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Generalized Models for Solar Sails

L. Rios-Reyes*and D. J. Scheeres†

Abstract

To successfully navigate a solar sail spacecraft requires that a precision propulsion model be defined. Such a model must incorporate all the relevant deviations of a sail from its ideal shape in a tractable form that will allow the model to be refined in-flight based on tracking data. As a first step in this direction we develop a new approach to model the force and moment generated by a solar sail as a function of the incident light direction. Analytical solutions for the force, moments, and center of pressure are found for an arbitrary shape, in terms of a number of computable coefficients. Using these results we compute models for a number of solar sails of increasing complexity.

1 Introduction

We define a new methodology for the general description of a solar sail. We find that the total force and moment generated by the sail can be completely defined by the computation of a number of constants of the sail which are only functions of the sail geometry and independent of the incident light. Given these constants, some standard sail properties, and the incident light pressure and direction, the total force and moment acting on the sail can be computed using simple formulae. Specifically, we can characterize the force generated by a solar sail of arbitrary sail geometry under general illumination conditions with 18 numbers. To characterize the moment requires an additional 36 numbers. The advantages of this description are that these constants can be computed off-line and are defined for arbitrarily shaped sails, meaning that only one formalism must be coded to deal with all sails.

There are a number of limits to this approach. First, this model tacitly assumes that the sail will not change shape as its attitude varies, thus their utility may be somewhat limited or constrained to relatively small angular motions. There are some pathways for addressing this limitation, however. Second, they assume that

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[*] Graduate Student, Department of Aerospace Engineering, The University of Michigan, 1320 Beal Avenue, Ann Arbor, MI 48105-2140, leonelr@umich.edu, Student Member AIAA

[†] Associate Professor, Department of Aerospace Engineering, The University of Michigan, 1320 Beal Avenue, Ann Arbor, MI 48105-2140, scheeres@umich.edu, Member AAS, Associate Fellow AIAA

there is no self-shadowing, i.e., that every element of the sail surface will “see” an incident ray. This is probably a reasonable assumption.

In this paper we first describe the forces acting on a solar sail, then derive a general approach for computing the forces and moments acting on a solar sail, and finally derive specific force and moment functions for various shape models.

2 Solar Radiation Pressure

McInnes [1] develops the solar pressure due to the sun’s finite disk on an ideal sail normal to the sun, which includes the force exerted on the sail due to impinging and reflected photons. The reflected force is not the same as the impinging force for a real sail. Thus, the radiation pressure at a distance r from the sun due to a finite solar disk is [1]:

$$P(r) = \frac{1}{c}\int_0^{\infty}\int_0^{2\pi}\int_0^{C_0} I_\nu \cos^2(C)\sin(C) dC d\delta d\nu \tag{1}$$

where c is the speed of light, C the sun’s apparent angular radius, C_0 the maximum apparent angular radius given by $\arcsin(R_s/r)$, δ the clock angle, ν the radiation wavelength, and I_ν the radiation specific intensity at a wavelength ν. Since I_ν does not depend on r, it can be averaged over the entire spectrum to yield [1]:

$$P(r) = \frac{2\pi I_0}{c}\int_0^{C_0} \cos^2(C)\sin(C) dC \tag{2}$$

where I_0 is the frequency integrated specific intensity. Performing the integration and substituting for C_0, the radiation pressure becomes [1]:

$$P(r) = \frac{2\pi I_0}{3c}\left\{1 - \left[1 - \left(\frac{R_s}{r}\right)^2\right]^{3/2}\right\} \tag{3}$$

Eq. (3) can be rearranged as [1]:

$$P(r) = P^*(r)F(r) \tag{4}$$

where $P^*(r)$ is the radiation pressure of a point source given by [1]:

$$P^*(r) = \frac{I_0\pi}{c}\left(\frac{R_s}{r}\right)^2 \tag{5}$$

and

$$F(r) = \frac{2}{3}\left(\frac{r}{R_s}\right)^2\left\{1 - \left[1 - \left(\frac{R_s}{r}\right)^2\right]^{3/2}\right\} \tag{6}$$

$F(r)$ is a correction function to account for the sun’s finite disk. With this formalism we assume the solar radiation travels in parallel rays when it reaches the sail.

3 Sail Optical Parameters and Differential Forces

The total force acting on the solar sail is due to a combination of forces that result from photons impinging on and reflecting from the sail surface, as shown in Fig. 1. Here, the sail is assumed to be opaque so that the transmissivity is zero. Then, the sum of the reflectivity ρ and the absorptivity a must be unity. Also, it must be noted that ρ and a might be dependent on the angle between the incident light source direction and the surface normal α and the wavelength ν. The differential force due to reflection $d\mathbf{F}_r$ is composed of two components: $d\mathbf{F}_{rs}$, a fraction s due to specular reflection acting along the normal and transverse directions, and $d\mathbf{F}_{rd}$, a fraction $B_f(1-s)$ due to diffuse or uniform reflection acting along the normal direction. B is a coefficient describing the deviation of the surface from a Lambertian surface while the subscript f denotes the front surface. According to Meyer-Arendt [2], a Lambertian surface has the same radiance in all directions.

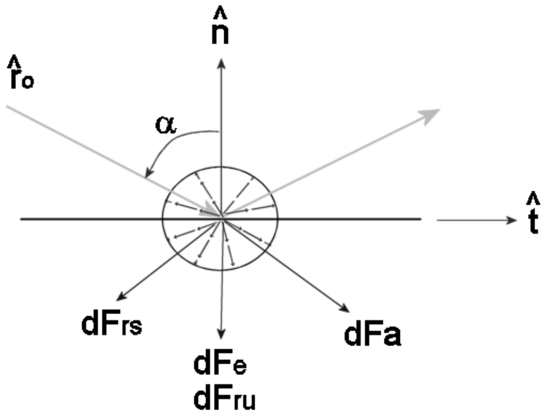

Figure 1: Force Directions.

The force caused by emission $d\mathbf{F}_e$ is due to absorbed photons that are now being radiated as heat and acts along the normal direction. When the sail absorbs photons its temperature increases up to an equilibrium temperature at which the absorbed energy is equal to the radiated energy. Performing an energy balance, it can be shown that the equilibrium temperature of the sail is given by [1]:

$$T^4 = \frac{(1-\rho)cP\cos(\alpha)}{\sigma(\epsilon_f+\epsilon_b)} \tag{7}$$

where ϵ is the surface emissivity, the subscripts f and b denote the front and back surfaces, respectively, and σ is the Stefan-Boltzmann constant.

Defining the unit normal vector $\hat{\mathbf{n}}$ perpendicular to a surface of an area dA and the transverse vector $\hat{\mathbf{t}}$, perpendicular to $\hat{\mathbf{n}}$ and in the plane of the incident light and the surface normal, the forces acting on a differential area of the sail along these directions are given by Eqs. (8)-(11). They represent the contribution from radiation impacting the sail $d\mathbf{F}_a$, reflected specularly $d\mathbf{F}_{rs}$ and diffusively $d\mathbf{F}_{ru}$ from it, and emitted by radiation from the sail $d\mathbf{F}_e$, respectively [1] .

$$d\mathbf{F}_a = P(r)\cos\alpha\left[-\cos\alpha\hat{\mathbf{n}} + \sin\alpha\hat{\mathbf{t}}\right]dA \tag{8}$$

$$d\mathbf{F}_{rs} = P(r)\cos\alpha\rho s\left[-\cos\alpha\hat{\mathbf{n}} - \sin\alpha\hat{\mathbf{t}}\right]dA \tag{9}$$

$$d\mathbf{F}_{ru} = -P(r)\cos\alpha B_f\rho(1-s)\hat{\mathbf{n}}dA \tag{10}$$

$$d\mathbf{F}_e = -P(r)\cos\alpha(1-\rho)\frac{\epsilon_f B_f - \epsilon_b B_b}{\epsilon_f + \epsilon_b}\hat{\mathbf{n}}dA \tag{11}$$

Decomposing the forces into their normal and transverse components, we obtain [1]:

$$\begin{aligned} dF_n = -P(r)\Big[&(1+\rho s)\cos^2(\alpha) + B_f(1-s)\rho\cos(\alpha) + \\ &(1-\rho)\frac{\epsilon_f B_f - \epsilon_b B_b}{\epsilon_f + \epsilon_b}\cos(\alpha)\Big]dA \end{aligned} \tag{12}$$

$$dF_t = P(r)(1-\rho s)\cos(\alpha)\sin(\alpha)dA \tag{13}$$

where the subscripts n and t denote the force magnitude along the normal and transverse vectors. Grouping the optical sail elements, the differential force normal to the sail element can be expressed as:

$$d\mathbf{F}_n = -P(r)[a_1\cos^2\alpha + a_2\cos\alpha]dA\hat{\mathbf{n}} \tag{14}$$

where $a_1 = 1 + \rho s$, $a_2 = B_f(1-s)\rho + (1-\rho)\frac{\epsilon_f B_f - \epsilon_b B_b}{\epsilon_f + \epsilon_b}$. The differential transverse force is given by:

$$d\mathbf{F}_t = P(r)a_3\cos\alpha\sin\alpha dA\hat{\mathbf{t}} \tag{15}$$

where $a_3 = 1 - \rho s$. The sun position unit vector is specified as $\hat{\mathbf{r}}_0$ and points from the sun to the sail. Thus, the angle α is defined by $\cos\alpha = -\hat{\mathbf{r}}_0 \cdot \hat{\mathbf{n}}$.

The total force due to these normal and transverse components are found by integrating these expressions over the sail surface:

$$\begin{aligned} \mathbf{F}_n &= \int_A d\mathbf{F}_n && (16) \\ \mathbf{F}_t &= \int_A d\mathbf{F}_t && (17) \\ \mathbf{F} &= \int_A (d\mathbf{F}_n + d\mathbf{F}_t) && (18) \end{aligned}$$

To derive this result we note that Eqs. (14) and (15) require knowledge of $\cos\alpha$, $\sin\alpha$ and $\hat{\mathbf{t}}$, which can be obtained from:

$$\begin{aligned} \cos\alpha &= -\hat{\mathbf{n}} \cdot \hat{\mathbf{r}}_0 && (19) \\ \sin\alpha &= \sqrt{1 - (\hat{\mathbf{n}} \cdot -\hat{\mathbf{r}}_0)^2} = \|\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times -\hat{\mathbf{r}}_0)\| && (20) \\ \hat{\mathbf{t}} &= \frac{(\hat{\mathbf{n}} \times \hat{\mathbf{r}}_0) \times \hat{\mathbf{n}}}{\|(\hat{\mathbf{n}} \times \hat{\mathbf{r}}_0) \times \hat{\mathbf{n}}\|} = -\frac{\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \hat{\mathbf{r}}_0)}{\|\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \hat{\mathbf{r}}_0)\|} && (21) \end{aligned}$$

The force equations stated in this form lead to difficulties when trying to carry out the surface integrations analytically. If the sail surface, and therefore its normal vector, is not simple, analytic solutions cannot be found in general. Also, the integrals are strongly dependent on the sun's position, apparently making it very difficult to generalize to any sail attitude.

4 Derivation of the Generalized Sail Force Equation

We have found that the integration of Eq. (18) can be reduced to an integration over the sail, independent of the incident light direction and magnitude (under an ideal solar sail assumption where the structure is fixed).

Let $\hat{\mathbf{n}} = [\hat{n}_1 \;\; \hat{n}_2 \;\; \hat{n}_3]^T$ and define the cross product as:

$$\hat{\mathbf{n}} \times -\hat{\mathbf{r}}_0 = -\tilde{\mathbf{n}} \cdot \hat{\mathbf{r}}_0 \tag{22}$$

where

$$\mathbf{n} = \begin{bmatrix} 0 & -\hat{n}_3 & \hat{n}_2 \\ \hat{n}_3 & 0 & -\hat{n}_1 \\ -\hat{n}_2 & \hat{n}_1 & 0 \end{bmatrix} \tag{23}$$

Then Eqs. 20 and 21 can be multiplied to obtain:

$$\sin\alpha\hat{\mathbf{t}} = -\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \hat{\mathbf{r}}_0) = -\tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} \cdot \hat{\mathbf{r}}_0 \tag{24}$$

Following similar simplifications, Eqs. (14) and (15) become:

$$d\mathbf{F}_n = -P(r)[a_1(\hat{\mathbf{r}}_0 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \hat{\mathbf{r}}_0) - a_2(\hat{\mathbf{r}}_0 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}]dA \tag{25}$$

$$d\mathbf{F}_t = -P(r)a_3(\hat{\mathbf{r}}_0 \cdot \hat{\mathbf{n}})\tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} \cdot \hat{\mathbf{r}}_0 dA \tag{26}$$

Some of the terms in the above expressions can be simplified by the introduction of a dyadic (and triadic) notation [3]. It is possible to define the dyadic of the normal vector as:

$$\overline{\overline{\mathbf{n}}} = \hat{\mathbf{n}}\hat{\mathbf{n}} \tag{27}$$

and the triadic as:

$$\overline{\overline{\overline{\mathbf{n}}}} = \hat{\mathbf{n}}\hat{\mathbf{n}}\hat{\mathbf{n}} \tag{28}$$

These are really just rank 2 and 3 tensors, and can be properly specified as $\overline{\overline{n}}_{ij} = \hat{n}_i\hat{n}_j$, and $\overline{\overline{\overline{n}}}_{ijk} = \hat{n}_i\hat{n}_j\hat{n}_k$, where the indices range from 1 to 3. Also, making use of the identity

$$\tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} = -\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}\overline{\overline{\mathbf{U}}} + \hat{\mathbf{n}}\hat{\mathbf{n}} = -\overline{\overline{\mathbf{U}}} + \overline{\overline{\mathbf{n}}} \tag{29}$$

where $\overline{\overline{\mathbf{U}}}$ is the unit dyadic, the differential forces can be stated as:

$$d\mathbf{F}_n = -P(r)a_1\hat{\mathbf{r}}_0 \cdot \overline{\overline{\overline{\mathbf{n}}}}dA \cdot \hat{\mathbf{r}}_0 + P(r)a_2\hat{\mathbf{r}}_0 \cdot \overline{\overline{\mathbf{n}}}dA \tag{30}$$

and

$$d\mathbf{F}_t \;=\; P(r)a_3\hat{\mathbf{r}}_0 \cdot \hat{\mathbf{n}}\overline{\overline{\mathbf{U}}}dA \cdot \hat{\mathbf{r}}_0 - P(r)a_3\hat{\mathbf{r}}_0 \cdot \overline{\overline{\overline{\mathbf{n}}}}dA \cdot \hat{\mathbf{r}}_0 \tag{31}$$

The products of these tensors and the sun's position unit vector can be stated in terms of the summation convention as:

$$\begin{aligned} \hat{\mathbf{r}}_0 \cdot \overline{\overline{\overline{\mathbf{n}}}} \cdot \hat{\mathbf{r}}_0 &= \hat{n}_i\hat{n}_j\hat{n}_k\hat{\mathbf{r}}_{0_j}\hat{\mathbf{r}}_{0_k} \\ \hat{\mathbf{r}}_0 \cdot \overline{\overline{\mathbf{n}}} &= \hat{n}_i\hat{n}_j\hat{\mathbf{r}}_{0_i} \\ \hat{\mathbf{r}}_0 \cdot \overline{\mathbf{n}} &= \hat{n}_i\hat{\mathbf{r}}_{0_i} \end{aligned}$$

where equal indices imply summation, i.e., $a_ib_i = \sum_{i=1}^{3} a_ib_i$.

Adding Eqs. (30) and (31), and integrating over the sail surface the total force is:

$$\mathbf{F} = P(r)\Big[\int_A a_2\overline{\overline{\mathbf{n}}}dA \cdot \hat{\mathbf{r}}_0 + \hat{\mathbf{r}}_0 \cdot \Big(-2\int_A \rho s\overline{\overline{\overline{\mathbf{n}}}}dA + \overline{\overline{\mathbf{U}}}\int_A a_3\hat{\mathbf{n}}dA\Big) \cdot \hat{\mathbf{r}}_0\Big] \tag{32}$$

The integrands of all these expressions are independent of the solar incidence, can be computed off-line for a given sail shape, re-used over a range of sail attitudes, and ideally can accommodate non-uniformities in the sail optical properties.

Now we will introduce a more systematic notation for these integrals. Let us define the surface normal distribution integrals as:

$$\mathbf{J}^m \;=\; \int_A \hat{\mathbf{n}}^m dA \tag{33}$$

$$\;=\; \int_A \hat{\mathbf{n}}\hat{\mathbf{n}}\ldots\hat{\mathbf{n}}dA \tag{34}$$

where $\mathbf{J}^m$ is a rank-m tensor, computed by integrating the product of the normal vectors over the surface area of the sail.

Assuming constant sail optical properties, the force can now be rewritten as:

$$\mathbf{F} \;=\; P(r)\Big[a_2\mathbf{J}^2 \cdot \hat{\mathbf{r}}_0 - 2\rho s(\mathbf{J}^3 \cdot \hat{\mathbf{r}}_0) \cdot \hat{\mathbf{r}}_0 + a_3(\mathbf{J}^1 \cdot \hat{\mathbf{r}}_0)\hat{\mathbf{r}}_0\Big] \tag{35}$$

Thus, we have arrived at a completely analytic formula for the force acting on a solar sail, which we believe, is an extremely general and new result.

To properly specify the $\mathbf{J}^m$ as a tensor, we can use the notation:

$$\mathbf{J}^m_{i_1 i_2 \ldots i_m} = \int_A \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_m} dA \tag{36}$$

$$i_j = 1, 2, 3 \tag{37}$$

where the entries $\hat{n}_i$ are just the elements of the normal vector evaluated at the surface element dA. We note that these tensors are completely symmetric in their indices, i.e., $J^m_{i_1 i_2 \ldots i_m} = J^m_{i_2 i_1 \ldots i_m}$, and so on for any two indices. Thus, for a rank-3 tensor, which could have up to 27 entries, we only need to compute 9 independent values. In general, a tensor $\mathbf{J}^m$ as defined above will only have $3m$ unique terms among its 3^m entries. Thus, the three integrals in Eq. (35) are specified by 3 + 6 + 9 = 18 numbers for the general case. If we are dealing with a simplified model, such as a flat-plate solar sail model where $\hat{\mathbf{n}} = [0, 0, 1]^T$, the number of independent numbers would reduce to 3, one for each $\mathbf{J}^m$.

Since these tensors are defined as integrations, it is always possible to add additional sail elements by adding the $\mathbf{J}^m$ term for that additional piece, so long as they are computed relative to the same coordinate frame.

Since we are dealing with tensors, it is also possible to transform a given $\mathbf{J}^m$ defined in one coordinate frame into a different coordinate frame. Suppose we have a $\mathbf{J}^m$ defined for a panel of our sail, computed in the panel-fixed frame. Also assume we have a transformation matrix T that takes a vector from the panel-fixed frame into the sail-fixed frame. Thus, to transform a normal vector $\hat{\mathbf{n}}$ from the panel-fixed frame to the sail-fixed frame we just need to perform a matrix multiply, $\hat{\mathbf{n}}' = T\hat{\mathbf{n}}$, where the $'$ signifies that the vector is specified in the new frame. Using tensor notation, this same transformation would be expressed as $\hat{n}'_j = T^i_j \hat{n}_i$, where the i index signifies the column number for the T matrix, and j signifies the row number, and the summation convention is assumed (equal indexes are summed over, i.e., $T^i_j \hat{n}_i = \sum_{i=1}^{3} T^i_j \hat{n}_i$). Then the following operations would transform the $\mathbf{J}^m$ tensor computed relative to the panel frame into the sail-fixed frame, where they could be directly added to obtain the sail's complete $\mathbf{J}^m$ tensors. As these transformation matrices are known in general, this would be a simple operation to define and extremely simple to carry out in an algorithm.

$$\mathbf{J}^{m'}_{j_1 j_2 \ldots j_m} = T^{i_1}_{j_1} T^{i_2}_{j_2} \ldots T^{i_m}_{j_m} \mathbf{J}^m_{i_1 i_2 \ldots i_m} \tag{38}$$

5 Derivation of the Generalized Sail Moment Equation

The total moment acting on the sail can be found by integrating the expression:

$$d\mathbf{M} = \vec{\varrho} \times d\mathbf{F} = \tilde{\varrho} \cdot d\mathbf{F} \tag{39}$$

over the entire sail. This is equivalent to:

$$d\mathbf{M} = P(r)\tilde{\varrho} \cdot \Big[a_2 \overline{\overline{\mathbf{n}}} dA \cdot \hat{\mathbf{r}}_0 + \hat{\mathbf{r}}_0 \cdot \Big(-2\rho s \overline{\overline{\overline{\mathbf{n}}}} dA + a_3 \hat{\mathbf{n}} \overline{\overline{\mathbf{U}}} dA \Big) \cdot \hat{\mathbf{r}}_0 \Big] \tag{40}$$

where $\vec{\varrho}$ is the position of the differential element dA with respect to a given reference frame on the sail. Integrating yields the total moment about the origin of the sail reference frame:

$$\begin{aligned}\mathbf{M} &= P(r)\Big[a_2 \int_A \tilde{\varrho} \cdot \overline{\overline{\mathbf{n}}} dA \cdot \hat{\mathbf{r}}_0 - 2\rho s(\int_A \tilde{\varrho} \cdot \overline{\overline{\overline{\mathbf{n}}}} dA \cdot \hat{\mathbf{r}}_0) \cdot \hat{\mathbf{r}}_0 && (41)\\ &\quad + a_3 \hat{\mathbf{r}}_0 \cdot \int_A \hat{\mathbf{n}} \tilde{\varrho} \cdot \overline{\overline{\mathbf{U}}} \cdot \hat{\mathbf{r}}_0 dA\Big] && (42)\end{aligned}$$

Defining the moment surface normal distribution integrals as:

$$\mathbf{K}^m = \int_A \tilde{\varrho} \cdot \hat{\mathbf{n}}^m dA \tag{43}$$

$$\mathbf{L} = \int_A \hat{\mathbf{n}} \vec{\varrho} dA \tag{44}$$

The moment can be rewritten as:

$$\mathbf{M} = P(r)\Big[a_2 \mathbf{K}^2 \cdot \hat{\mathbf{r}}_0 - 2\rho s(\mathbf{K}^3 \cdot \hat{\mathbf{r}}_0) \cdot \hat{\mathbf{r}}_0 + a_3 \hat{\mathbf{r}}_0 \cdot \mathbf{L} \cdot \tilde{\hat{\mathbf{r}}}_0\Big] \tag{45}$$

The tensors $\mathbf{K}^m$ and $\mathbf{L}$ (which are rank-m and rank-2 tensors, respectively) do not have the same complete symmetry as the $\mathbf{J}^m$ do. Thus, there are more numbers needed to specify them. For example, $\mathbf{L}$ requires at least 9 numbers, $\mathbf{K}^2$ requires 9 numbers and $\mathbf{K}^3$ requires 18. Despite this, they are well defined and can be used to characterize the moments acting on the body as the sail attitude changes. Transformations from different coordinate frames might be necessary in some cases. For these situations the use of Eq. (38) will be still appropriate, so long as the transformation is a pure rotation and does not involve translation. If the panel is to be translated as well, an additional term $\vec{\varrho}_t \times \mathbf{F}$ must be added, where $\vec{\varrho}_t$ is the translation vector, and $\mathbf{F}$ is the total force acting on that panel.

Center of Pressure

One particular application of the current result is to find the center of pressure of the sail, $\mathbf{r}_p$. In general this vector is defined by the condition:

$$\begin{aligned}\mathbf{M} &= \mathbf{r}_p \times \mathbf{F} && (46)\\ &= \mathbf{r}_p \cdot \tilde{\mathbf{F}} && (47)\end{aligned}$$

where $\mathbf{F}$ is the total computed force and $\mathbf{M}$ is the total computed moment for a given origin. Let us dot both sides on the right with $-\tilde{\mathbf{F}}$ to obtain:

$$-\mathbf{r}_p \cdot \tilde{\mathbf{F}} \cdot \tilde{\mathbf{F}} = -\mathbf{M} \cdot \tilde{\mathbf{F}} = \tilde{\mathbf{F}} \cdot \mathbf{M} \tag{48}$$

Dividing by the total force magnitude, F, this equation becomes:

$$\mathbf{r}_p \cdot \Big[\overline{\overline{\mathbf{U}}} - \hat{\mathbf{F}}\hat{\mathbf{F}}\Big] = \frac{1}{F^2} \tilde{\mathbf{F}} \cdot \mathbf{M} \tag{49}$$

The terms in the brackets are dyads (rank-2 tensors) that project the center of pressure vector into a vector perpendicular to the force line. Taking the pseudo-inverse of this operator yields the center of pressure vector and the associated line of action for the sail force:

$$\mathbf{r}_p = \frac{1}{F^2}\mathbf{F} \times \mathbf{M} + \sigma\hat{\mathbf{F}} \tag{50}$$

where σ is an arbitrary distance.

Finally, if we are also given the center of mass of the sail, $\mathbf{r}_{CM}$, and the center of pressure, $\mathbf{r}_p$, we can compute the total moment acting on the sail about its center of mass:

$$\begin{aligned} \mathbf{M}_{CM} &= (\mathbf{r}_p - \mathbf{r}_{CM}) \times \mathbf{F} && (51) \\ &= \mathbf{M} - \mathbf{r}_{CM} \times \mathbf{F} && (52) \end{aligned}$$

6 Example Cases

In this section we compute force and moment coefficients of sails of increasing complexity. These include a flat sail model, a symmetric circular sail with billow, and more complex four-panel sail with billow in each panel.

6.1 Flat Sail

For the case of a flat sail, the normal vector is invariant with location on the sail, thus, the unit normal can be chosen to be $\hat{\mathbf{n}} = [0, 0, 1]^T$. The $\mathbf{J}$ tensors reduce to:

$$\mathbf{J}^m = \hat{\mathbf{n}}^m \int_A dA = \hat{\mathbf{n}}^m A \tag{53}$$

and the force equation becomes:

$$\mathbf{F} = P(r)A\Big[a_2\hat{\mathbf{n}}^2 \cdot \hat{\mathbf{r}}_0 - 2\rho s(\hat{\mathbf{n}}^3 \cdot \hat{\mathbf{r}}_0) \cdot \hat{\mathbf{r}}_0 + a_3(\hat{\mathbf{n}} \cdot \hat{\mathbf{r}}_0) \cdot \hat{\mathbf{r}}_0\Big] \tag{54}$$

A more familiar equation is obtained if Eq. (19) is used in the above expression to obtain:

$$\mathbf{F} = -P(r)A\Big[a_2 \cos\alpha\hat{\mathbf{n}} + 2\rho s \cos^2\alpha\hat{\mathbf{n}} + a_3 \cos\alpha\hat{\mathbf{r}}_0\Big] \tag{55}$$

For a uniform shape, the products $\tilde{\varrho} \cdot \hat{\mathbf{n}}^m$ and $\hat{\mathbf{n}}\vec{\varrho}$ tensor are odd functions, implying that the $\mathbf{K}^m$ and $\mathbf{L}$ tensors will be zero about the geometric center of the sail. Hence, the total moment is zero at the sail geometric center.

6.2 Circular Sail with Billow

Next consider a circular sail. Let us assume that the radiation pressure causes a small deformation on the surface that is invariant with attitude and given by:

$$z_b = -\frac{\alpha_{max}}{2R_0}(x_b^2 + y_b^2) + \frac{\alpha_{max} R_0}{2} \tag{56}$$

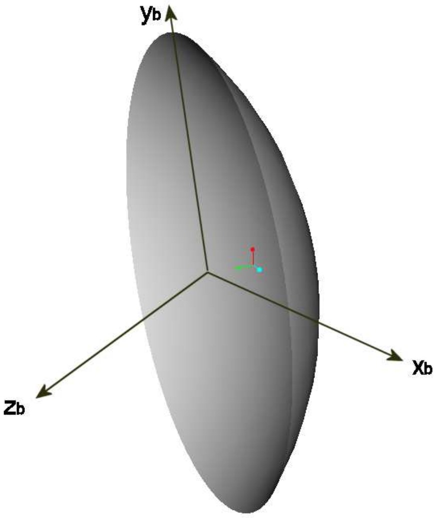

Figure 2: Circular Sail Geometry.

where R_0 is the sail radius, α_{max} the surface slope at the rim (and must be negative for this case), and x_b,y_b,z_b are the sail coordinates in the body-fixed frame as shown in Fig. 2. If polar coordinates are used, it can be noted that the slope varies linearly with distance from center. The surface function $\phi(x_b, y_b, z_b) = 0$ is obtained by setting Eq. (56) to zero and the surface normal is obtained by taking the gradient of $\phi(x_b, y_b, z_b)$:

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{1 + (\frac{\alpha_{max}}{R_0})^2 x_b^2 + (\frac{\alpha_{max}}{R_0})^2 y_b^2}} \begin{bmatrix} \frac{\alpha_{max}}{R_0} x_b \\ \frac{\alpha_{max}}{R_0} y_b \\ 1 \end{bmatrix} \tag{57}$$

Introducing polar coordinates by letting $x_b = r_b \cos\delta$ and $y_b = r_b \sin\delta$, the surface normal can also be stated as:

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{1 + (\frac{\alpha_{max}}{R_0} r_b)^2}} \begin{bmatrix} \frac{\alpha_{max}}{R_0} r_b \cos\delta \\ \frac{\alpha_{max}}{R_0} r_b \sin\delta \\ 1 \end{bmatrix} \tag{58}$$

and the differential area is given by:

$$dA = \sqrt{1 + (\frac{\alpha_{max}}{R_0} r_b)^2} r_b dr_b d\delta \tag{59}$$

With these terms defined, the coefficient integrals can be computed analytically. The results are:

$$\mathbf{J}^1 \;=\; \begin{bmatrix} 0 \\ 0 \\ \pi R_0^2 \end{bmatrix} \tag{60}$$

$$\mathbf{J}^2 = \begin{bmatrix} \frac{\pi R_0^2 \left(2+(-2+\alpha_{max})\sqrt{1+\alpha_{max}^2}\right)}{3\alpha_{max}^2} & 0 & 0 \\ 0 & \frac{\pi R_0^2 \left(2+(-2+\alpha_{max})\sqrt{1+\alpha_{max}^2}\right)}{3\alpha_{max}^2} & 0 \\ 0 & 0 & \frac{2\pi R_0^2 \left(-1+\sqrt{1+\alpha_{max}^2}\right)}{\alpha_{max}^2} \end{bmatrix} \tag{61}$$

and the non-zero elements of the $\mathbf{J}^3$ tensor are:

$$\mathbf{J}^3_{113} = \frac{\pi R_0^2}{2\alpha_{max}^2}(\alpha_{max}^2 - \log(1 + \alpha_{max}^2)) \tag{62}$$

$$\mathbf{J}^3_{333} = \frac{\pi R_0^2}{\alpha_{max}^2} \log(1 + \alpha_{max}^2) \tag{63}$$

and

$$\mathbf{J}^3_{131} = \mathbf{J}^3_{311} = \mathbf{J}^3_{223} = \mathbf{J}^3_{232} = \mathbf{J}^3_{322} = \mathbf{J}^3_{113} \tag{64}$$

The moment acting on the sail can also be computed analytically. First, let the position of a differential area on the sail be defined as $\vec{\varrho} = [r_b \cos\delta, r_b \sin\delta, z]$. The $\mathbf{K}^2$ tensor is found from Eq. (43) and given by:

$$\mathbf{K}^2 = \frac{\pi R_0^3 \left(6 - 5\alpha_{max}^2 - \sqrt{1 + \alpha_{max}^2}(6 - 8\alpha_{max}^2 + \alpha_{max}^4)\right)}{15\alpha_{max}^3} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{65}$$

The $\mathbf{K}^3$ tensor non-zero elements are:

$$\mathbf{K}^3_{213} = \mathbf{K}^3_{312} = -\frac{\pi R_0^3}{8\alpha_{max}} \left(\alpha_{max}^2(-2 + \alpha_{max}^2) - 2(-1 + \alpha_{max}^2)\log(1 + \alpha_{max}^2)\right) \tag{66}$$

and

$$\mathbf{K}^3_{123} = \mathbf{K}^3_{321} = -\mathbf{K}^3_{213} \tag{67}$$

where the subscript denotes the entry position. The final moment surface normal distribution integral is given by:

$$\mathbf{L} = \frac{1}{4}\pi R_0^3 \alpha_{max} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{68}$$

6.3 Four-Panel Sail with Billow

A square solar sail with beams along its main diagonals can be modeled by combining four panels as shown in Fig. 3; each section being of triangular form. The billow due to radiation pressure acting on the sail membrane will be modeled by approximating each quadrant as a section of an oblique circular cone [4]. Due to the complexity of this shape, the coefficients will be found using numerical integration. The equation for an oblique cone with its base centered at the origin and vertex positioned at $(h_c, 0, z_{co})$, as shown in Fig. 4, is given by:

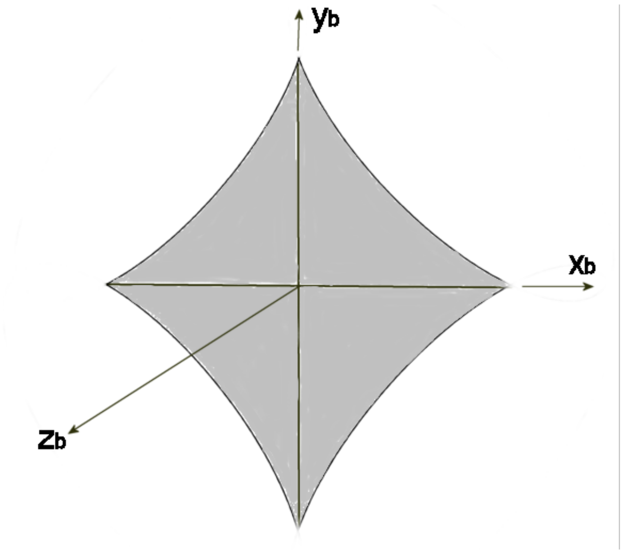

Figure 3: Square Sail Modeling.

$$(x_c - h_c)^2 = -\frac{h_c^2\Big((r_c\cos\theta - z_{co})^2 + (-r_c\sin\theta)^2\Big)}{(R_c\cos\theta - z_{co})^2 + (-R_c\sin\theta)^2} \tag{69}$$

where the polar coordinates $z_c = r_c\cos\theta$, $y_c = -r_c\sin\theta$ have been substituted, R_c is the radius at the base of the cone, and h_c is the height of the cone. The cone-fixed coordinates are related to the body-fixed coordinates by $x_c = 2(x_b - y_b)/\sqrt{2}$ and

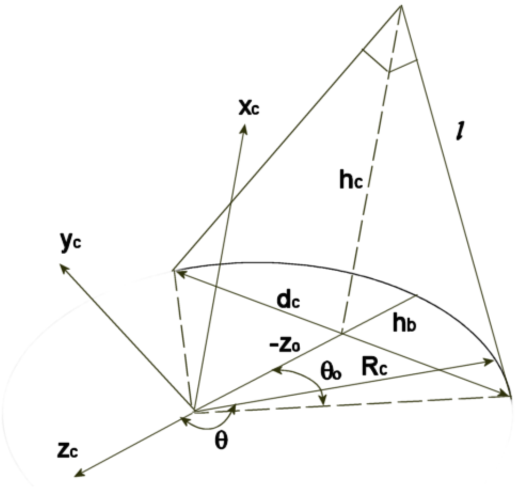

Figure 4: Sail Area Modeling.

$y_c = 2(y_b + x_b)/\sqrt{2}$. The sail surface is modeled as a section of the cone as shown in Fig. 4.

It is assumed that the sail beams are perpendicular with half-length l and the sail billow is described by h_b as shown in Fig. 4. With this information the oblique cone can be fully described. Notice that the cone height and the distance between the tip of the beams are related to l by:

$$h_c = \frac{l}{\sqrt{2}} \tag{70}$$

$$d_c = \sqrt{2}l \tag{71}$$

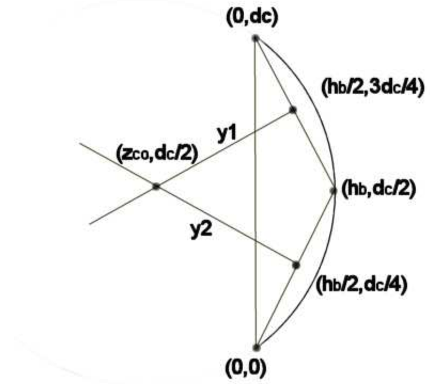

Figure 5: Cone Radius.

Using a top view, the base centered at (0,0,0) can be described as shown in Fig. 5, a view looking down the panel along the x_c axis. If three points are known to lie on the circumference of a circle, the circle radius and center can be found by finding the intersection of lines perpendicular to and passing through the center of the lines joining the three points as shown in Fig. 5. In order to find the cone radius at the base for the sail, the three points chosen on the cone base are the points defined by the tip of the beams and the third point is equidistant from the first two located on the sail rim. Let y_1 and y_2 be two lines that meet the restrictions mentioned, then they are described by:

$$y_1 = \frac{\sqrt{2}h_b}{l} z_c + \frac{3l}{2\sqrt{2}} - \frac{h_b^2}{l\sqrt{2}} \tag{72}$$

$$y_2 = -\frac{\sqrt{2}h_b}{l} z_c + \left(\frac{\sqrt{2}l}{2} + \frac{\sqrt{2}h_b^2}{l}\right) \tag{73}$$

Since these two equations intersect at the center, the value of z_{co} can be obtained by equating the two lines, thus the radius is found from:

$$R_c = h_b - z_{co} = \frac{h_b}{2} + \frac{l^2}{4h_b} \tag{74}$$

The angle θ_0 can be found from:

$$\cos\theta_0 = \frac{|d|}{R_c} \tag{75}$$

The normal vector is found in the usual way, by taking the gradient of $\phi(x_c, y_c, z_c) = 0$ in cartesian coordinates and dividing by its magnitude. The gradient for the oblique cone equation in polar coordinates is given by:

$$\nabla\phi = \begin{bmatrix} -h_c\sqrt{\frac{(-z_{c0}+r\cos\theta)^2+(r\sin\theta)^2}{(-z_{c0}+R_c\cos\theta)^2+(R_0\sin\theta)^2}} \\ \frac{h_c^2 r_c \sin\theta}{(R_c\cos\theta - z_{c0})^2+(R_c\sin\theta)^2} \\ -\frac{h_c^2(r_c\cos\theta - z_{c0})}{(R_c\cos\theta - z_{c0})^2+(R_c\sin\theta)^2} \end{bmatrix} \tag{76}$$

The next step is to find an equation for computing the surface area of an oblique cone. Let a differential area be given by a small triangle with one of its vertex at the cone apex, and the other two at the cone base separated by a distance $r_c d\theta$. The height of the triangle l_c, the distance form the cone apex to the point on the surface, is given by:

$$l_c = \sqrt{(R_c\cos\theta - z_{c0})^2 + (-R_c\sin\theta)^2 + h_c^2} \tag{77}$$

the base of the triangle is given by $r_c d\theta \sin d$, where d is the angle between $\vec{l_c}$ and the tangent line at the circular base. Then, the differential area is given by:

$$dA = \frac{1}{2}\sqrt{(r - z_{co}\cos\theta)^2 + h_c^2}\, r_c d\theta \tag{78}$$

When computing the $\mathbf{J}^m$, $\mathbf{K}^m$, and $\mathbf{L}$ tensors, the limits of integration go from $\pi - \theta_0$ to $\pi + \theta_0$ if z_{c0} is negative. There is no known analytical solution for any of these integrals, so numerical integration is necessary for computing the tensors. The computation of the $\mathbf{K}^m$ and $\mathbf{L}$ tensors require the knowledge of the area element position with respect to a specific point. For a triangle, the position can be defined at its center of mass given by:

$$\vec{\varrho} = \frac{1}{3}(\vec{\varrho_1} + \vec{\varrho_2} + \vec{\varrho_3}) \tag{79}$$

where the vectors $(\vec{\varrho_1}\ \vec{\varrho_2}\ \vec{\varrho_3})$ define the position of the triangle vertices relative to a chosen reference point. If, for instance, the reference point is chosen to be the oblique cone apex, which represents the sail center, then the element triangle will have its vertices given by:

$$\vec{\varrho_1} = \vec{v_3} - \vec{v_1} \tag{80}$$

$$\vec{\varrho_2} = \vec{v_3} - \vec{v_2} \tag{81}$$

$$\vec{\varrho_3} = 0 \tag{82}$$

where

$$\vec{v_1} = \begin{bmatrix} 0 \\ -R_c \sin\theta \\ R_c\cos\theta - z_{c0} \end{bmatrix} \tag{83}$$

$$\vec{v_2} = \begin{bmatrix} 0 \\ -R_c \sin(\theta + d\theta) \\ R_c\cos(\theta + d\theta) - z_{c0} \end{bmatrix} \tag{84}$$

$$\vec{v_3} = \begin{bmatrix} z_{c0} \\ 0 \\ h_c \end{bmatrix} \tag{85}$$

Let us apply these results to a 100 m by 100 m sail. One triangular quadrant approximated by an oblique cone will have the following values: $h_c = 50m$, $l = 70.71m$. Furthermore, if h_b is assumed to be 10% of l, then $h_b = 7.071m$, $Rc = 180.3m$, $z_{c0} = -173.2m$, and $\theta_0 = 16.1^o$.

The results were obtained by integration of the equations numerically with a step size of $d\theta = 0.0001$ radians. The $\tilde{\mathbf{J}}^m$ tensors are:

$$\tilde{\mathbf{J}}^1 \;=\; \begin{bmatrix} -1.2774e+002 \\ 3.8892e-002 \\ 2.5645e+003 \end{bmatrix} \tag{86}$$

$$\tilde{\mathbf{J}}^2 = \begin{bmatrix} 1.0772e+001 & -5.4980e-003 & -1.2643e+002 \\ -5.4980e-003 & 1.7545e+001 & 3.8107e-002 \\ -1.2643e+002 & 3.8107e-002 & 2.5504e+003 \end{bmatrix} \tag{87}$$

$$\tilde{\mathbf{J}}^3_{ij1} = \begin{bmatrix} -1.0884e+000 & 7.7722e-004 & 1.0634e+001 \\ 7.7722e-004 & -1.5130e+000 & -5.3870e-003 \\ 1.0634e+001 & -5.3870e-003 & -1.2514e+002 \end{bmatrix} \tag{88}$$

$$\tilde{\mathbf{J}}^3_{ij2} = \begin{bmatrix} 7.7722e-004 & -1.5130e+000 & -5.3870e-003 \\ -1.5130e+000 & 7.7738e-004 & 1.7360e+001 \\ -5.3870e-003 & 1.7360e+001 & 3.7338e-002 \end{bmatrix} \tag{89}$$

$$\tilde{\mathbf{J}}^3_{ij3} = \begin{bmatrix} 1.0634e+001 & -5.3870e-003 & -1.2514e+002 \\ -5.3870e-003 & 1.7360e+001 & 3.7338e-002 \\ -1.2514e+002 & 3.7338e-002 & 2.5365e+003 \end{bmatrix} \tag{90}$$

Following the same procedure, the $\tilde{\mathbf{K}}^m$ and $\tilde{\mathbf{L}}$ tensors are:

$$\tilde{\mathbf{K}}^2 = \begin{bmatrix} -7.6024e-001 & 2.2922e-004 & 1.5333e+001 \\ -9.1614e-006 & 1.7806e-002 & 6.3499e-005 \\ -6.2945e-002 & 3.1749e-005 & 7.4243e-001 \end{bmatrix} \tag{91}$$

$$\tilde{\mathbf{K}}^3_{ij1} = \begin{bmatrix} 6.3950e-002 & -3.2404e-005 & -7.5247e-001 \\ 1.2951e-006 & -1.8072e-003 & -8.9765e-006 \\ 6.3432e-003 & -4.4882e-006 & -6.2143e-002 \end{bmatrix} \tag{92}$$

$$\tilde{\mathbf{K}}^3_{ij2} = \begin{bmatrix} -3.2404e-005 & 1.0439e-001 & 2.2459e-004 \\ -1.8072e-003 & 1.2954e-006 & 1.7577e-002 \\ -4.4882e-006 & 8.8400e-003 & 3.1108e-005 \end{bmatrix} \tag{93}$$

$$\tilde{\mathbf{K}}^3_{ij3} = \begin{bmatrix} -7.5247e-001 & 2.2459e-004 & 1.5249e+001 \\ -8.9765e-006 & 1.7577e-002 & 6.2217e-005 \\ -6.2143e-002 & 3.1108e-005 & 7.3489e-001 \end{bmatrix} \tag{94}$$

$$\tilde{\mathbf{L}} = \begin{bmatrix} 0 & -3.8167e-004 & -1.1011e-004 \\ 0 & -3.8181e-004 & -1.1015e-004 \\ 0 & 2.6469e-003 & 7.6361e-004 \end{bmatrix} \tag{95}$$

The results for the single quadrant must now be rotated to account for the complete sail geometry. The objective is achieved by performing the rotations about the z_b body-fixed axis, which is parallel to the z_c cone-fixed axis, and performing the transformation outlined in Eq. (38). The transformation T is given by:

$$T = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{96}$$

Since the x_c and y_c cone axes are not aligned to the x_b and y_b sail body-fixed axes, the initial rotation is through 45^0. The subsequent rotation angles are 135^0, 225^0 and 315^0. After performing these transformations to the **J**, **K**, and **L** integrals and adding the results, the complete sail integrals obtained are:

$$\mathbf{J}^1 \quad = \quad \begin{bmatrix} 2.8422e-014 \\ -2.8422e-014 \\ 1.0258e+004 \end{bmatrix} \tag{97}$$

$$\mathbf{J}^2 = \begin{bmatrix} 5.6634e+001 & 8.8818e-016 & 2.8422e-014 \\ 8.8818e-016 & 5.6634e+001 & -2.8422e-014 \\ 2.8422e-014 & -2.8422e-014 & 1.0202e+004 \end{bmatrix} \tag{98}$$

$$\mathbf{J}^3_{ij1} = \begin{bmatrix} 4.4409e-016 & -3.3307e-016 & 5.5988e+001 \\ -2.2204e-016 & 5.5511e-016 & 8.8818e-016 \\ 5.5988e+001 & 8.8818e-016 & 2.8422e-014 \end{bmatrix} \tag{99}$$

$$\mathbf{J}^3_{ij2} = \begin{bmatrix} -3.3307e-016 & 4.4409e-016 & 8.8818e-016 \\ 5.5511e-016 & 0 & 5.5988e+001 \\ 8.8818e-016 & 5.5988e+001 & -2.8422e-014 \end{bmatrix} \tag{100}$$

$$\mathbf{J}^3_{ij3} = \begin{bmatrix} 5.5988e+001 & 8.8818e-016 & 2.8422e-014 \\ 8.8818e-016 & 5.5988e+001 & -2.8422e-014 \\ 2.8422e-014 & -2.8422e-014 & 1.0146e+004 \end{bmatrix} \tag{101}$$

$$\mathbf{K}^2 = \begin{bmatrix} -1.4849e+000 & 4.7676e-004 & -3.5527e-015 \\ -4.7676e-004 & -1.4849e+000 & 3.5527e-015 \\ 1.3878e-017 & -1.3878e-017 & 2.9697e+000 \end{bmatrix} \tag{102}$$

$$\mathbf{K}^3_{ij1} = \begin{bmatrix} -6.9389e-018 & 0 & -1.4698e+000 \\ 2.0817e-017 & -5.2042e-018 & -4.6713e-004 \\ 3.0366e-002 & 0 & 1.3878e-017 \end{bmatrix} \tag{103}$$

$$\mathbf{K}^3_{ij2} = \begin{bmatrix} 0 & -3.4694e-017 & 4.6713e-004 \\ -5.2042e-018 & 1.3878e-017 & -1.4698e+000 \\ 0 & 3.0366e-002 & -1.3878e-017 \end{bmatrix} \tag{104}$$

$$\mathbf{K}^3_{ij3} = \begin{bmatrix} -1.4698e+000 & 4.6713e-004 & -3.5527e-015 \\ -4.6713e-004 & -1.4698e+000 & 3.5527e-015 \\ 1.3878e-017 & -1.3878e-017 & 2.9396e+000 \end{bmatrix} \tag{105}$$

$$\mathbf{L} = \begin{bmatrix} -7.6362e-004 & -7.6334e-004 & 5.4210e-020 \\ 7.6334e-004 & -7.6362e-004 & 2.7105e-020 \\ -2.1684e-019 & -4.3368e-019 & 3.0544e-003 \end{bmatrix} \tag{106}$$

6.4 Generic Sail

For a generic sail, or numerically defined sail, such as what might be defined by a Finite Element program, a simple approach can be outlined. Assume the sail is defined by a set of triangular facets denoted by an index a, each of which with an area A_a, position (to its center of mass) $\vec{\varrho}_a$, and unit vector normal $\hat{\mathbf{n}}_a$. Then the coefficients for a sail are simply defined as summations over these quantities:

$$\mathbf{J}^m = \sum_a \hat{\mathbf{n}}_a^m A_a \tag{107}$$

$$\mathbf{L} = \sum_a \hat{\mathbf{n}}_a \vec{\varrho}_a A_a \tag{108}$$

$$\mathbf{K}^m = \sum_a \tilde{\varrho} \cdot \hat{\mathbf{n}}_a^m A_a \tag{109}$$

with these formulae it is simple to compute a sail's coefficients at any step of the modeling process.

6.5 Test Results

Now that the coefficients for different sail models have been computed, let us compare the forces generated by each sail. This will be done by pitching the sail about the y_b body-fixed axis. The sail optical parameters were chosen as $\rho = 0.9$, $s = 1$, $B_f = 0.8$, $B_b = 0.5$, $\epsilon_f = 0.05$, $\epsilon_b = 0.3$. The flat sail area was chosen to be $10000m^2$.

The circular sail is specified by $R_0 = 56.42m$ and $\alpha_{max} = 0.1$. The resultant forces, along the z_b and x_b body-fixed axes are normalized by the solar radiation pressure and their respective surface areas and shown in Figs. 6 and 7.

Fig 6. shows that all the sails produce a similar force along their z_b body fixed axis. The flat sail has the best performance followed by the circular sail. Fig. 7 shows that the force along the x_b body-fixed axis is very similar for the flat and four-panel sail, while the force by the circular force is higher due to its concavity.

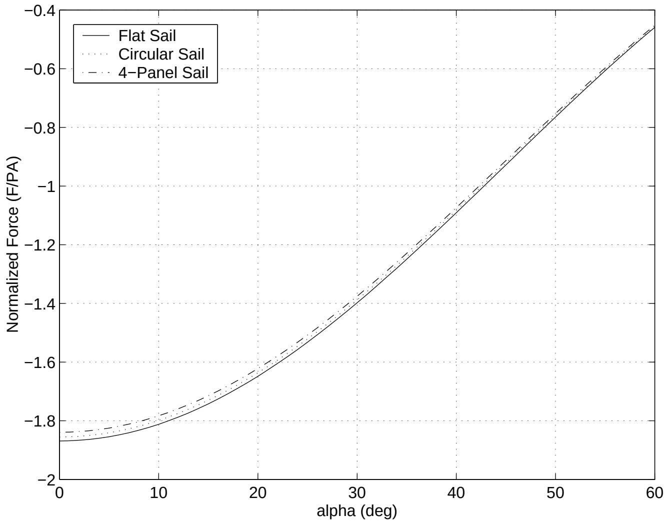

Figure 6: Normalized Force along z_b-axis.

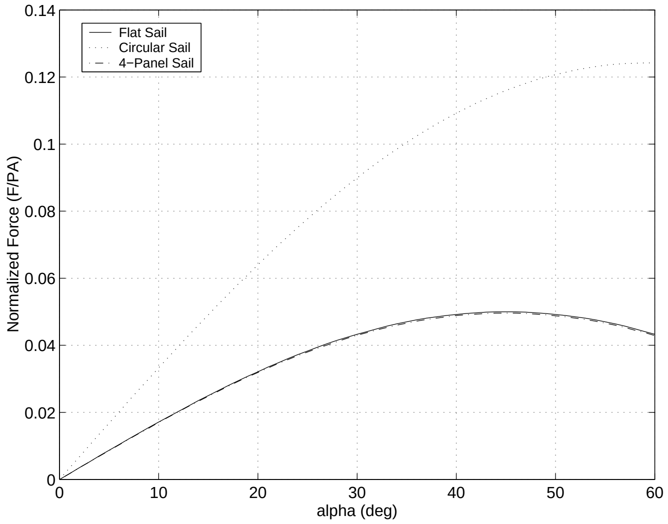

Figure 7: Normalized Force along x_b-axis.

7 Conclusions

The forces and moments acting on a solar sail of arbitrary shape and optical properties can be determined analytically by computing a series of coefficients. If the optical properties are constant, then the surface integrals $\mathbf{J}^1$, $\mathbf{J}^2$, $\mathbf{J}^3$, $\mathbf{L}$, $\mathbf{K}^2$, and $\mathbf{K}^3$ define the sail specific coefficients for forces and moments. With this formalism, completely different sail models can be handled by using a single generic approach. It should be noted that these distributions are nominally constant in the sail-fixed frame, and thus cannot directly represent the effect of changing sail geometry. However, it is not inconceivable that we could generate look-up tables for these tensors as a function of changing solar illumination conditions. The pathway to this would be to parameterize the surface geometry in terms of boom bend angles, or some similar approach.

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