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Solving optimal control problems with generating functions

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Abstract

The optimal control of a spacecraft as it transitions between specified states in a fixed amount of time is studied. We approach the solution to our optimal control problem with a novel technique, treating the resulting system for the state and adjoints as a Hamiltonian system. We show that the optimal control for this system can be found once the F_1 generating function for the Hamiltonian system is found. A solution procedure for this generating function is posed and applied to the non-linear control of a spacecraft in the vicinity of a circular orbit.

1 Introduction

This paper presents a novel approach to evaluating optimal low-thrust trajectories and feedback control laws for a spacecraft subject to a general gravity field. This approach is derived by relying on the Hamiltonian nature of the necessary conditions associated with optimal control, and utilizing certain properties of generating functions and canonical transformations. In particular, we show that certain solutions to the Hamilton-Jacobi(HJ) equation, associated with canonical transformations of Hamiltonian systems, can directly yield optimal control laws for a general system. Typically, application of Pontryagin's principle changes the non-linear optimal rendezvous problem to a two point boundary value problem (TPBVP), for which one generally requires an initial estimate for the initial (or final) adjoint variables followed by an iterative solution procedure. Our approach provides an algorithm to compute the initial (or final) values of the adjoints without requiring an initial estimate, and for arbitrary boundary conditions, simply by algebraic manipulations of the generating function. Our approach not only satisfies the TPBVP found from the necessary conditions, by definition, but it also satisfies the Hamilton-Jacobi-Bellman equation, which is a sufficient condition for optimality. Most importantly, we have derived and applied a general solution procedure for this problem to a non-linear dynamical system of interest to astrodynamics. To develop and apply this algorithm requires certain conditions on the dynamics and cost function, which we detail in this paper.

Since Lawden [1] initially introduced primer vector theory, the problem of low-thrust optimal rendezvous has been a topic of continual interest. Much work has been done on this topic, so in the following we only give a brief review of work that has direct relation to analytical work on the optimal control problem for space trajectories. Billik [2] applied differential game theory to rendezvous problems subject to linearized dynamics. London [3] and Antony and Sasaki [4] studied the uncontrolled motion subject to second order

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approximation. Jezewski and Stoolz [5] considered minimum-time problems subject to the inverse square field and evaluated an analytic solution under highly restricted assumptions. Later, Marec [6] extended Lawden's primer vector theory graphically with the Contensou principle. Various types of low thrust optimal rendezvous problems subject to a linearized gravity field have been extensively explored by Carter [7, 8, 9, 10], Carter and Brient [11], Carter and Humi [12, 13, 14], Humi [15], Carter and Pardis [16, 17], etc. Also, Lembeck and Prussing [18] solved a combined problem of high-thrust intercept and low-thrust rendezvous subject to linearized dynamics.

As is seen, except for the very basic works of London, Antony et al, and Jezewski et al, all of the above researches consider linearized dynamics, which clearly limits the applicability and utility of this problem. Thus it is desirable to find the optimal trajectory subject to the original nonlinear dynamics. However, to do so in general requires one to solve the TPBVP for the adjoints for each boundary condition of interest, a challenging problem. Additionally, it is even more difficult to find a non-linear optimal feedback control, generally found by solving the Hamilton-Jacobi-Bellman (HJB) equation.

As an alternative, Park and Scheeres[19] suggested an indirect approach for evaluating the initial adjoints (without guess) and for obtaining optimal feedback control via generating functions. This work is a direct product of applying the previous work by Guibout and Scheeres on the solution of boundary value problems in Hamiltonian systems [20]. To solve the resultant Hamilton-Jacobi (HJ) equations, they use a practical and straightforward series approximation approach also developed in the same reference.

This document defines this method and demonstrates a direct application of this algorithm to low-thrust optimal rendezvous problems subject to inverse-square central gravity fields. The discussion is structured as follows. First we give a brief review of classical optimal control theory as applied to a specific class of problems. Next we motivate our current approach and show how it satisfies the necessary conditions by default, and how it can be used to derive an optimal feedback control law. Then we apply the theory to the problem of optimally transfering from one state to another, using non-linear dynamics relative to a circular orbit. Finally, we discuss the uniqueness of our solutions and contrast the current results with our previous work.

2 A General Solution of the Optimal Control Problem

2.1 Classical Necessary Conditions for Optimal Control

Assume we have a dynamical system stipulated as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$. The state $\mathbf{x}$ is n-dimensional in general and could consist of position and velocity vectors. The goal is to transfer from an initial state to a final state in a specified time span while minimizing some cost function. The application envisioned is for a spacecraft in a specified state (consisting of a specific orbit, hence position and velocity) to transition to another state while minimizing a measure of fuel consumption. For this initial study we restrict ourselves to the simplest system:

$$J = \int_{t_o}^{t_f} L(\mathbf{u}, \mathbf{x}, t) dt \qquad (1)$$

where $\mathbf{u}$ is the 3-dimensional thrust vector. For a more comprehensive introduction to the theory of optimal control we cite [6]. The Hamiltonian of the system can be stated as:

$$H = \mathbf{p} \cdot \mathbf{f}(\mathbf{x}, \mathbf{u}, t) + L(\mathbf{u}, \mathbf{x}, t) \qquad (2)$$

where $\mathbf{p}$ are the adjoint variables. Applying the Pontryagin principle we find the optimal control:

$$\mathbf{u}^* = \arg\min_{\mathbf{u}} H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t) \qquad (3)$$

Substituting this control back into H leads to the new hamiltonian $H^*(\mathbf{x}, \mathbf{p}, t)$ and to the dynamical system:

$$\dot{\mathbf{x}} = \frac{\partial H^*}{\partial \mathbf{p}} \tag{4}$$

$$\dot{\mathbf{p}} = -\frac{\partial H^*}{\partial \mathbf{x}} \tag{5}$$

with specified boundary conditions:

$$\mathbf{x}(t_o) = \mathbf{x}_o \tag{6}$$

$$\mathbf{x}(t_f) = \mathbf{x}_f \tag{7}$$

With our stated problem of fixed initial and final states and time there are no transversality conditions[1].

The fundamental difficulty in this approach is, as is well known, finding the initial or final conditions for the adjoints, $\mathbf{p}_o$ or $\mathbf{p}_f$, which will satisfy this boundary value problem. If given these initial or final adjoints, along with the specified initial or final conditions on the state, we can directly solve the optimal control problem by integrating the associated differential equations, solving for the optimal control from the Pontryagin principle at each point along the trajectory.

2.2 Motivation of the Proposed Method

The drawback of the approach described above is that solution procedures for the TPBVP generally require an initial estimate for the adjoints, which usually have no physical interpretations. Furthermore, we must repetitively solve the TPBVP for each boundary condition of interest, which is time-consuming, lacks definiteness, and is subject to numerical divergence. The conventional alternative method is to solve the Hamilton-Jacobi-Bellman (HJB) equation, and thus to evaluate the optimal cost and the corresponding optimal control law. However, the HJB is a first order partial differential equation and is extremely difficult to solve in general. Furthermore, for the type of boundary conditions we are considering, the HJB cost function has a singularity at the terminal condition, which makes the problem even more difficult (this is discussed in more detail in [19]).

In an attempt to overcome these disadvantages, we suggest an *indirect* method, which specifically utilizes the theory of canonical transformations and their associated generating functions. This method provides a way to compute the initial (or final) adjoints as a function of the initial and final states, and thus to evaluate the optimal trajectory by simple forward (or backward) integration. It also enables us to systematically construct the optimal feedback control by avoiding (or detouring around) the fundamental singularity found in the HJB equation at its terminal condition. The next section is dedicated to the discussion of our approach.

2.3 Solution of the Boundary Value Problem using Generating Functions

Recall the theory of canonical transformations and generating functions in Hamiltonian dynamics (c.f. [21]). In addition to generating canonical transformations between Hamiltonian systems, generating functions also solve boundary value problems between Hamiltonian coordinate and momentum states for a single flow field. See the Appendix for a more detailed derivation of the results we present in the following. In particular, the generating function $F_1(\mathbf{x}_o, \mathbf{x}_f, t_o, t_f)$ can be used to find the initial and final momentum vectors from the

[1] There is nothing that does not allow them to be added to our approach, however we reserve the detailed derivation of this application for the future

relationship:

$$\mathbf{p}_o = \frac{\partial F_1}{\partial \mathbf{x}_o} \tag{8}$$

$$\mathbf{p}_f = -\frac{\partial F_1}{\partial \mathbf{x}_f} \tag{9}$$

The generating function F_1 also satisfies a partial differential equation, the Hamilton-Jacobi equation:

$$\frac{\partial F_1}{\partial t} + \mathcal{H}^*(\mathbf{x}, \frac{\partial F_1}{\partial \mathbf{x}}, t) = 0 \tag{10}$$

Analogous relations and definitions exist for the generating functions $F_2(\mathbf{x}_o, \mathbf{p}_f, t_o, t_f)$, $F_3(\mathbf{p}_o, \mathbf{x}_f, t_o, t_f)$, and $F_4(\mathbf{p}_o, \mathbf{p}_f, t_o, t_f)$ with results:

$$\mathbf{p}_o = \frac{\partial F_2}{\partial \mathbf{x}_o} \quad ; \quad \mathbf{x}_f = \frac{\partial F_2}{\partial \mathbf{p}_f} \tag{11}$$

$$\mathbf{x}_o = -\frac{\partial F_3}{\partial \mathbf{p}_o} \quad ; \quad \mathbf{p}_f = -\frac{\partial F_3}{\partial \mathbf{x}_f} \tag{12}$$

$$\mathbf{x}_o = -\frac{\partial F_4}{\partial \mathbf{p}_o} \quad ; \quad \mathbf{x}_f = \frac{\partial F_4}{\partial \mathbf{p}_f} \tag{13}$$

These functions all solve their own version of the Hamilton-Jacobi equation:

$$\frac{\partial F_2}{\partial t} + \mathcal{H}^*(\mathbf{x}, \frac{\partial F_2}{\partial \mathbf{x}}, t) = 0 \tag{14}$$

$$\frac{\partial F_3}{\partial t} + \mathcal{H}^*(-\frac{\partial F_3}{\partial \mathbf{p}}, \mathbf{p}, t) = 0 \tag{15}$$

$$\frac{\partial F_4}{\partial t} + \mathcal{H}^*(-\frac{\partial F_4}{\partial \mathbf{p}}, \mathbf{p}, t) = 0 \tag{16}$$

A final key property of the generating functions is that they can be transformed into each other via the Legendre transformation. Specifically, we find the following relations between the generating functions:

$$F_2(\mathbf{x}_o, \mathbf{p}_f, t_o, t_f) = F_1(\mathbf{x}_o, \mathbf{x}_f, t_o, t_f) + \mathbf{p}_f \cdot \mathbf{x}_f \tag{17}$$

$$F_3(\mathbf{p}_o, \mathbf{x}_f, t_o, t_f) = F_1(\mathbf{x}_o, \mathbf{x}_f, t_o, t_f) - \mathbf{p}_o \cdot \mathbf{x}_o \tag{18}$$

$$F_4(\mathbf{p}_o, \mathbf{p}_f, t_o, t_f) = F_2(\mathbf{x}_o, \mathbf{p}_f, t_o, t_f) - \mathbf{p}_o \cdot \mathbf{x}_o \tag{19}$$

The key observation we make is that solving for F_1 solves the boundary value problem and hence the optimal control problem. Indeed, if we have an analytical form for F_1 we can directly take its partial derivatives, fix $\mathbf{x}_o$ and $\mathbf{x}_f$, and find the appropriate momentum to generate our control. It can also be shown that F_1 satisfies the Hamilton-Jacobi-Bellmann equation, and thus is also a sufficient condition for optimality [19]. Furthermore, using Eq. 3 and the desired F_1 function we can define a feedback control law:

$$\mathbf{u}^* = \arg\min \mathcal{H}(\mathbf{x}, (\partial F_1(\mathbf{x}, \mathbf{x}_f)/\partial \mathbf{x}), \mathbf{u}, t) \tag{20}$$

where we fix the final boundary condition $\mathbf{x}_f$ and allow the initial condition to equal the current state.

2.4 Implementing a solution for F_1

The difficulty, of course, is in finding the generating function F_1. This problem is directly addressed in [20], where they show that the generating functions can be solved as power series expansions in their respective arguments, the coefficients of these power series satisfying a set of ordinary differential equations derived from the Hamilton-Jacobi equation.

To carry out this method, however, requires that some restrictions be placed on the system dynamics and cost function. The approach developed in [20] is based on constructing local solutions to the generating functions, i.e., expanding them as a Taylor series about a nominal trajectory that is known. This implies that a solution to the optimal control problem has already been found, and our specific method operates in the vicinity of this solution. It is important to note that this includes the case of no control, i.e., if the dynamics of the system carry a state between two points, $\mathbf{x}_o$ to $\mathbf{x}_f$, then the optimal control for this transition is simply stated as $\mathbf{u} \equiv 0$, and our method can be used on such a system.

Thus, to formally apply that method to the current system requires that the system dynamics satisfy $\mathbf{f}(\mathbf{0},\mathbf{u},t) = \mathbf{0}$. Furthermore, as we similarly expand the Hamiltonian function as a Taylor series expansion about a nominal solution we also require analyticity of this function. This, in turn, places a requirement on the analyticity of the cost function (since this becomes part of the Hamiltonian function through the Pontryagin principle). This particular constraint prevents the current approach from addressing the minimum fuel problem. There are indications, however, that this particular problem can be circumvented, but that is reserved for future research.

The process derived in [20] consists of expanding the Hamiltonian function as a Taylor series in the states and adjoints, and the F_1 generating function as Taylor series in the initial and final states. Then the series for F_1 is substituted into the Hamilton-Jacobi equation (Eq. 10). A balancing technique is then used to equate all like powers of the states to zero, which in turn defines a set of differential equations for the coefficients of the F_1 Taylor series expansion. A major problem in this approach, however, is that initial conditions for F_1 at time $t_o = t_f$ and $\mathbf{x}_o \neq \mathbf{x}_f$ are undefined, making it impossible to initiate the integration of the coefficients. Furthermore, these coefficients are not known *a priori* at any other time. This problem can be circumvented, however, by solving for a different generating function and then transforming back to the F_1 function, using the Legendre transformation, at some later time. Given a power series expansion for a given F_i, it is always possible to transform to a different generating function using the transformations in Eqs. 17-19 along with the fundamental results given in Eqs. 8, 9, 11-13.

The generating function F_2, it turns out, can be solved using our initial value approach. This is due, essentially, to the fact that it generates the identity transformation when $t_o = t_f$. Thus, we solve for the F_2 generating function as a function of time by integrating the differential equations for the coefficients and, when needed, transform to the F_1 function via the Legendre transformation to solve the boundary value problem, which in turn solves the optimal control problem.

This approach was succesfully applied to solve non-linear boundary value problems in a Hamiltonian dynamical system in [20]. In this paper we investigate the application of this approach to the optimal control of a spacecraft in the vicinity of a nominal trajectory, incorporating dynamical non-linearities. For definiteness we will develop and apply this approach for a specific example.

3 A Specific Formulation of the Optimal Rendezvous Problem

Consider a spacecraft subject to the central gravity field. Its equations of motion are given by

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} + \frac{\vec{F}}{m}$$

where

- $\vec{r}$ is the position vector of the spacecraft from the center of gravity and $r = |\vec{r}|$ is the distance of the spacecraft from the center of gravity.
- $\mu = GM$ is the gravitational parameter of the central body where G is the universal gravity constant and M is the mass of the central body.
- $\vec{F}$ is the non-gravitational control force.
- m is the mass of the spacecraft, which is assumed constant.

We first locate the origin of the inertial frame at the center of gravity and introduce another coordinate frame which is rotating along a circular orbit at a constant angular velocity. Hereafter

- $\delta\vec{r} = [x\, y\, z]^T$ is the position vector of the spacecraft whose components represent radial, tangential, and normal displacements from the origin of the rotating frame, respectively.
- The triple of (i, j, k) represents the corresponding unit vector which are mutually orthogonal.
- $\vec{R} = Ri$ is the position vector of the origin of the rotating frame from the origin of the inertial frame.
- $\vec{\omega} = \omega k$ is the constant angular velocity vector equal to the mean motion of the circular orbit.
- $\vec{u} = \vec{F}/m = u_x i + u_y j + u_z k$ is the control acceleration whose components are given in the rotating frame.

The position, velocity, and acceleration vectors are, respectively,

$$\begin{aligned} \vec{r} &= \vec{R} + \delta\vec{r} = (R+x)i + yj + zk \\ \Rightarrow \dot{\vec{r}} &= (\dot{x} - \omega y)i + (\dot{y} + \omega(R+x)) j + \dot{z}k \\ \Rightarrow \ddot{\vec{r}} &= (\ddot{x} - 2\omega\dot{y} - \omega^2(R+x))i + (\ddot{y} + 2\omega\dot{x} - \omega^2 y) j + \ddot{z}k. \end{aligned}$$

From Newton's law, we obtain the following component-wise equations of motion in the rotating frame:

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} - \omega^2(R+x) &= -\frac{\mu}{r^3}(R+x) + u_x \\ \ddot{y} + 2\omega\dot{x} - \omega^2 y &= -\frac{\mu}{r^3} y + u_y \\ \ddot{z} &= -\frac{\mu}{r^3} z + u_z \end{aligned}$$

where $r = \sqrt{(R+x)^2 + y^2 + z^2}$. If non-dimensionalized with reference length R and reference time $1/\omega$, they are simplified as

$$\begin{aligned} \ddot{x} - 2\dot{y} + (1+x)\left(\frac{1}{r^3} - 1\right) &= u_x \\ \ddot{y} + 2\dot{x} + y\left(\frac{1}{r^3} - 1\right) &= u_y \\ \ddot{z} + \frac{1}{r^3} z &= u_z \end{aligned}$$

where now $r = \sqrt{(x+1)^2 + y^2 + z^2}$. We consider planar motion henceforth[2]. With the definition of states as $\vec{x} = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]^T = [x \;\; y \;\; \dot{x} \;\; \dot{y}]^T$, the equations of planar motion in state space form are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 2x_4 - (1+x_1)\left(\frac{1}{r^3} - 1\right) + u_x \\ -2x_3 - x_2\left(\frac{1}{r^3} - 1\right) + u_y \end{bmatrix} \tag{21}$$

where $r = \sqrt{(x_1+1)^2 + x_2^2}$. Note that linearization about the circular reference trajectory leads to the in-plane dynamics of the well-known Clohessy-Wiltshire equation:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \\ \Leftrightarrow \;\; \dot{x} &= Ax + Bu \end{aligned} \tag{22}$$

Furthermore the RHS of (21) is analytic and can be approximated by Taylor series expansion about the circular reference trajectory as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 3x_1 + 2x_4 - 3x_1^2 + 1.5x_2^2 + 4x_1^3 - 6x_1x_2^2 + \cdots + u_x \\ -2x_3 + 3x_1x_2 - 6x_1^2x_2 + 1.5x_2^3 + \cdots + u_y \end{bmatrix},$$

a result which will be used later.

Finally, the objective is to minimize the quadratic performance index[3]

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^T(t)u(t)dt$$

subject to the nonlinear dynamics (21). According to the definition of the rendezvous problem, the initial and terminal boundary conditions are assumed to be completely specified as

$$x(t_0) = x_0 \quad , \quad x(t_f) = x_f$$

4 A Non-linear Analytical Solution to the Optimal Rendezvous Problem

To summarize the previous discussion, let us consider minimization of the quadratic cost

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^T(t)u(t)dt$$

[2]This is only for simplicity's sake. All of these approaches can be directly extended into 3-dimensional formulation.

[3]The original motivation for our approach was to evaluate the optimal trajectory and the optimal feedback control law for the minimum fuel problem with linear cost function subject to upper and lower limits on thrust. The application of Pontryagin's principle, however, leads to a switching structure of optimal control logic between maximum and minimum thrust, which makes our approach much more challenging. Hence we decided to modify the problem and consider the quadratic cost function without control constraints. The original fuel optimal problem will be a topic of our future research.

subject to the system dynamics in central gravity field

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 3x_1 + 2x_4 - 3x_1^2 + 1.5x_2^2 + 4x_1^3 - 6x_1x_2^2 + \cdots + u_x \\ -2x_3 + 3x_1x_2 - 6x_1^2x_2 + 1.5x_2^3 + \cdots + u_y \end{bmatrix}.$$

with the boundary condition

$$x(t_0) = x_0 \quad , \quad x(t_f) = x_f.$$

Note that the right hand side of the equations of motion have been expanded as a Taylor series about the nominal circular orbit. With the Hamiltonian H defined as

$$H = \frac{1}{2}u^T u + \sum_{i=1}^{4} \lambda_i \dot{x}_i \tag{23}$$

it too can be expanded as a Taylor series to find

$$\begin{aligned} H &= \frac{1}{2}(u_x^2 + u_y^2) + \lambda_1 x_3 + \lambda_2 x_4 + \lambda_3(3x_1 + 2x_4 - 3x_1^2 + 1.5x_2^2 + 4x_1^3 - 6x_1x_2^2 + \cdots + u_x) \\ &+ \lambda_4(-2x_3 + 3x_1x_2 - 6x_1^2x_2 + 1.5x_2^3 + \cdots + u_y), \end{aligned} \tag{24}$$

The corresponding costate equations are

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \\ \dot{\lambda}_4 \end{bmatrix} = \begin{bmatrix} -3\lambda_3 + 6x_1\lambda_3 - 3x_2\lambda_4 - 12x_1^2\lambda_3 + 6x_2^2\lambda_3 + 12x_1x_2\lambda_4 \cdots \\ -3x_2\lambda_3 - 3x_1\lambda_4 + 12x_1x_2\lambda_3 + 6x_1^2\lambda_4 - 4.5x_2^2\lambda_4 \cdots \\ -\lambda_1 + 2\lambda_4 \cdots \\ -\lambda_2 - 2\lambda_3 \cdots \end{bmatrix}$$

From the optimality condition $H_u = 0$ we find the optimal control law:

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} -\lambda_3 \\ -\lambda_4 \end{bmatrix} \tag{25}$$

Introducing (25) into (24) yields the Hamiltonian as a function of states and costates only:

$$\begin{aligned} H(x, \lambda) &= \frac{1}{2}(\lambda_3^2 + \lambda_4^2) + \lambda_1 x_3 + \lambda_2 x_4 + \lambda_3(3x_1 + 2x_4 - 3x_1^2 + 1.5x_2^2 + 4x_1^3 - 6x_1x_2^2 + \cdots - \lambda_3) \\ &\quad + \lambda_4(-2x_3 + 3x_1x_2 - 6x_1^2x_2 + 1.5x_2^3 + \cdots - \lambda_4) \end{aligned}$$

As discussed earlier, we evaluate $F_2(x, \lambda_0, t; t_0)$ as a power series instead of $F_1(x, x_0, t; t_0)$. For illustration purposes, in the following we only derive the equations to the linear order, in the actual analysis we kept terms up to higher orders. With this restriction, the Hamiltonian is reduced to

$$H_2(x, \lambda, t) = \frac{1}{2} \begin{bmatrix} x \\ \lambda \end{bmatrix}^T \begin{bmatrix} 0 & A^T \\ A & -BB^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

In keeping with this quadratic form of the Hamiltonian, we also expand F_2 in a quadratic form:

$$F_2(x, \lambda_0, t; t_0) = \frac{1}{2} \begin{bmatrix} x \\ \lambda_0 \end{bmatrix}^T \begin{bmatrix} F_{xx}(t; t_0) & F_{x\lambda_0}(t; t_0) \\ F_{\lambda_0 x}(t; t_0) & F_{\lambda_0\lambda_0}(t; t_0) \end{bmatrix} \begin{bmatrix} x \\ \lambda_0 \end{bmatrix} \tag{26}$$

Now recalling the relation

$$\lambda = \frac{\partial F_2}{\partial x} = \begin{bmatrix} F_{xx} & F_{x\lambda_0} \end{bmatrix} \begin{bmatrix} x \\ \lambda_0 \end{bmatrix}$$

we can express the Hamiltonian as

$$\mathcal{H} = \frac{1}{2} \begin{bmatrix} x \\ \lambda_0 \end{bmatrix}^T \begin{bmatrix} I & F_{xx} \\ 0 & F_{\lambda_0 x} \end{bmatrix} \begin{bmatrix} 0 & A^T \\ A & -BB^T \end{bmatrix} \begin{bmatrix} I & 0 \\ F_{xx} & F_{x\lambda_0} \end{bmatrix} \begin{bmatrix} x \\ \lambda_0 \end{bmatrix} \tag{27}$$

Introduction of (26) and (27) into the HJ equation (14) yields the following set of differential equations for $F_{xx}(t;t_0)$, $F_{x\lambda_0}(t;t_0) = F^T_{\lambda_0 x}(t;t_0)$, and $F_{\lambda_0\lambda_0}(t;t_0)$:

$$\begin{aligned} 0 &= \dot{F}_{xx} + F_{xx}A + A^T F_{xx} - F_{xx}BB^T F_{xx} \\ 0 &= \dot{F}_{x\lambda_0} + A^T F_{x\lambda_0} - F_{xx}BB^T F_{x\lambda_0} \\ 0 &= \dot{F}_{\lambda_0\lambda_0} - F_{\lambda_0 x}BB^T F_{x\lambda_0} \end{aligned}$$

Also, the corresponding initial conditions are derived from the identity transformation, $F_2(x, \lambda_0, t = t_0; t_0) = x^T \lambda_0$, as

$$\begin{aligned} F_{xx}(t_0;t_0) &= 0 \\ F_{x\lambda_0}(t_0;t_0) &= I = F_{\lambda_0 x}(t_0;t_0) \\ F_{\lambda_0\lambda_0}(t_0;t_0) &= 0. \end{aligned}$$

Note that $F_{xx} \equiv 0$ for all time due to the zero initial conditions. Generalizing this method, we can solve recursively for the remaining higher order terms[4]. We do not show the higher order terms here, due to space limitations. The symbolic and numerical computations and results reported here have all been carried out using Matlab.

Once this system of differential equations is solved up to as high an order as desired, we can construct F_2. Then, the Legendre transformation (17) enables us to compute F_1, which provides λ_0 from the relation (8). To carry out this solution essentially requires that a reversion of series technique be applied to the relation: $x_0 = \partial F_2/\partial \lambda_0$. Now that λ_0 has been found, the optimal trajectory can be evaluated by simple forward integration for the corresponding boundary conditions. Furthermore, the optimal feedback control can be obtained from (20).

The following plots show the optimal state trajectory and control history for two different boundary conditions and time intervals[5]. Figure 1 shows a general offset in initial conditions of $[0.2, 0.2, 0.1, 0.1]$ in position and velocity, transitioning to the origin $[0, 0, 0, 0]$ in 1 unit of time. The next two show the optimal trajectory starting from a circular orbit displaced in downtrack direction, and then transitioning to a circular orbit at the coordinate frame origin. Figure 2 shows the trajectory and controls for transitioning from an offset of 0.1 units in the y direction to the origin in 1 unit of time. Figure 3 shows the trajectory and controls for transitioning from an offset of 0.003 in the y direction to the origin in 2π units of time. As the time of the transfer is increased, more terms are needed in the control to converge to the true solution. In the future we plan to implement this algorithm for much higher orders, in the current paper we only develop the Taylor Series up to the third order, and thus for the longer time span we must shrink the initial offset from the orbit.

[4] Refer to Guibout and Scheeres[20] for detailed explanation.

[5] Note that all variables are non-dimensionalized ones, so that 2π time units equals one period of the reference orbit.

For the control histories, the solid line, dashed line, and dotted line indicate those computed from nonlinear systems, linear systems, and 3rd order approximated systems, respectively. The nonlinear open loop control(solid line) has been evaluated for comparison by solving the TPBVP using a forward shooting method. For the state trajectories, each line represents the application of each control history to the original nonlinear system. It is clear that the 3rd order control yields a better approximation than the linear control. This observation also holds as long as the boundary condition is close enough to the reference trajectory. By introducing additional higher order terms in the system dynamics, we can approximate the original system to as high an order as desired.

5 Singularities of Generating Functions

So far we have demonstrated a step-by-step procedure for evaluating optimal trajectory as well as optimal feedback control *indirectly* via generating functions. Also it has been shown that once one kind of generating function is computed, the others can also be obtained by Legendre transformation. This section is dedicated to a discussion on the possibility of singularities in the generating functions (and how to avoid them, if any) and their relationship to optimal trajectories. This is a potentially important issue. In [20] it was found that all the generating functions considered became "singular" at different times, and that this formed a fundamental barrier to the construction of long-term solutions for the generating functions (which was ultimately overcome). Thus, it is of interest to consider the possibility of singularities in the generating functions we are computing here.

In terms of the boundary value problem, the presence of singularities are usually associated with the existence of multiple solutions to the problem. In the case of Lambert-type problems in astrodynamics, a familiar situation where this arises concerns 180° transfers about a point-mass in a fixed time, as an infinity of possible transfer trajectories exist. The singularities arise in our approach as soon as there is more than one possible solution to the boundary value problem, as then the linear order terms in our expansion for F_i become degenerate and cannot represent the true solutions. As documented in [20], this leads to a divergence in these linear terms, and serves as a barrier for continued integration of the coefficients. It is important to note, however, that not all the generating functions can become singular at one time, and thus it is always possible to transform to a different generating function using the Legendre transformation, and continue computation of that generating function in time until the "singularity" in the other generating function has been passed. This has the drawback of complicating the solution procedure, however.

Fortunately, with our approach, singularities in the generating functions are easily identified, as they are associated with singularities in the state transition matrix associated with the Hamiltonian system. Once the generating function F_2 is found, λ and x_0 can be derived from the necessary conditions associated with F_2:

$$\lambda = \frac{\partial F_2}{\partial x} = F_{xx}x + F_{x\lambda_0}\lambda_0 \tag{28}$$

$$x_0 = \frac{\partial F_2}{\partial \lambda_0} = F_{\lambda_0 x}x + F_{\lambda_0\lambda_0}\lambda_0 \tag{29}$$

Let us express x and λ as a function of x_0 and λ_0. From (29)

$$x = F_{\lambda_0 x}^{-1}x_0 - F_{\lambda_0 x}^{-1}F_{\lambda_0\lambda 0}\lambda_0$$

Substituting this expression into (28),

$$\lambda = F_{xx}F_{\lambda_0 x}^{-1}x_0 + (F_{x\lambda_0} - F_{xx}F_{\lambda_0 x}^{-1}F_{\lambda_0\lambda 0})\lambda_0$$

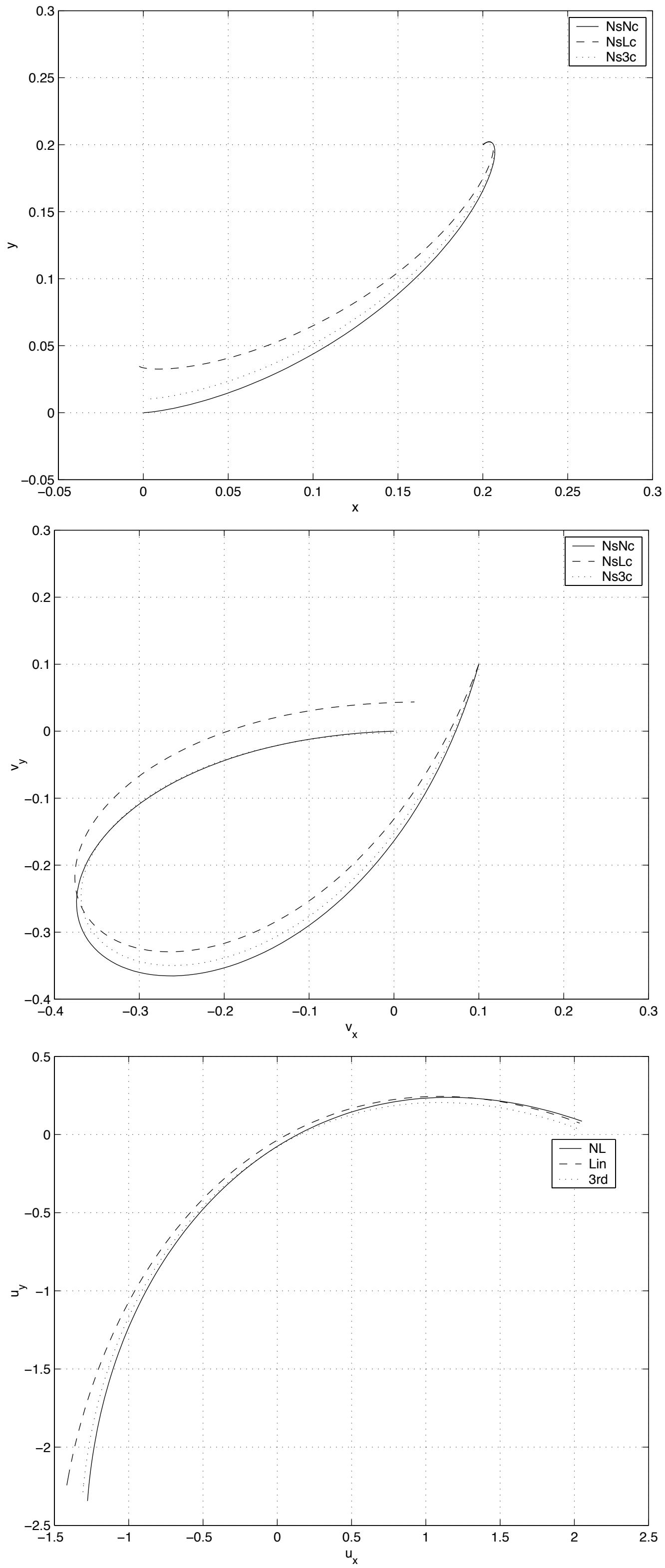

Figure 1: Radial and Tangential positions, velocities, and controls for a transfer from an arbitrary point in position and velocity space to the origin in one unit of time.

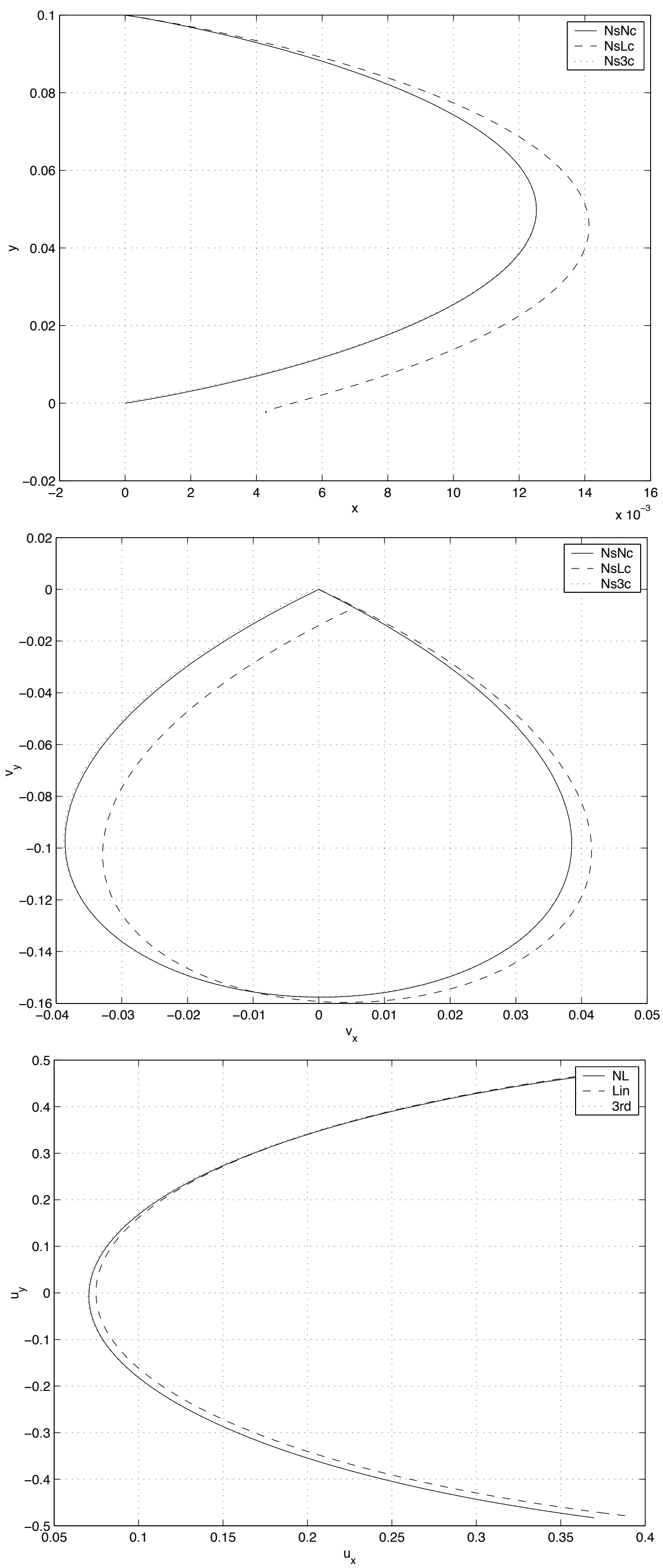

Figure 2: Radial and Tangential positions, velocities, and controls for a transfer from a neighboring circular orbit to the origin in one unit of time.

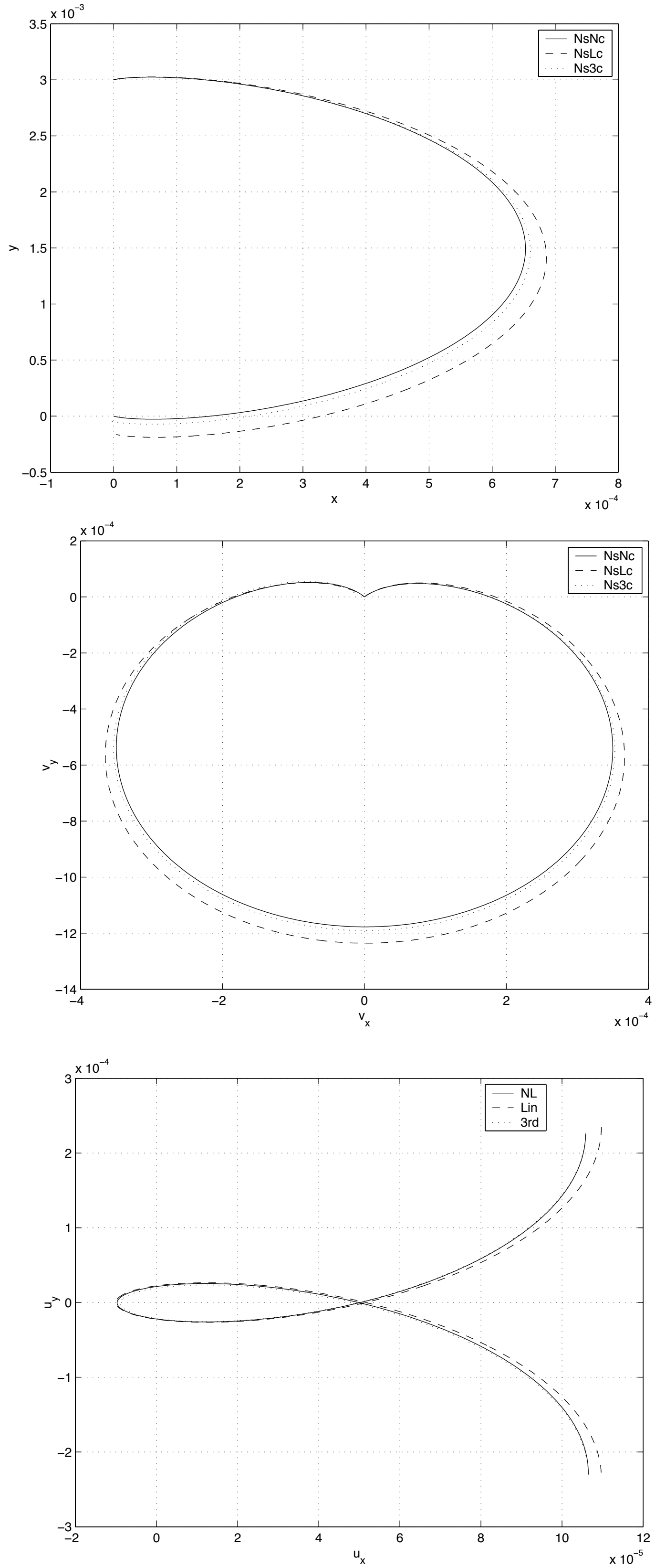

Figure 3: Radial and Tangential positions, velocities, and controls for a transfer from a neighboring circular orbit to the origin in 2π units of time.

These two equations can be combined into the matrix form:

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} F_{\lambda_0 x}^{-1}(t,t_0) & -(F_{\lambda_0 x}^{-1}F_{\lambda_0\lambda 0})(t,t_0) \\ (F_{xx}F_{\lambda_0 x}^{-1})(t,t_0) & (F_{x\lambda_0} - F_{xx}F_{\lambda_0 x}^{-1}F_{\lambda_0\lambda 0})(t,t_0) \end{bmatrix} \begin{bmatrix} x(t_0) \\ \lambda(t_0) \end{bmatrix} \tag{30}$$

Also if we define the state transition matrix as

$$\Phi(t,t_0) = \begin{bmatrix} \phi_{xx}(t,t_0) & \phi_{x\lambda}(t,t_0) \\ \phi_{\lambda x}(t,t_0) & \phi_{\lambda\lambda}(t,t_0) \end{bmatrix},$$

then the following expression holds:

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} \phi_{xx}(t,t_0) & \phi_{x\lambda}(t,t_0) \\ \phi_{\lambda x}(t,t_0) & \phi_{\lambda\lambda}(t,t_0) \end{bmatrix} \begin{bmatrix} x(t_0) \\ \lambda(t_0) \end{bmatrix}. \tag{31}$$

From the fact that (30) and (31) should be equivalent, we can easily find the following relation between them:

$$\begin{aligned} F_{xx} &= \phi_{\lambda x}\phi_{xx}^{-1} \\ F_{x\lambda_0} &= \phi_{xx}^{-T} \\ F_{\lambda_0 x} &= \phi_{xx}^{-1} \\ F_{\lambda_0\lambda_0} &= -\phi_{xx}^{-1}\phi_{x\lambda} \end{aligned}$$

These results indicate that F_2 is singular when ϕ_{xx} is singular. Also with the aid of Legendre transformation, it can be shown that F_1 is singular whenever $\phi_{x\lambda}$ is singular[6].

Consider the linear dynamics for our specific system, from which we can derive the state transition matrix:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BB^T \\ 0 & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \bar{A} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

where

$$\bar{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 \end{bmatrix}$$

Here we note that the $\bar{A}_{21}$ sub-matrix in $\bar{A}$ is always zero for the class of problems we consider here, namely problems where the "nominal" solution is no control ($\lambda \equiv 0$ for $\mathbf{x} \equiv 0$). For expansions about an existing optimal control problem, the following observations are no longer true in general.

Now the state transition matrix is defined as

$$\Phi(t,t_0) = e^{\bar{A}(t-t_0)}$$

[6]Refer to Guibout and Scheeres[20] for more comprehensive analysis about singularities.

whose explicit solution is given by

$$\Phi_{xx} = \begin{bmatrix} 4-3\cos t & 0 & \sin t & -2\cos t+2 \\ 6\sin t-6t & 1 & -2+2\cos t & -3t+4 sint \\ 3\sin t & 0 & \cos t & 2\sin t \\ 6\cos t-6 & 0 & -2\sin t & -3+4\cos t \end{bmatrix}$$

$$\Phi_{x\lambda} = \begin{bmatrix} -2.5t\cos t+6.5\sin t-4t & -16\cos t-5t\sin t+16-3t^2 & -4\cos t-2.5t\sin t+4 & \cdots \\ 16\cos t+5t\sin t-16+3t^2 & -10t\cos t+1.5t^3+38\sin t-28t & -5t\cos t+11\sin t-6t & \cdots \\ 4\cos t-4+2.5t\sin t & -5t\cos t+11\sin t-6t & 1.5\sin t-2.5t\cos t & \cdots \\ 5t\cos -11\sin t+6t & 28\cos t+4.5t^2+10t\sin t-28 & 6\cos t+5t\sin t-6 & \cdots \\ \cdots & 5t\cos t-11\sin t+6t & & \\ \cdots & -4.5t^2+28-10t\sin t-28\cos t & & \\ \cdots & -6\cos t+6-5t\sin t & & \\ \cdots & -9t+18\sin t-10t\cos t & & \end{bmatrix}$$

$$\Phi_{\lambda x} = 0_{4\times 4}$$

$$\Phi_{\lambda\lambda} = \begin{bmatrix} 4-3\cos t & -6\sin t+6t & -3\sin t & 6\cos t-6 \\ 0 & 1 & 0 & 0 \\ -\sin t & -2+2\cos t & \cos t & 2\sin t \\ 2-2\cos t & 3t-4 sint & -2\sin t & -3+4\cos t \end{bmatrix}$$

Computing the determinants symbolically, using Matlab, we find

$$\begin{aligned} |\Phi_{xx}(t)| &\equiv \cos^2 t+\sin^2 t = 1 \\ |\phi_{x\lambda}(t)| &= 1536-30t^4\cos t-250.5t^2-2048\cos t-912t\sin t-18t^3\sin 2t \\ &\quad +456t\sin 2t+\frac{27}{32}t^4\cos 2t-139.5t^2\cos 2t+\frac{75}{16}t^6+512\cos 2t \\ &\quad +400t^2\cos t+222t^3\sin t-\frac{1147}{32}t^4 \\ |\Phi_{\lambda x}(t)| &\equiv 0 \\ |\Phi_{\lambda\lambda}(t)| &\equiv (\cos^2 t+\sin^2 t)^2 = 1 \end{aligned}$$

For optimal control problems of this class, however, the ϕ_{xx} matrix is the state transition matrix of the dynamical system, and for well-defined dynamical systems this matrix is never singular. We see this explicitly above. In fact, this should hold for all optimal control problems for which we expand the generating functions about a zero solution, as the $\bar{A}_{21}$ sub-matrix will always be zero and allow the ϕ_{xx} sub-matrix dynamics to decouple from the other sub-matrices. For the applications in [20], the corresponding matrix was only a sub-element of the state transition matrix, and hence could be singular without violating singularity of the entire state transition matrix (and indeed, was singular at certain times). Thus, we see that F_2 can never suffer this sort of singularity.

Of more interest, however, is whether or not the matrix $\phi_{x\lambda}$ is ever singular. Here we know that it is at the initial epoch, when it equals zero, and that it may be singular at some future time. The time history of this determinant is shown in Fig. 4. From this, for our particular system, it is clear that $\phi_{x\lambda}$ is never singular after the initial epoch, and thus that our optimal control is well defined and unique.

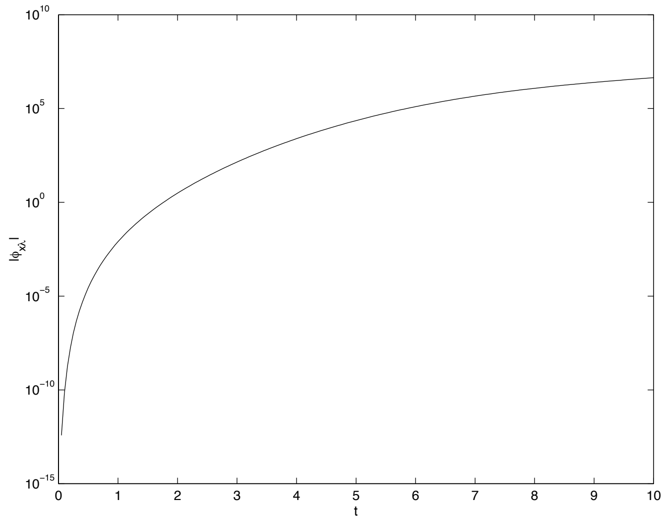

Figure 4: Determinant of $\phi_{x\lambda}(t)$

6 Conclusion

Our proposed *indirect* method of evaluating an optimal trajectory as well as an optimal feedback control via generating functions has been described and successfully applied to the low-thrust optimal rendezvous problem relative to a circular orbit. In contrast to the prevalent results in the literature based on linearized dynamics, we considered the nonlinear system by performing a Taylor series expansion of the system dynamics, and were able to show that the introduction of higher order terms results in convergence of our approach to the (unapproximated) nonlinear solution. Moreover, the proposed optimal feedback control law can be used as an improved guidance law. Finally, we considered the possibility of singularities existing in our control procedure, and were able to show that they are absent in general for the particular application we are considering.

Our future research will be focused on generalizing our procedure to solve the optimal-fuel problem which, in general, has a switching structure that introduces discontinuities in the system Hamiltonian. In addition to this, we will continue to explore the application of this procedure to classical optimal control problems in order to better understand the benefits of our approach.

Acknowledgements

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Appendix

Hamiltonian System and Canonical Transformation

This appendix briefly reviews Hamiltonian dynamical systems. See Greenwood [21] for a more comprehensive discussion. Suppose we have a system whose equations of motion can be represented using Hamilton's canonical form

$$\begin{bmatrix} \dot{q}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial p}(q(t),p(t),t) \\ -\frac{\partial H}{\partial q}(q(t),p(t),t) \end{bmatrix}$$

where

- $H = H(q(t), p(t), t)$ is the Hamiltonian of the system,
- $q(t) = [q_1(t) \;\; q_2(t) \;\; \cdots q_n(t)]^T$ is the generalized coordinate vector,
- $p(t) = [p_1(t) \;\; p_2(t) \;\; \cdots p_n(t)]^T$ is the generalized momentum vector conjugate to $q(t)$.

Suppose furthermore that there exists a canonical transformation from (q, p) to a new set of coordinate(Q,P) which is related by

$$Q = Q(q,p,t) \tag{32}$$
$$P = P(q,p,t). \tag{33}$$

Then there exists a Hamiltonian $K = K(Q(t),P(t),t)$ in the new set of coordinates such that the equations of motion is of the form

$$\begin{bmatrix} \dot{Q} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} \frac{\partial K}{\partial P}(Q,P,t) \\ -\frac{\partial K}{\partial Q}(Q,P,t) \end{bmatrix}$$

In order to relate K with H, let us recall Hamilton's principle

$$\delta I = \delta \int_{t_0}^{t_f} L dt = 0 \tag{34}$$

where the Lagrangian L is defined as $L(q,\dot{q},t) = p^T \dot{q} - H(q,p,t)$. From (34) we have, both in old and new coordinates,

$$\begin{aligned} \delta \int_{t_0}^{t_f} \left(p^T \dot{q} - H(q,p,t)\right) dt &= 0 \\ \delta \int_{t_0}^{t_f} \left(P^T \dot{Q} - K(Q,P,t)\right) dt &= 0, \end{aligned}$$

which implies that the integrands of the two integrals differ at most by a total time derivative of an arbitrary function F,i.e.,

$$p^T \dot{q} - H(q,p,t) = P^T \dot{Q} - K(Q,P,t) + \frac{dF}{dt} \tag{35}$$

Such a function is called a generating function and is a function of both the old and new coordinates and time. However, from the $2n$ relation (32) and (33) it turns out that F is a function of $2n+1$ variables instead of $4n+1$ variables. Let us assume that F is dependent upon n old coordinates and n new coordinates. Then the generating function has one of the following forms [21]

$$F_1(q,Q,t;t_0), \quad F_2(q,P,t;t_0), \quad F_3(p,Q,t;t_0), \quad F_4(p,P,t;t_0)$$

If, for instance, q and Q are independent variables, then F_1 should be used. Expanding the total time derivative of F_1 yields

$$\frac{d}{dt}F_1(q,Q,t;t_0) = \frac{\partial F_1^T}{\partial q}\dot{q} + \frac{\partial F_1^T}{\partial Q}\dot{Q} + \frac{\partial F_1}{\partial t}. \tag{36}$$

Substitution of (36) into (35) leads to

$$\left(p - \frac{\partial F_1}{\partial q}\right)^T \dot{q} - H = \left(P + \frac{\partial F_1}{\partial Q}\right)^T \dot{Q} - K + \frac{\partial F_1}{\partial t},$$

which is equivalent to

$$p = \frac{\partial F_1}{\partial q}(q,Q,t;t_0) \tag{37}$$

$$P = -\frac{\partial F_1}{\partial Q}(q,Q,t;t_0) \tag{38}$$

$$K(Q,P,t) = H(q,p,t) + \frac{\partial F_1}{\partial t}(q,Q,t;t_0) \tag{39}$$

Similarly, if q and P are independent variables, then (35) can be rewritten as a function of two independent variables q and P

$$p^T\dot{q} - H(q,p,t) = -Q^T\dot{P} - K(Q,P,t) + \frac{dF_2}{dt},$$

which yields

$$p = \frac{\partial F_2}{\partial q}(q,P,t;t_0) \tag{40}$$

$$Q = \frac{\partial F_2}{\partial P}(q,P,t;t_0) \tag{41}$$

$$K(Q,P,t) = H(q,p,t) + \frac{\partial F_2}{\partial t}(q,P,t;t_0). \tag{42}$$

Furthermore, it can be verified that the Legendre transformation

$$F_2(q,P,t;t_0) = F_1(q,Q,t;t_0) + P^TQ$$

relates F_1 with F_2. The same procedure leads to the similar results for $F_3(p,Q,t;t_0)$ and $F_4(p,P,t;t_0)$.

Application to Boundary Value Problems

Consider a canonical transformation

$$\begin{aligned} Q &= Q(q,p,t) \\ P &= P(q,p,t). \end{aligned}$$

where (q,p) and (Q,P) satisfy the following canonical equations of motion subject to the Hamiltonian $H = H(q,p,t)$ and $K = K(Q,P,t)$ respectively

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{bmatrix} \quad , \quad \begin{bmatrix} \dot{Q} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} \frac{\partial K}{\partial P} \\ -\frac{\partial K}{\partial Q} \end{bmatrix}$$

Here the new variables (Q, P) can be chosen to be constants by setting the new Hamiltonian $K \equiv 0$. That is

$$\begin{bmatrix} \dot{Q} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} \frac{\partial K}{\partial P} \\ -\frac{\partial K}{\partial Q} \end{bmatrix} \equiv 0 \quad \Rightarrow \quad \begin{bmatrix} Q \\ P \end{bmatrix} \equiv \text{constant}$$

And Eqs. (39) and (42) become

$$\begin{aligned} \frac{\partial F_1}{\partial t} + H\left(q, \frac{\partial F_1}{\partial q}, t\right) &= 0 \\ \frac{\partial F_2}{\partial t} + H\left(q, \frac{\partial F_2}{\partial q}, t\right) &= 0, \end{aligned}$$

both of which are often referred to as the Hamilton-Jacobi (HJ) equation. Indeed, they are equivalent; the only difference between the two is in their "initial boundary conditions". This difference, however, leads to a very different time evolution and even leads to the functions becoming singular at different epochs (reviewed and discussed in [20]).

For an application to the boundary value problem, let us simply choose the initial conditions of the trajectory to be the constants of motion and solve the Hamilton-Jacobi equation. In order to solve the Hamilton-Jacobi equation, the value of the generating function needs to be specified at some epoch. At $t = 0$, both old and new coordinates are equal, therefore the generating function must define an identity transformation. F_1 cannot generate such a transformation since the initial and final positions are equal and not independent at $t = t_0$, thus F_1 is undefined at $t = t_0$. On the contrary, F_2 is well defined at $t = t_0$, in fact

$$F_2(q, P, t = t_0; t_0) = q^T P$$

defines the identity transformation at $t = t_0$. Therefore, given the Hamiltonian of a system we can solve the HJ equation for F_2 from the initial time[7]. F_1 can only be solved if it is known at some other epoch than the initial time.

The main advantage of this approach is that once the generating function has been found, the unknown boundary conditions are simply evaluated from the algebraic manipulation of (37)- (38) and (40)- (41) without solving a differential equation.

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