radial and tangential control input, respectively[6]. The boundary condition is given by

$$x(t_0) = x_0 \quad , \quad x(t_f) = x_f.$$

Expanded as a polynomial series about the zero equilibrium point $[x_1 \;\; x_2 \;\; x_3 \;\; x_4]_e^T = [0 \;\; 0 \;\; 0 \;\; 0]^T$, the system can be re-written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 3x_1 + 2x_4 - 3x_1^2 + 1.5x_2^2 + \cdots + u_1 \\ -2x_3 + 3x_1x_2 + \cdots + u_2 \end{bmatrix} \tag{20}$$

Pontryagin's principle (or simply the optimality condition $H_u = 0$) provides the well-known Lawden's primer vector optimal control strategy[22]

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\lambda_3 \\ -\lambda_4 \end{bmatrix} \tag{21}$$

If we introduce (21) into the Hamiltonian, it becomes a function of states and costates only:

$$\begin{aligned} H &= -\frac{1}{2}(u_x^2 + u_y^2) + \lambda_1 x_3 + \lambda_2 x_4 \\ &+ \lambda_3(3x_1 + 2x_4 - 3x_1^2 + 1.5x_2^2 + \cdots) \\ &+ \lambda_4(-2x_3 + 3x_1x_2 + \cdots) \end{aligned} \tag{22}$$

while the costate equations are

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \\ \dot{\lambda}_4 \end{bmatrix} = \begin{bmatrix} -3\lambda_3 + 6x_1\lambda_3 - 3x_2\lambda_4 + \cdots \\ -3x_2\lambda_3 - 3x_1\lambda_4 + \cdots \\ -\lambda_1 + 2\lambda_4 \cdots \\ -\lambda_2 - 2\lambda_3 \cdots \end{bmatrix} \tag{23}$$

[6]See Scheeres, Park, and Guibout[21] for a more detailed review and derivation of this problem

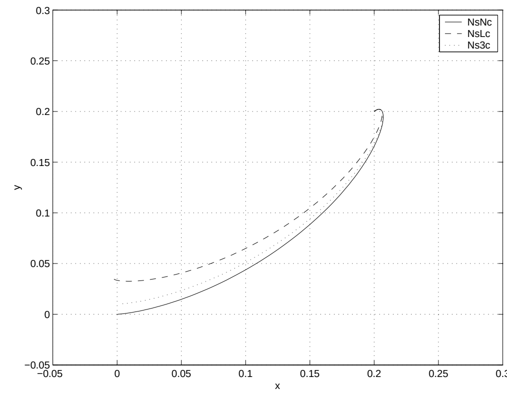

Fig. 1. Radial Position (x_1) vs. Tangential Position (x_2)

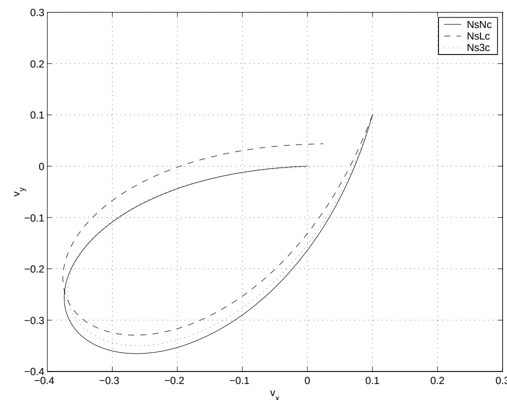

Fig. 2. Radial Velocity (x_3) vs. Tangential Velocity (x_4)

Now (20)–(23) constitute the Hamiltonian canonical system to which our approach has been applied. In our implementation of the method we expand H, F_1, and F_2 to the third order using Matlab. Higher order expansions will be investigated in the future.

Figures 1–3 show the state and control trajectory for an arbitrarily chosen boundary condition and time interval chosen as

$$x(t_0 = 0) = [0.2 \;\; 0.2 \;\; 0.1 \;\; 0.1]^T \quad , \quad x(t_f = 1) = [0 \;\; 0 \;\; 0 \;\; 0]^T$$

For the control histories, the solid line, dashed line, and dotted line indicate the solution of the nonlinear TPBVP using a shooting method (which is our reference "true" solution), a linear systems solution, and the 3rd order analytical solution described here, respectively. For the state trajectories, each line represents the application of each control history to the original nonlinear system. It is clear that the 3rd order control is a better approximation than the linear control and is close to the true solution. By introducing additional higher order terms in the system dynamics, we can approximate the original system to as high a degree as desired.

As another example, Figure 4 shows the 3rd order approximated cost function for zero initial velocity ($x_3 = x_4 = 0$) and

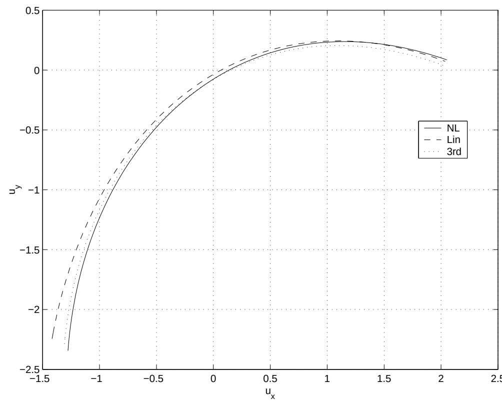

Fig. 3. Radial Control (u_1) vs. Tangential Control (u_2)

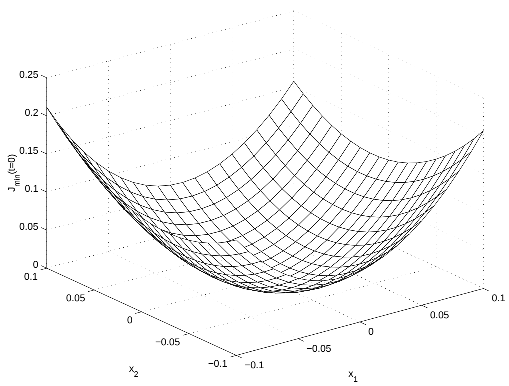

Fig. 4. Optimal Cost Function (3rd order approximation)

zero terminal boundary condition ($x_1 = x_2 = x_3 = x_4 = 0$) when $t_0 = 0$ and $t_f = 1$. It is important to note that our algorithm provides us with an analytical solution of the cost function as a general function of the initial and final states, and is only limited by the convergence properties of this function[7].

VI. Conclusion

We have introduced a new approach for finding the optimal feedback control by solving for a generating function of the Hamiltonian system defined by the necessary conditions for optimality. We have applied our approach to the hard constraint problem to show the generality of our method. By establishing that the optimal cost function is a certain type of generating function, the optimal feedback control problem can be included within the more fundamental problem of canonical transformations for Hamiltonian systems.

These results also reflect the connection between the necessary conditions and the sufficient conditions, which are usually derived independently. We have shown that the necessary conditions derived from variational calculus and Pontryagin's principle, once the control is removed and viewed as a Hamiltonian system, recovers the optimal feedback control which satisfies the sufficiency conditions derived from dynamic programming (i.e., the HJB equation).

Our future research will be directed toward the application of this approach to general nonlinear systems with a variety of boundary conditions. We will also explore the group properties of Hamiltonian systems to search for additional advantages of our approach.

[7] Some convergence properties of this algorithm were investigated in [18]

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