A Generating Function for Optimal Feedback Control Laws that Satisfies the General Boundary Conditions of A System *

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September 19, 2003

Abstract

Given a system and performance index to be minimized, we present a general approach to evaluating optimal feedback control laws for this system that can be easily modified to satisfy different types of boundary conditions. This work rests on recent observations that, for a class of optimal control problems, the optimal cost function that satisfies the Hamilton-Jacobi-Bellman (HJB) equation is a generating function for a class of canonical transformations of the Hamiltonian system defined by the necessary conditions for optimality. This result allows us to circumvent the final time singularity in the HJB equation for a class of finite time problems and thus to analytically construct a nonlinear optimal feedback control and cost function. Furthermore, once this optimal feedback control is found for a given set of boundary conditions, our methodology allows us to obtain the feedback control for a different set of boundary conditions only using a series of algebraic manipulations, partial differentiations, and solutions of implicit algebraic equations. This methodology provides an advantage over the conventional method based on dynamic programming, which requires one to solve the HJB PDE repetitively for each type of boundary condition.

Key Words. Optimal Feedback Control, Hamilton-Jacobi-Bellman Equation, Boundary Condition, Canonical Transformation, Hamiltonian System, Generating Function, Hamilton-Jacobi Equation, Legendre Transformation

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*This document has been deliberately extended, widely spaced, and formatted into single column for reviewers' convenience. If accepted, it will be condensed and reformatted into double column.

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1 Introduction

For a general nonlinear system with arbitrary performance criteria, optimal state feedback control laws can be derived from the solution to the Hamilton-Jacobi-Bellman (HJB) equation. The HJB equation does not have closed-form solutions in general, thus much research has been performed to find practical approaches for obtaining sub-optimal feedback controls.

Based on the time duration of interest, optimal control formulations can be classified as infinite horizon regulator or finite-horizon targeting problems[1]. In the former case, the time variable does not explicitly appear in the optimal feedback control. Due to this simplification, many algorithms have been proposed and tested (See Durbeck[1], Lukes[2], Huang et al[3], Beard et al[4], Cloutier[5], Tsiotras et al[6], Curtis et al[7], Beeler et al[8], etc.).

In the finite horizon targeting problem, the optimal feedback control is an explicit function of both spatial and time variables in general. Due to this difficulty, relatively few results exist in the literature, some of them can be found in Burghart[9], Garrard et al[10], Saridis et al[11], Fax et al [12]. All the methodologies discussed in these papers are applicable to both finite and infinite horizon problems in theory, however only Fax et al[12] treats examples of finite horizon problems and considers the case where the terminal condition is fixed to a point (hard constraint problem). In spite of the limited applicability of their method due to underlying assumptions, they did not provide the solution to the problem they considered. All this implies that the finite horizon problem is more difficult to solve than the infinite horizon problem due to the explicit dependence of the HJB equation on time. Furthermore, when the final condition is completely specified and there is no control constraint, the optimal control law becomes singular at the final time, which adds to the problem difficulty.

Independent of these results, there have been studies which have taken advantage of the theory of canonical transformations for Hamiltonian systems in an optimal control context[13, 14, 15]. These studies were very limited, however, and only considered the control of Hamiltonian systems during periods of null control and did not take full advantage of the Hamiltonian nature of the necessary conditions.

Recently Park and Scheeres[16] studied the optimal feedback control problem in the context of Hamiltonian dynamical systems. They observed that the optimal cost function for hard constraint problems is a generating function for a class of canonical transformations, which allowed them to devise a systematic methodology to evaluate the optimal feedback control and cost function. This approach was applied to a nonlinear optimal rendezvous problem in a central gravity field and was able to demonstrate the superiority of their method.

This work is an extension of these previous results and considers a wider range of optimal feedback control problems with various types of boundary conditions. We show that our method can be applied to the

[1]Alternatively, the infinite horizon problem can be viewed as a special case of finite horizon problem by forcing the final time to be infinite.

optimal control of a given system with a different type of boundary conditions. Furthermore, by exploiting fundamental links between generating functions, we present an algorithm to evaluating optimal feedback controls for different types of boundary conditions without solving the HJB equation repetitively. We first formulate the optimal feedback control problem and introduce the sufficient and necessary conditions for optimality (section 2). After a short introduction to Hamiltonian dynamical systems and canonical transformations, the core discussion of how to employ these results to obtain optimal feedback control for a variety of boundary conditions follows (section 3). Then we propose a systematic numerical implementation based on series approximation, and justify our approach by applying it to a linear quadratic example (section 4). More extensive applications to the optimal control of nonlinear orbital maneuvers in a central gravity field follows (section 5).

2 Problem Statement

Consider the minimization of a general performance index for an arbitrary initial point (x,t)[2]

$$J(x,t) = \phi(x(t_f),t_f) + \int_t^{t_f} L(x(\tau),u(\tau),\tau)d\tau$$

subject to the following system with final time constraints

$$\dot{x}(t) = F(x(t),u(t),t) \quad , \quad \psi(x(t_f),t_f) = 0.$$

Here $x \in \Re^n$, $t \in \Re$, $u \in \Re^m$, and $\psi \in \Re^{p \le n}$. We assume that there exist no constraints on state and control trajectories. Our objective is to evaluate the optimal trajectory satisfying the final time constraints and to find the optimal feedback control for an arbitrary initial point $(x,t) \in \Re^{n+1}$.

We consider two representative problem formulations, which are characterized by the types of terminal boundary conditions:[3]

- **Hard Constraint Problem (HCP)** Terminal boundary condition for states is pre-specified a to a fixed point in $\Re^n$.

- **Soft Constraint Problem (SCP)** Terminal boundary condition for states is not pre-specified but indirectly affected by the final time performance index $\phi(x,t)$.

Given the problem statement, the optimal trajectory and associated optimal control are determined by the following sufficient and necessary conditions.

[2]In this document, x represents the initial state as well as the time history of the state, as does u, and t stands for the initial time as well as the general time variable for an interval of interest. This is conventional in the context of classical optimal feedback control theory.

[3]This classification is just for simplicity of arguments. It does not imply that the applicability of our approach is confined to these two kinds of boundary conditions. There exists some mixed type of boundary conditions, which are specified by terminal hyper plane. Later we will discuss, briefly, how to manipulate them.

Sufficient Conditions for Optimality

We first define the Hamiltonian H as

$$H(x,\lambda,u,t) = L(x,u,t) + \lambda^T F(x,u,t) \tag{1}$$

According to the classical derivation from dynamic programming[17][18], if

1. In the domain considered for (x,t), the Hamiltonian has a unique minimizer with respect to u such that
$$u = arg\min_{\bar{u}} H\left(x, \frac{\partial J}{\partial x}, \bar{u}, t\right)$$

2. $J(x,t)$ is sufficiently smooth (or analytic) and satisfies the Hamilton-Jacobi-Bellman (HJB) equation with the given boundary condition
$$\begin{aligned} -\frac{\partial J}{\partial t}(x,t) &= \min_u H\left(x, \frac{\partial J}{\partial x}, u, t\right) \\ J(x(t_f),t_f) &= \phi(x(t_f),t_f) \quad \text{on} \quad \psi(x(t_f),t_f) = 0 \end{aligned}$$

then J is the optimal cost and u is the corresponding optimal feedback control law.

For both of our formulations classified above, the HJB equation is the same nonlinear first order partial differential equation (PDE) for the spatial variables x and the time variable t. The mathematical expressions for the boundary conditions, however, become distinct from each other:

- HCP: $\phi(x(t_f),t_f) \equiv 0$, $\psi(x(t_f),t_f) = x(t_f) - x_f$. The pair $(x(t_f),t_f)$ is given a priori.
- SCP: $\psi(x(t_f),t_f)$ does not exist and $\phi(x(t_f),t_f) \in \Re$. The pair $(x(t_f),t_f)$, in contrast to HCP, is not given a priori but is indirectly constrained by minimizing $\phi(x(t_f),t_f)$.

Necessary Conditions for Optimality

Now re-consider the Hamiltonian (1) defined above. The standard derivation from the variational calculus and Pontryagin's principle provides the well-known 1st order necessary conditions[17]:

$$\dot{x} = H_\lambda(x,\lambda,u,t) \tag{2}$$
$$\dot{\lambda} = -H_x(x,\lambda,u,t) \tag{3}$$
$$u = arg\min_{\bar{u}} H(x,\lambda,\bar{u},t) \tag{4}$$

where λ is the costate. Substituting (4) into (1)- (3) yields

$$H(x,\lambda,t) = L(x,t) + \lambda^T F(x,t) \tag{5}$$
$$\dot{x} = H_\lambda(x,\lambda,t) \tag{6}$$
$$\dot{\lambda} = -H_x(x,\lambda,t) \tag{7}$$

which is a Hamiltonian canonical system for states and costates.

Evaluating the optimal trajectory corresponds to solving this system of ordinary differential equations (ODEs) satisfying the given boundary conditions. For HCP, the initial states x_0 and terminal states x_f are given and the initial costates λ_0 and terminal costates λ_f are to be determined. For SCP, the initial states are given whereas terminal states, initial costates, and terminal costates are to be determined. Note however that the well-known transversality condition for this case applies[17]:

$$\lambda(t_f) = \partial\phi(x(t_f), t_f)/\partial x(t_f) \tag{8}$$

and relates the terminal states and costates and provides an additional boundary condition for the SCP[4]. Since we need to solve this system of ODEs with split boundary conditions, the optimal control problem is again reduced to a two point boundary value problem (TPBVP).

Our alternative approach to solving the optimal control problem evaluates the optimal trajectory and the associated optimal feedback control using properties of Hamiltonian dynamical systems. The core procedural scheme we use employs the theory of canonical transformations to solve the TPBVP, which is elucidated in the next section.

3 Boundary Value Problems in Hamiltonian Systems

This section introduces the application of canonical transformation theory to solve boundary value problems in Hamiltonian systems. A more detailed description of this theory can be found in Guibout and Scheeres[19]. For a general review of Hamiltonian Dynamical systems see Greenwood [20].

Hamiltonian Systems and Canonical Transformations

Suppose we have a system whose equations of motion can be represented by Hamilton's canonical form

$$\begin{bmatrix} \dot{q}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial H(q(t),p(t),t)}{\partial p} \\ -\frac{\partial H(q(t),p(t),t)}{\partial q} \end{bmatrix}$$

where

- $H = H(q(t), p(t), t)$ is the Hamiltonian of the system,
- $q(t) = [q_1(t) \;\; q_2(t) \;\; \cdots \;\; q_n(t)]^T$ is the generalized coordinate vector,
- $p(t) = [p_1(t) \;\; p_2(t) \;\; \cdots \;\; p_n(t)]^T$ is the generalized momentum vector conjugate to $q(t)$.

[4]Other than HCP and SCP, if there exists terminal hyper plane $\psi(x(t_f), t_f) \in \Re^{p<n}$, then we have more general type of terminal boundary conditions where terminal states are partially determined and the transversality conditions relates the undetermined terminal states with terminal costates. Refer to [17] in detail.

In our application we restrict ourselves to canonical transformations with time as independent parameter, i.e., solutions to the given dynamical system.

Consider a transformation between (q, p, t) and (Q, P, T) defined by

$$Q(T) = Q(q(t), p(t), t, T) \tag{9}$$

$$P(T) = P(q(t), p(t), t, T). \tag{10}$$

If the transformation is canonical, there exists a new Hamiltonian $K = K(Q(T), P(T), T)$ such that the equations of motion have the form:

$$\begin{bmatrix} \dot{Q} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} \frac{\partial K(Q,P,T)}{\partial P} \\ -\frac{\partial K(Q,P,T)}{\partial Q} \end{bmatrix}$$

We note that for this class of transformations $K \equiv H(Q(T), P(T), T)$, however we retain the K notation for convenience in the following. In order to relate K and H recall Hamilton's principle

$$\delta I = \delta \int_{t_0}^{t_f} L dt = 0 \tag{11}$$

where the Lagrangian L is defined as $L(q, \dot{q}, t) = p^T \dot{q} - H(q, p, t)$.

Then, by application of Hamilton's principle (11) for the old and new coordinates, we find that the Lagrangians differ by at most the total time derivative of an arbitrary function F:

$$p^T \dot{q} - H(q, p, t) = P^T \dot{Q} - K(Q, P, T) + \frac{dF}{dt} \tag{12}$$

The function F is called a *generating function* and depends on the old and new states and times (i.e., $4n+2$ variables). Using the $2n$ relations from (9) and (10), we see that F is reduced to a function of $2n+2$ variables. Assume that F is dependent upon n old states and n new states. Then the generating function can have one of the classical forms:

$$F_1(q, Q, t, T), \quad F_2(q, P, t, T), \quad F_3(p, Q, t, T), \quad F_4(p, P, t, T)$$

If, for instance, q and Q are independent variables, then F_1 should be used.

When treating the transformations as solutions we generally choose the states (Q, P) to be the initial conditions at time T, which are constants of motion and hence have a constant Hamiltonian K (which is equivalent to setting $K \equiv 0$). Treating t as the independent variable and expanding the total time derivative of the generating function in (12), we can find the standard results for each generating function:[20]

$$p = \frac{\partial F_1(q, Q, t, T)}{\partial q} \tag{13}$$

$$P = -\frac{\partial F_1(q, Q, t, T)}{\partial Q} \tag{14}$$

$$0 = H(q, p, t) + \frac{\partial F_1(q, Q, t, T)}{\partial t} \tag{15}$$

$$
\begin{aligned}
p &= \frac{\partial F_2(q,P,t,T)}{\partial q} && (16)\\
Q &= \frac{\partial F_2(q,P,t,T)}{\partial P} && (17)\\
0 &= H(q,p,t)+\frac{\partial F_2(q,P,t,T)}{\partial t}. && (18)
\end{aligned}
$$

$$
\begin{aligned}
q &= -\frac{\partial F_3(p,Q,t,T)}{\partial p} && (19)\\
P &= -\frac{\partial F_3(p,Q,t,T)}{\partial Q} && (20)\\
0 &= H(q,p,t)+\frac{\partial F_3(p,Q,t,T)}{\partial t} && (21)
\end{aligned}
$$

$$
\begin{aligned}
q &= \frac{\partial F_4(p,P,t,T)}{\partial p} && (22)\\
Q &= -\frac{\partial F_4(p,P,t,T)}{\partial P} && (23)\\
0 &= H(q,p,t)+\frac{\partial F_4(p,P,t,T)}{\partial t}. && (24)
\end{aligned}
$$

The generating functions satisfy a partial differential equation found by substituting for p in (15)and (18), and for q in (21) and (24):

$$
\begin{aligned}
\frac{\partial F_1(q,Q,t,T)}{\partial t} &+ H\left(q,\frac{\partial F_1(q,Q,t,T)}{\partial q},t\right)=0\\
\frac{\partial F_2(q,P,t,T)}{\partial t} &+ H\left(q,\frac{\partial F_2(q,P,t,T)}{\partial q},t\right)=0\\
\frac{\partial F_3(p,Q,t,T)}{\partial t} &+ H\left(-\frac{\partial F_3(p,Q,t,T)}{\partial p},p,t\right)=0\\
\frac{\partial F_4(p,P,t,T)}{\partial t} &+ H\left(-\frac{\partial F_4(p,P,t,T)}{\partial p},p,t\right)=0,
\end{aligned}
$$

which are usually referred to as the Hamilton-Jacobi (HJ) equation.

A crucial property of the generating functions related to a given transformation is that they are linked to each other via Legendre transformations, which can be represented by the following identities:

$$
F_2(q,P,t,T) = F_1(q,Q,t,T)+P^TQ \qquad (25)
$$

$$F_3(p,Q,t,T) = F_1(q,Q,t,T) - p^T q \tag{26}$$

$$F_4(p,P,t,T) = F_2(q,P,t,T) - p^T q \tag{27}$$

Given an analytical solution to any generating function, it is then possible to evaluate the analytical form of any other generating function as long as some uniqueness conditions are satisfied[19].

Solving Boundary Value Problems with Generating Functions

The choice of the appropriate generating function depends on the type of boundary condition of TPBVP. For the hard constraint problem (HCP), $F_1(x,x_0,t,t_0)$ is the appropriate choice as we know the initial and terminal states. Indeed if we can find F_1, we can directly evaluate the initial and final costates from (13)- (14)[5]:

$$\lambda_f = \left.\frac{\partial F_1}{\partial x}\right|_{t=t_f, x=x_f} = \frac{\partial F_1(x_f,x_0,t_f,t_0)}{\partial x_f} \tag{28}$$

$$\lambda_0 = -\left.\frac{\partial F_1}{\partial x_0}\right|_{t=t_f, x=x_f} = -\frac{\partial F_1(x_f,x_0,t_f,t_0)}{\partial x_0} \tag{29}$$

Furthermore since any time $t \leq t_f$ can be the initial time, the equation (29) should hold for arbitrary initial conditions $x = x(t)$ and $\lambda = \lambda(t)$:

$$\lambda = -\frac{\partial F_1(x_f,x,t_f,t)}{\partial x}. \tag{30}$$

Substitution of (30) into (4) yields the optimal feedback control for the hard constraint problem:

$$u = arg\min_{\bar{u}} H\left(x, -\frac{\partial F_1(x_f,x,t_f,t)}{\partial x}, \bar{u}, t\right)$$

For the soft constraint problem (SCP), a similar argument suggests that $F_3(\lambda,x_0,t,t_0)$ is the appropriate generating function. However, now note that we have $3n$ unknown boundary conditions $(\lambda_0,x_f,\lambda_f)$ and $2n$ corresponding necessary conditions (19)- (20). In order to determine the unknown boundary conditions completely, we also need to consider the n transversality conditions (8). We therefore have

$$x_f = -\left.\frac{\partial F_3}{\partial \lambda}\right|_{t=t_f, x=x_f} = -\frac{\partial F_3(\lambda_f,x_0,t_f,t_0)}{\partial \lambda_f} \tag{31}$$

$$\lambda_0 = -\left.\frac{\partial F_3}{\partial x_0}\right|_{t=t_f, x=x_f} = -\frac{\partial F_3(\lambda_f,x_0,t_f,t_0)}{\partial x_0} \tag{32}$$

$$\lambda_f = \frac{\partial \phi(x_f,t_f)}{\partial x_f} \tag{33}$$

Introducing (33) into (31) and solving for the terminal states yields x_f as a function of pre-specified variables (x_0,t_f,t_0)

$$x_f = x_f(x_0,t_f,t_0). \tag{34}$$

[5] The 1st order necessary condition for optimality can be mapped into the notation of a Hamiltonian system by making the identifications: $x \equiv q$ and $\lambda \equiv p$.

Now substituting (34) into (32)-(33) results in

$$\lambda_0 = \lambda_0(x_0, t_f, t_0) \tag{35}$$

$$\lambda_f = \lambda_f(x_0, t_f, t_0) \tag{36}$$

Note that we have determined all the unknown boundary conditions using (34)-(36) as the solutions of a set of implicit equations. Finally, a similar procedure leads to the optimal feedback control for the SCP[6]:

$$u = arg\min_{\bar{u}} H\left(x, -\frac{\partial F_3(\lambda_f(x,t_f,t),x,t_f,t)}{\partial x}, \bar{u}, t\right)$$

As is shown, once the appropriate generating function has been found, the unknown boundary conditions are simply evaluated by a series of partial differentiations and algebraic manipulations without solving a differential equation. Furthermore the evaluation of the initial costates λ_0 enables us to develop the optimal trajectory by simple forward integration.

Note that we do not need to solve the HJ PDE twice for both F_1 and F_3. In fact, given one kind of generating function enables us to evaluate the rest of the generating functions via the Legendre transformations (25)-(27). This observation is at the heart of our application and provides a substantial advantage over the classical dynamic programming approach. Unlike the dynamic programming approach, we can initially choose any one kind of generating function which may be easier to solve than others. Then without solving the HJ PDE repetitively, a series of partial differentiations, algebraic manipulations, and solutions of implicit functions, provides the other generating functions, which ultimately yields the optimal feedback control for the relevant optimal control formulations. For example, if we have computed $F_1(x, x_0, t, t_0)$, we can directly find $F_3(\lambda, x_0, t, t_0)$ from (13) and (26). Indeed, we have

$$\lambda = \frac{\partial F_1(x, x_0, t, t_0)}{\partial x} \tag{37}$$

$$F_3(\lambda, x_0, t, t_0) = F_1(x, x_0, t, t_0) - \lambda^T x \tag{38}$$

Assuming the uniqueness of inversion for the terminal states x in (37), we can express x as a function of the initial states and terminal costates (x_0, λ):

$$x = x(x_0, \lambda, t, t_0)$$

Then, introducing this to (38) yields F_3 as a function of the desired variables:

$$F_3(\lambda, x_0, t, t_0) = F_1(x(x_0, \lambda, t, t_0), x_0, t, t_0) - \lambda^T x(x_0, \lambda, t, t_0)$$

The next section is dedicated to the numerical implementation of our method based on these favorable properties.

[6]For the terminal constraint given by a hyper plane $\psi(x(t_f), t_f) = 0$ in $\Re^{p<n}$, we will have mixed terminal conditions for both states and costates in general. In this case, a more generalized kind of generating function is required, which would mix all 4 kinds of variables (initial and terminal states and costates).

4 Solution of Optimal Feedback Control Problem via Generating Functions

We have shown that the generating functions can be used to evaluate the optimal trajectory and optimal feedback controls. This result suggests that the optimal feedback control problem can be considered as a part of more comprehensive field of canonical transformations for Hamiltonian dynamical systems.

Now in order to solve optimal feedback control problems for different types of boundary conditions, we need to solve the HJ PDE at least and at most once for one kind of generating function. This requires, at the least, that the functional form of the generating function be specified at some epoch. Now recall that for our canonical transformation the old and new coordinates are equal when $t = T$, and thus the generating function solution to the HJ equation must define an identity transformation at $t = T$. F_1 and F_4 cannot generate such a transformation since the initial and final positions are equal and not independent at $t = T$. On the other hand, F_2 and F_3 are well defined at $t = T$ and generates the identity transformation[7]. Indeed, $F_2(x, \lambda_0, t = t_0, t_0) = x^T \lambda_0$ generates the identity transformation

$$x_0 = \frac{\partial F_2}{\partial \lambda_0} = x \quad , \quad \lambda = \frac{\partial F_2}{\partial x} = \lambda_0.$$

Also similar arguments hold for $F_3(\lambda, x_0, t = t_0, t_0) = -x_0^T \lambda$. Therefore, given the Hamiltonian of a system, we can solve the HJ equation for F_2 or F_3 from the initial time and then evaluate the rest of the others through the Legendre transformations at a later time. In [19] this fact is used to derive a specific solution algorithm for a class of problems where:

1. the system $\dot{x} = F(x(t), u(t), t)$ is analytic and has a zero equilibrium, i.e., $F(x = 0, u = 0, t) = 0$
2. the integrand of the cost function $L(x(t), u(t), t)$ is analytic

In this approach we expand the Hamiltonian H and the generating functions F_2 as Taylor series in their spatial arguments about the zero condition, and these expansions can be made to arbitrarily high order[8]. Then, using the HJ equation, we find a series of ODEs for the coefficients of the series expansion F_2. We solve these equations for F_2, using our initial boundary conditions to generate initial conditions for the ODEs. Then we use the Legendre transformations and inversion of series to compute the coefficients of F_1 and F_3[9]. See [19] for a complete description of this solution method.

[7] Note that this identity transformation is invariant and does not depend on the boundary conditions.

[8] Since F_3 also defines the identity transformation at $t = T$ and we are interested in F_1 and F_3, it is more effective to first evaluate F_3 instead of F_2. For the example and application in this document, however, we first consider F_2 since we have already developed it for previous use. This again suggests the advantage of our approach since obtaining F_1 and F_3 from F_2 by algebraic manipulations of polynomial is easier than solving HJ PDE for them twice.

[9] In case a singularity occurs for a generating function at some epoch, we need to solve for another generating function which is not singular at the same epoch. In this case, we can use the Legendre transformation and inversion of series to compute the coefficients of the latter from the former, and use these to initiate the ODEs that define the latter generating function. However, it turns out that there do not exist any singularities for our special Hamiltonian dynamical systems except the fundamental singularities in F_1 and F_4 when the initial time coincides with the final time. See [19] for a complete description of this solution method.

Example: Specialization to Linear Quadratic Terminal Controller

As an introduction to this approach, we first analyze the well-known linear quadratic terminal controller, as we can explicitly detail the solution procedure. For the accepted paper we will instead focus on the non-linear example given in the next example.

Consider minimization of the following performance index

$$J(x_0,t_0) = \frac{1}{2}x^T(t_f)Q_f x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right] dt$$

subject to the linear system with initial and terminal conditions

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad x(t_0) = x_0 \quad , \quad Mx(t_f) - \beta = 0. \tag{39}$$

The mathematical expression for the terminal condition varies by problem formulation:

- **HCP** : $Q_f = 0_{n\times n},\ M = I_{n\times n},\ \beta = x_f$ where x_f is assumed to be known.
- **SCP** : $M \equiv 0_{n\times n},\ \beta \equiv 0_{n\times 1}$, which implies that the terminal boundary condition is not pre-specified.

The Hamiltonian and the corresponding necessary conditions are independent of boundary conditions and are given by

$$H(x,\lambda,u,t) = \frac{1}{2}(x^T Qx + u^T Ru) + \lambda^T(Ax + Bu) \tag{40}$$

$$\dot{\lambda} = -Qx - A^T\lambda \tag{41}$$

$$u = -R^{-1}B^T\lambda. \tag{42}$$

Substituting (42) into (39), (40), and (41) thus eliminating the control variable u, we have the following Hamiltonian dynamical system:

$$H(x,\lambda,t) = \frac{1}{2}\begin{bmatrix} x \\ \lambda \end{bmatrix}^T \begin{bmatrix} Q & A^T \\ A & -BR^{-1}B^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{43}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{44}$$

As previously suggested, we first evaluate $F_2(x,\lambda_0,t,t_0)$. The quadratic form of the Hamiltonian (43) enables us to express F_2 also in quadratic form:

$$F_2(x,\lambda_0,t;t_0) = \frac{1}{2}\begin{bmatrix} x \\ \lambda_0 \end{bmatrix}^T \begin{bmatrix} F_{xx}(t;t_0) & F_{x\lambda_0}(t;t_0) \\ F_{\lambda_0 x}(t;t_0) & F_{\lambda_0\lambda_0}(t;t_0) \end{bmatrix} \begin{bmatrix} x \\ \lambda_0 \end{bmatrix} \tag{45}$$

Recalling (16)

$$\lambda = \frac{\partial F_2}{\partial x} = \begin{bmatrix} F_{xx} & F_{x\lambda_0} \end{bmatrix} \begin{bmatrix} x \\ \lambda_0 \end{bmatrix}$$

we can express the Hamiltonian as a function of (x, λ_0):

$$H = \frac{1}{2} \begin{bmatrix} x \\ \lambda_0 \end{bmatrix}^T \begin{bmatrix} I & F_{xx} \\ 0 & F_{\lambda_0 x} \end{bmatrix} \begin{bmatrix} Q & A^T \\ A & -BR^{-1}B^T \end{bmatrix} \begin{bmatrix} I & 0 \\ F_{xx} & F_{x\lambda_0} \end{bmatrix} \begin{bmatrix} x \\ \lambda_0 \end{bmatrix} \tag{46}$$

Introduction of (45) and (46) into the HJ PDE (18) yields

$$0 = \begin{bmatrix} x \\ \lambda_0 \end{bmatrix}^T \left\{ \begin{bmatrix} \dot{F}_{xx} & \dot{F}_{x\lambda_0} \\ \dot{F}_{\lambda_0 x} & \dot{F}_{\lambda_0\lambda_0} \end{bmatrix} + \begin{bmatrix} I & F_{xx} \\ 0 & F_{\lambda_0 x} \end{bmatrix} \begin{bmatrix} Q & A^T \\ A & -BR^{-1}B^T \end{bmatrix} \begin{bmatrix} I & 0 \\ F_{xx} & F_{x\lambda_0} \end{bmatrix} \right\} \begin{bmatrix} x \\ \lambda_0 \end{bmatrix},$$

whose sub-matrix components provide the following set of matrix ODEs for $F_{xx}(t;t_0), F_{x\lambda_0}(t;t_0) = F^T_{\lambda_0 x}(t;t_0)$, and $F_{\lambda_0\lambda_0}(t;t_0)$:

$$\begin{aligned} 0 &= \dot{F}_{xx} + Q + F_{xx}A + A^T F_{xx} - F_{xx}BR^{-1}B^T F_{xx} \\ 0 &= \dot{F}_{x\lambda_0} + A^T F_{x\lambda_0} - F_{xx}BR^{-1}B^T F_{x\lambda_0} \\ 0 &= \dot{F}_{\lambda_0\lambda_0} - F_{\lambda_0 x}BR^{-1}B^T F_{x\lambda_0} \end{aligned} \tag{47}$$

Also, the corresponding initial conditions are derived from the identity transformation, $F_2(x, \lambda_0, t = t_0; t_0) = x^T \lambda_0$, as

$$\begin{aligned} F_{xx}(t_0;t_0) &= 0_{n\times n} \\ F_{x\lambda_0}(t_0;t_0) &= I_{n\times n} \\ F_{\lambda_0\lambda_0}(t_0;t_0) &= 0_{n\times n}. \end{aligned} \tag{48}$$

Now we can solve the ODEs (47) with the initial conditions (48), which finally evaluates F_2 by (45). With these results at hand, we now consider for HCP and SCP separately.

Hard Constraint Problem

In order to solve the HCP, recall the Legendre transformation (25)

$$\begin{aligned} F_1(x_f, x_0, t_f, t_0) &= F_2(x_f, \lambda_0, t_f, t_0) - x_0^T \lambda_0 \\ &= \frac{1}{2} \begin{bmatrix} x_f \\ \lambda_0 \end{bmatrix}^T \begin{bmatrix} F_{xx}(t_f, t_0) & F_{x\lambda_0}(t_f, t_0) \\ F_{\lambda_0 x}(t_f, t_0) & F_{\lambda_0\lambda_0}(t_f, t_0) \end{bmatrix} \begin{bmatrix} x_f \\ \lambda_0 \end{bmatrix} - x_0^T \lambda_0. \end{aligned} \tag{49}$$

With the aid of (17)

$$x_0 = \frac{\partial F_2}{\partial \lambda_0} = F_{\lambda_0 x} x_f + F_{\lambda_0\lambda_0} \lambda_0,$$

equation (49) becomes after some algebraic manipulations

$$F_1(x_f,x_0,t_f,t_0)=\frac{1}{2}\begin{bmatrix} x_f \\ x_0 \end{bmatrix}^T \begin{bmatrix} (F_{xx}-F_{x\lambda_0}F^{-1}_{\lambda_0\lambda_0}F_{\lambda_0 x})(t_f,t_0) & (F_{x\lambda_0}F^{-1}_{\lambda_0\lambda_0})(t_f,t_0) \\ (F^{-1}_{\lambda_0\lambda_0}F_{\lambda_0 x})(t_f,t_0) & -(F^{-1}_{\lambda_0\lambda_0})(t_f,t_0) \end{bmatrix} \begin{bmatrix} x_f \\ x_0 \end{bmatrix}.$$

Then from the corresponding necessary condition (29), we have

$$\lambda_0 = F^{-1}_{\lambda_0\lambda_0}(t_f,t_0)(x_0 - F_{\lambda_0 x}(t_f,t_0)x_f).$$

This relation determines the initial costates as a function of the initial and terminal states, which are known by assumption. Hence the optimal trajectory can be obtained by simple forward integration of (44) from the initial time. Furthermore, since this relation holds for any initial time $t \leq t_f$, we have

$$\lambda(t) = F^{-1}_{\lambda_0\lambda_0}(t_f,t)(x(t) - F_{\lambda_0 x}(t_f,t)x_f).$$

Finally from (42) we can evaluate the optimal control in feedback form for hard constraint formulation:

$$u(t) = -R^{-1}(t)B^T(t)F^{-1}_{\lambda_0\lambda_0}(t_f,t)(x(t) - F_{\lambda_0 x}(t_f,t)x_f)$$

Note that we have evaluated the optimal feedback control not by solving HJB PDE directly, but by manipulating the Legendre transformations algebraically.

The Figure 1–2 below compares the results from our method based on generating functions with the exact solution provided by [17]. Here the specific numerical data are

$$A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ B=\begin{bmatrix} 0 \\ 1 \end{bmatrix},\ Q=0_{2\times 2},\ R=I_{2\times 2},\ x_0=\begin{bmatrix} 1 \\ 1 \end{bmatrix},\ x_f=\begin{bmatrix} 0 \\ 0 \end{bmatrix},\ t_0=0,\ t_f=2\pi$$

It is obvious that both results coincide with each other.

Soft Constraint Problem

In order to solve the SCP, we again do not resort to HJB PDE, but start from the appropriate Legendre transformation and derive the unknown boundary conditions and optimal feedback control algebraically. From the relation (26), we have

$$\begin{aligned} F_3(\lambda_f,x_0,t_f,t_0) &= F_1(x_f,x_0,t_f,t_0) - x_f^T\lambda_f \\ &= \frac{1}{2}\begin{bmatrix} x_f \\ x_0 \end{bmatrix}^T \begin{bmatrix} (F_{xx}-F_{x\lambda_0}F^{-1}_{\lambda_0\lambda_0}F_{\lambda_0 x})(t_f,t_0) & (F_{x\lambda_0}F^{-1}_{\lambda_0\lambda_0})(t_f,t_0) \\ (F^{-1}_{\lambda_0\lambda_0}F_{\lambda_0 x})(t_f,t_0) & -(F^{-1}_{\lambda_0\lambda_0})(t_f,t_0) \end{bmatrix} \begin{bmatrix} x_f \\ x_0 \end{bmatrix} - x_f^T\lambda_f \end{aligned} \quad (50)$$

Similarly as in HCP, with the aid of (13)

$$\begin{aligned} \lambda_f &= \frac{\partial F_1}{\partial x_f} = (F_{xx}-F_{x\lambda_0}F^{-1}_{\lambda_0\lambda_0}F_{\lambda_0 x})x_f + F_{x\lambda_0}F^{-1}_{\lambda_0\lambda_0}x_0 \\ \Rightarrow x_f &= (F_{xx}-F_{x\lambda_0}F^{-1}_{\lambda_0\lambda_0}F_{\lambda_0 x})^{-1}\lambda_f - (F_{\lambda_0\lambda_0}F^{-1}_{x\lambda_0}F_{xx} - F_{\lambda_0 x})^{-1}x_0, \end{aligned}$$

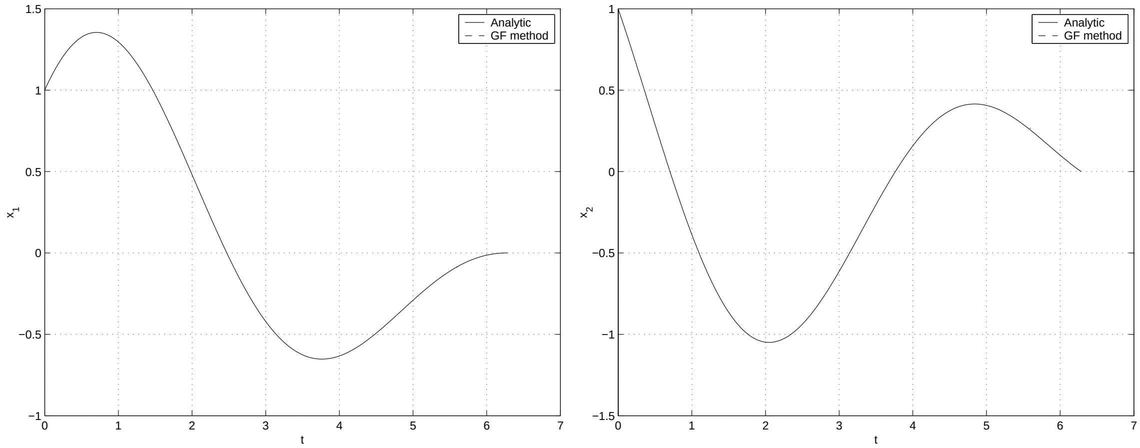

Figure 1: State Trajectory (Hard Constraint Problem)

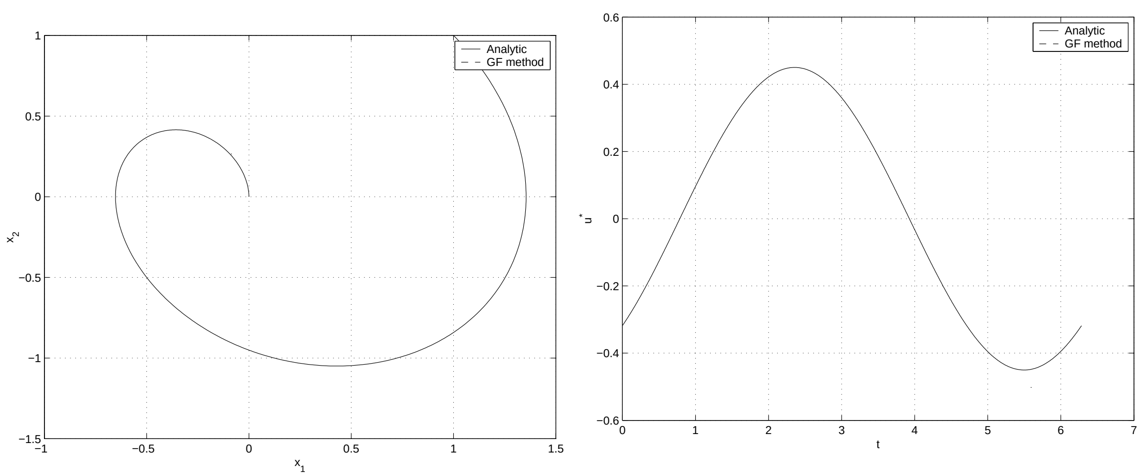

Figure 2: Phase Plane and Control History (Hard Constraint Problem)

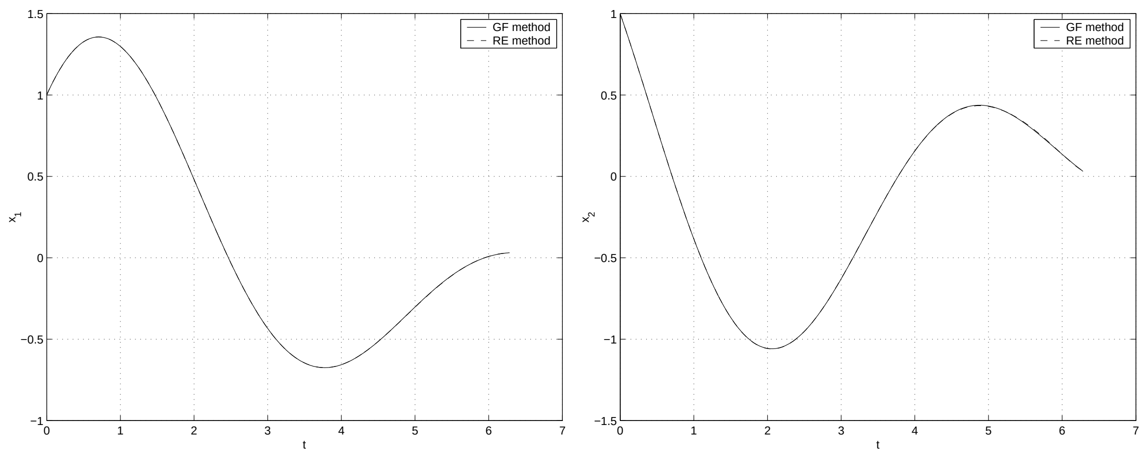

Figure 3: State Trajectory (Soft Constraint Problem)

equation (50) becomes after some algebraic manipulations

$$F_3(\lambda_f, x_0, t_f, t_0) = \frac{1}{2} \begin{bmatrix} \lambda_f \\ x_0 \end{bmatrix}^T \begin{bmatrix} \Gamma_1(t_f,t_0) & \Gamma_2(t_f,t_0) \\ \Gamma_2^T(t_f,t_0) & \Gamma_3(t_f,t_0) \end{bmatrix} \begin{bmatrix} \lambda_f \\ x_0 \end{bmatrix}$$

where

$$\begin{aligned} \Gamma_1 &= -(F_{xx} - F_{x\lambda_0} F_{\lambda_0\lambda_0}^{-1} F_{\lambda_0 x})^{-1} \\ \Gamma_2 &= (F_{\lambda_0\lambda_0} F_{x\lambda_0}^{-1} F_{xx} - F_{\lambda_0 x})^{-1} \\ \Gamma_3 &= -F_{\lambda_0\lambda_0}^{-1} - (F_{\lambda_0\lambda_0} F_{x\lambda_0}^{-1} F_{xx} F_{\lambda_0 x}^{-1} F_{\lambda_0\lambda_0} - F_{\lambda_0\lambda_0})^{-1} \end{aligned}$$

Then from the corresponding necessary conditions and transvarsality condition (31)-(33), the initial costates λ_0 can be obtained as a function of the initial states x_0, which are given by assumption:

$$\lambda_0 = (\Gamma_2^T(t_f,t_0) Q_f (I + \Gamma_1(t_f,t_0) Q_f)^{-1} \Gamma_2(t_f,t_0) - \Gamma_3(t_f,t_0)) x_0$$

Hence the optimal trajectory can be obtained by simple forward integration of (44) from the initial time. Finally similar arguments, as in HCP, leads to the optimal control in feedback form for soft constraint formulation:

$$u(t) = -R^{-1}(t) B^T(t) (\Gamma_2^T(t_f,t) Q_f (I + \Gamma_1(t_f,t) Q_f)^{-1} \Gamma_2(t_f,t) - \Gamma_3(t_f,t)) x(t)$$

The Figure 3–4 below compares the results from our method with the classical solution by Ricatti Equation[17]. All the numerical data are the same as those of HCP except that $Q_f = 10$ and x_f is not given in advance. It is obvious that both results coincide with each other. Also we see that if the final time weight Q_f is large enough, the solution of SCP recovers that of HCP.

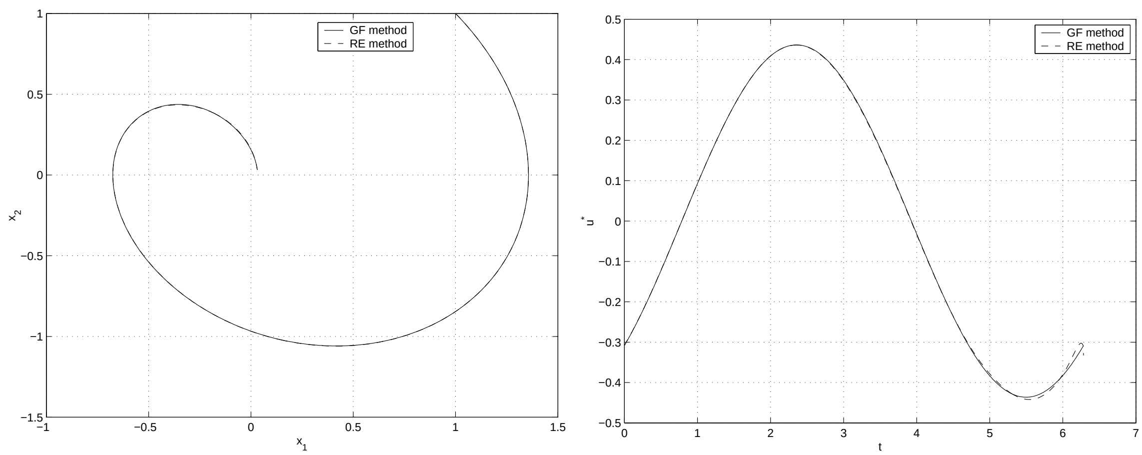

Figure 4: Phase Plane and Control History (Soft Constraint Problem)

5 Application to Nonlinear Optimal Maneuvers in a Central Gravity Field

We apply this specific algorithm to the following problem: consider minimizing

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^T(t)u(t)dt$$

subject to the system dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 2x_4 - (1+x_1)\left(\frac{1}{r^3}-1\right) + u_1 \\ -2x_3 - x_2\left(\frac{1}{r^3}-1\right) + u_2 \end{bmatrix}$$

where $r = \sqrt{(x_1+1)^2 + x_2^2}$. This system represents the planar motion of a particle in a central gravity field, expressed in a rotating coordinate frame. The origin of this frame corresponds to a circular orbit, the coordinates (x_1, x_2, x_3, x_4) represent radial displacement, tangential displacement, radial velocity, and tangential velocity deviations from the circular orbit, and (u_1, u_2) represent the radial and tangential control input, respectively[10]. Expanded as a polynomial series about the zero equilibrium point $[x_1 \;\; x_2 \;\; x_3 \;\; x_4]_e^T = [0 \;\; 0 \;\; 0 \;\; 0]^T$, the system can be re-written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 3x_1 + 2x_4 - 3x_1^2 + 1.5x_2^2 + \cdots + u_1 \\ -2x_3 + 3x_1x_2 + \cdots + u_2 \end{bmatrix} \tag{51}$$

[10] See Scheeres, Park, and Guibout[21] for a more detailed review and derivation of this problem

Pontryagin's principle (or simply the optimality condition $H_u = 0$) provides the well-known Lawden's primer vector optimal control strategy[22]

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\lambda_3 \\ -\lambda_4 \end{bmatrix} \tag{52}$$

If we introduce (52) into the Hamiltonian, it becomes a function of states and costates only:

$$\begin{aligned} H &= -\frac{1}{2}(u_x^2 + u_y^2) + \lambda_1 x_3 + \lambda_2 x_4 \\ &+ \lambda_3(3x_1 + 2x_4 - 3x_1^2 + 1.5x_2^2 + \cdots) \\ &+ \lambda_4(-2x_3 + 3x_1 x_2 + \cdots) \end{aligned} \tag{53}$$

while the costate equations are

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \\ \dot{\lambda}_4 \end{bmatrix} = \begin{bmatrix} -3\lambda_3 + 6x_1\lambda_3 - 3x_2\lambda_4 + \cdots \\ -3x_2\lambda_3 - 3x_1\lambda_4 + \cdots \\ -\lambda_1 + 2\lambda_4 \cdots \\ -\lambda_2 - 2\lambda_3 \cdots \end{bmatrix} \tag{54}$$

Now (51)–(54) constitute the Hamiltonian canonical system to which our approach has been applied. In our implementation of the method we expand H, F_1, F_2, and F_3 to the third order using Matlab.

Figures 5–6 show the state and control trajectories of hard constraint problem for an arbitrarily chosen boundary condition and time interval chosen as

$$x(t_0 = 0) = [0.2 \;\; 0.2 \;\; 0.1 \;\; 0.1]^T \quad , \quad x(t_f = 1) = [0 \;\; 0 \;\; 0 \;\; 0]^T$$

For the control histories, the solid line, dashed line, and dotted line indicate the solution of the nonlinear TPBVP using a shooting method (which is our reference "true" solution), a linear systems solution, and the 3rd order analytical solution described here, respectively. For the state trajectories, each line represents the application of each control history to the original nonlinear system. It is clear that the 3rd order control is a better approximation than the linear control and is close to the true solution. By introducing additional higher order terms in the system dynamics, we can approximate the original system to as high a degree as desired.

Finally given the F_2 and F_1 solutions computed for the HCP, we can use the Legendre transformation to find F_3 and complete the solution process for the SCP. Figures 7–8 show the state and control trajectories of soft constraint problem for the same numerical data except that x_f is not given in advance and that the terminal performance weight $Q_f = diag(10, 0, 10, 10)$. The difference between trajectories by application of the linear controller and the 3rd order controller is even more apparent.

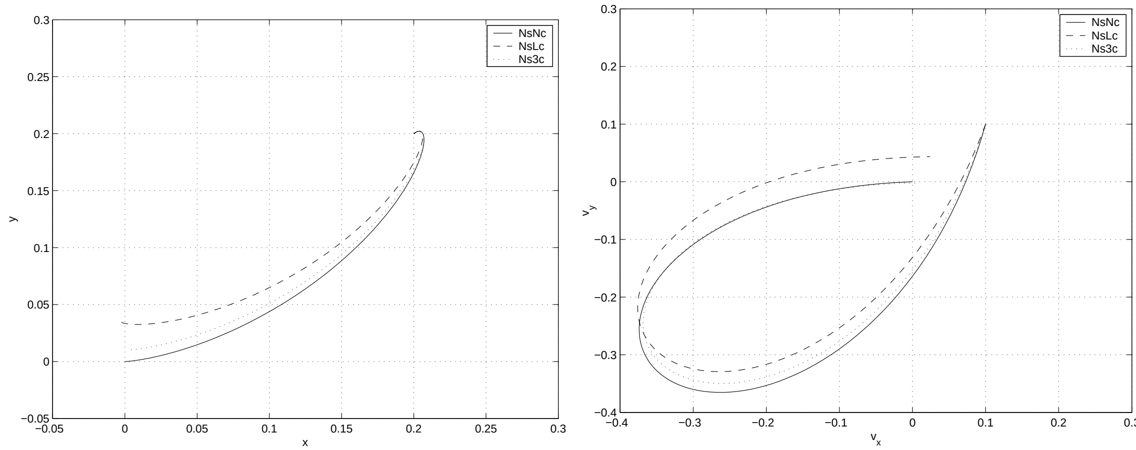

Figure 5: Radial vs. Tangential Position (x_1 vs. x_2) and Radial vs. Tangential Velocity(x_3 vs. x_4) (Hard Constraint Problem)

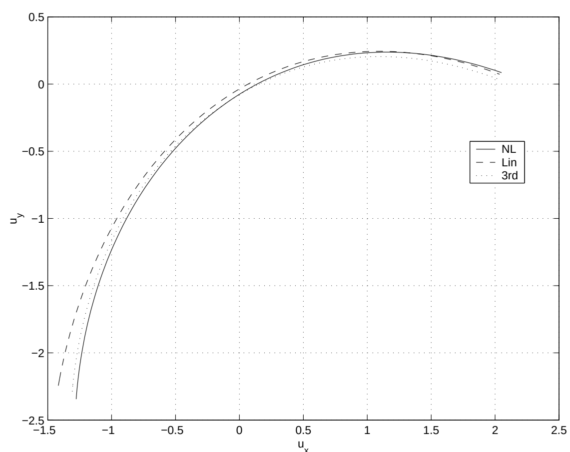

Figure 6: Radial Control (u_1) vs. Tangential Control (u_2) (Hard Constraint Problem)

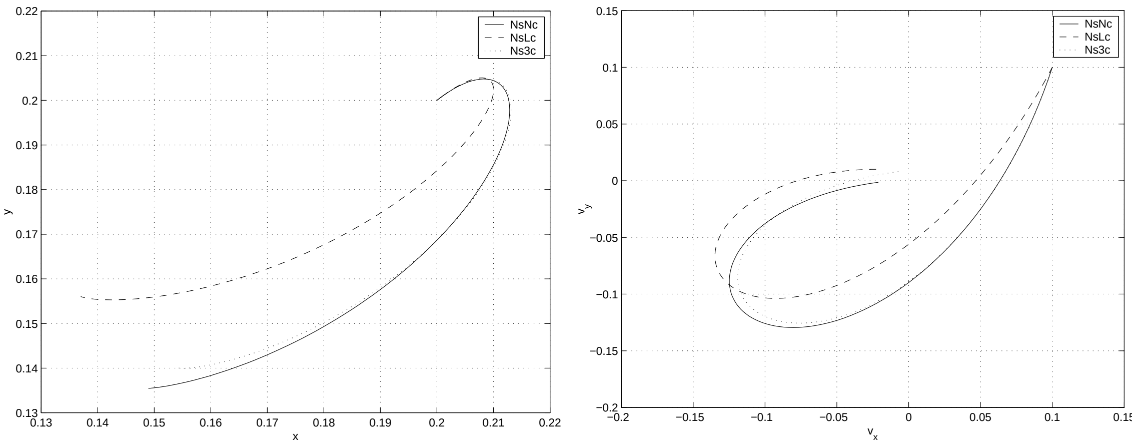

Figure 7: Radial vs. Tangential Position (x_1 vs. x_2) and Radial vs. Tangential Velocity(x_3 vs. x_4) (Soft Constraint Problem)

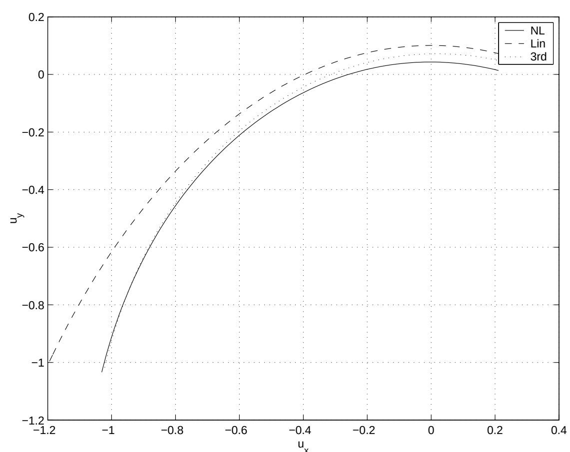

Figure 8: Radial Control (u_1) vs. Tangential Control (u_2) (Soft Constraint Problem)

6 Conclusion

We have introduced a new approach for finding the optimal feedback control for different types of boundary conditions. Then we have applied our approach to two extreme cases of boundary conditions (hard constraint problem and soft constraint problem). Our method is advantageous over the classical dynamical programming approach in that 1) we do not need to solve the HJ-type PDE repetitively for different types of boundary conditions and 2) the final time singularity for hard constraint problem can be circumvented to evaluate the optimal feedback control and cost function. All this implies that the optimal feedback control problem can be included within the more fundamental problem of canonical transformations for Hamiltonian systems.

Our future research will be directed toward 1) the application of this approach to more general types of boundary conditions 2) more systematic and efficient numerical implementation of our approach. We will also explore the group properties of Hamiltonian systems to search for additional advantages of our approach.

References

[1] R. C. Durbeck. An approximation technique for suboptimal control. 10:144–149, 1965.

[2] D. L. Lukes. Optimal regulation of nonlinear dynamical systems. *Siam Journal on Control*, 7(1):75–100, 1966.

[3] Y. Huang and W. Lu. Nonlinear optimal control:alternatives to hamilton jacobi equation. In *Proceedings of the American Control Conference*, pages 3942–3947, 1996. Kobe, Japan.

[4] R. Beard, G. Saridis, and J. Wen. Galerkin approximation of the Generalized Hamilton-Jacobi-Bellman equation. *Automatica*, 33(12):2159–2177, 1997.

[5] J. R. Cloutier. State-dependent Riccati equation techniques: An overview. In *Proceedings of the American Control Conference*, pages 932–936, 1997. Alberquerque, NM.

[6] P. Tsiotras, M. Corless, and M. Rotea. An $\mathcal{L}_2$ disturbance attenuation solution to the nonlinear benchmark problem. *International Journal of Robust and Nonlinear Control*, 8:311–330, 1998.

[7] J. W. Curtis and R. W. Beard. Successive collocation: An approximation to optimal nonlinear control. In *Proceedings of the American Control Conference*, pages 3481–3485, 2001. Arlington, VA.

[8] S. C. Beeler, H. T. Tran, and H. T. Banks. Feedback control methodologies for nonlinear systems. *Journal of Optimization Theory and Applications*, 107(1):1–33, 2000.

[9] J. H. Burghart. A technique for suboptimal feedback control of nonlinear systems. *IEEE Transactions on Automatic Control*, 14:530–533, 1969.

[10] W. L. Garrard, N.H. McClamroch, and L. G. Clark. An approach to sub-optimal feedback control of non-linear systems. *International Journal of Control*, 5(5):425–435, 1967.

[11] G. N. Saridis and G. L. Chun-Sing. An approximation theory of optimal control for trainable manipulators. *IEEE Transactions on Systems, Man, and Cybernetics*, 9(3):152–159, 1979.

[12] J. A. Fax and R. M. Murray. Finite-horizon optimal control and stabilization of time-scalable systems. In *Proceedings of the 39st IEEE Conference on Decision and Control*, pages 748–753, 2000. Sydney, Australia.

[13] B. Fraeijs de Veubeke. Canonical transformations and the thrust-coast-thrust optimal transfer problem. *Astronautica Acta*, 11(4):271–282, 1965.

[14] W. F. Powers and B. D. Tapley. Canonical transformation applications to optimal trajectory analysis. *AIAA Journal*, 7(3):394–399, 1969.

[15] M. Popescu. Applications of canonical transformations in optimizing orbital transfers. *Journal of Guidance, Control, and Dynamics*, 20(4):774–779, 1997.

[16] C. Park and D. J. Scheeres. Solutions of optimal feedback control problem using hamiltonian dynamics and generating functions. In *IEEE conference on Decision and Control*, page accepted, 2003. Maui, Hawaii.

[17] A. E. Bryson and Y. Ho. *Applied Optimal Control*. Hemisphere Publishing Corp., London, 1975.

[18] Athans M and P. L. Falb. *Optimal Control: An Introduction to the Theory and its Applications*. McGraw-Hill, New York, U.S.A., 1966.

[19] V. M. Guibout and D. J. Scheeres. Formation flight with generating functions: Solving the relative boundary value problem. In *AIAA Astrodynamics Specialist Meeting*, 2002. Monterey, California. Paper AIAA 2002-4639.

[20] D. T. Greenwood. *Classical Dynamics*. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1977.

[21] D. J. Scheeres, C. Park, and V. M. Guibout. Solving optimal control problems with generating functions. In *AAS/AIAA Astrodynamics Specialist Meeting*, pages Paper AAS 03–575, 2003. Big Sky, Montana.

[22] D. F. Lawden. *Optimal Trajectories for Space Navigation*. Butterworths, London, England, 1963.