AAS 03-566

Periodic orbits from generating functions*

V.M. Guibout† and D.J. Scheeres ‡

Abstract

Periodic orbits are studied using generating functions that yield canonical transformations induced by the phase flow as defined by the Hamilton-Jacobi theory. By posing the problem as a two-point boundary value problem, we are able to develop necessary and sufficient conditions for the existence of periodic orbits of a given period, or going through a given point in space. These conditions reduce the search for periodic orbits to either solving a set of implicit equations, which can often be handled graphically, or to finding the roots of an equation of one variable only. We present an algorithm that solves this problem locally in space for any Hamiltonian dynamical environment. Specific examples of finding periodic orbits in the vicinity of other periodic orbits and around the Libration points in the three-body problem are studied.

Introduction

Periodic orbits have been widely studied over the last century and are still a topic of great interest. Poincaré[1] already realized their importance for understanding the dynamics of non-integrable Hamiltonian systems when he claimed that they are "the only opening through which we can try to penetrate the stronghold". Indeed, he conjectured that periodic orbits are dense on typical energy surfaces. Though the Poincaré conjecture is not true for every system (e.g., for a product of harmonic oscillator with incommensurate frequencies), many systems have the property predicted by Poincaré. MacKay[2] provides conditions under which the Poincaré conjecture holds.

Many techniques have been developed to find periodic orbits. For instance, in the restricted three-body problem one may use perturbation methods[3]. Such a method allows one to find families of periodic orbits very efficiently, but does not provide a systematic procedure to find a periodic orbit of either a given period or going through a given point. By using the generating functions from the Hamilton-Jacobi theory, we can solve such a problem. Indeed, we can reduce the search for periodic orbits to either solving a set of implicit equations, which can often be done graphically, or to finding the roots of an equation of one variable only. The method we propose is independent of the Hamiltonian system considered.

Nevertheless, this method is only suitable for systems for which the generating functions are known, that is, for which we can solve the Hamilton-Jacobi equation. This equation is usually very hard to solve. Some algorithms have been developed, but each of them has its own peculiarities[4–6].

In this paper, we will be using the algorithm developed by Guibout and Scheeres[6]. It allows us to compute the generating functions for any Hamiltonian system describing nonlinear relative motion of particles moving in a Hamiltonian field. Hence, we will be able to study periodic orbits in the vicinity of any trajectory, and will be focusing on periodic orbits in the vicinity of other periodic orbits and around the Libration points in the three-body problem.

†Graduate Research Assistant, PhD Candidate, Aerospace Engi neering Department, FXB Building, 1320 Beal Avenue, Ann Arbor, MI 48109-2140, guibout@umich.edu

‡Associate Professor, Senior member AIAA, Aerospace Engineering Departm ent, FXB Building, 1320 Beal Avenue, Ann Arbor, MI 48109-2140, scheeres@umich.ed u

Hamilton's principle and Classical dynamics

Hamilton-Jacobi theory

In this section we provide the necessary background needed for our results from Hamilton-Jacobi theory. For more details on generating functions, canonical transformations and Hamilton-Jacobi theory we refer the reader to[6–12].

Let H be the Hamiltonian function of a Hamiltonian dynamical system with n degrees of freedom. The equations of motion are given by Hamilton's canonical equations:

$$\dot{q} = \frac{\partial H}{\partial p} \tag{1}$$

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{2}$$

Definition 0.1. (**Canonical transformation***) A canonical transformation is a transformation in phase space from one set of coordinates (q, p) to a new set (Q, P) such that if Eqns. 1 and 2 hold, then the following equations also hold:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \tag{3}$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} \tag{4}$$

where $K = K(Q, P, t)$ is the Hamiltonian of the dynamical system expressed as a function of the new coordinates.

Definition 0.2. Given a canonical transformation which maps (q, p) into (Q, P), we define four kinds of generating functions:

1. The generating function of the first kind, $F_1(q, Q, t)$, which satisfies:

$$p = \frac{\partial F_1}{\partial q}(q, Q, t) \tag{5}$$

$$P = -\frac{\partial F_1}{\partial Q}(q, Q, t) \tag{6}$$

2. The generating function of the second kind, $F_2(q, P, t)$, which satisfies:

$$p = \frac{\partial F_2}{\partial q}(q, P, t) \tag{7}$$

$$Q = \frac{\partial F_2}{\partial P}(q, P, t) \tag{8}$$

3. The generating function of the third kind, $F_3(p, Q, t)$, which satisfies:

$$q = -\frac{\partial F_3}{\partial p}(p, Q, t) \tag{9}$$

$$P = -\frac{\partial F_3}{\partial Q}(p, Q, t) \tag{10}$$

4. The generating function of the fourth kind, $F_4(p, P, t)$, which satisfies:

$$q = -\frac{\partial F_4}{\partial p}(p, P, t) \tag{11}$$

$$Q = \frac{\partial F_4}{\partial P}(p, P, t) \tag{12}$$

*Authors disagree on the exact definition of a canonical transformation. Our definition differs from the one of Arnold[8] but is in agreement with Abraham[7], Goldstein[9] and Greenwood[10]. Arnold's definition considers the canonical transformations as being a subclass of maps preserving Hamilton's equations.

We now introduce the Legendre transformation which allows one to relate generating functions with each other.

Theorem 0.1. ***(Legendre Transformation)***

$$F_2(q, P, t) = F_1(q, Q, t) + \sum_i P_i Q_i \tag{13}$$

where the Q's are expressed as a function of the q's and P's using Eq. 6.

Equation 13 defines F_2 as a function of F_1 implicitly, since we need to solve Eq. 6 for the P's. The Legendre transformation also applies to generating functions of the third and fourth kinds and reads as follows:

$$\begin{aligned} F_3(p, Q, t) &= F_1(q, Q, t) - \sum_i q_i p_i \\ F_4(p, P, t) &= F_1(q, Q, t) - \sum_i P_i Q_i - \sum_i p_i q_i \end{aligned} \tag{14}$$

We have chosen to express F_2, F_3 and F_4 as a function of F_1, but similar relations exist to express F_i as a function of F_j.

To conclude, canonical transformations can be seen as transformations defining a change of coordinates which preserves Hamilton's equations, and generating functions as functions of $2n+1$ independent variables whose knowledge is sufficient to fully recover the canonical transformation. As the four generating functions are associated to the same transformation, they can be derived from each other. The transformation which maps one generating function to another is the Legendre transformation.

In this paper, we only focus on one specific canonical transformation, the one induced by the phase flow, mapping one system state to another, with time as a free parameter.

Proposition 0.2. *The transformation of phase space induced by the phase flow is canonical*[8].

Let (q, p) be a trajectory such that $(q(t = 0) = Q, p(t = 0) = P)$. Then, the transformation induced by the phase flow maps the set of initial positions and momenta (Q, P) to the set of positions and momenta at a later time, (q, p). For such a transformation, $F_1(q, Q)$ is a function of initial and current positions, $F_2(q, P)$ is a function of initial momentum and current position, $F_3(p, Q)$ is a function of initial position and current momentum and $F_4(p, P)$ is a function of initial and current momenta. In addition, the generating functions for this canonical transformation verify the Hamilton-Jacobi equation:

$$\frac{\partial F_1}{\partial t}(q, Q, t) + H(q, \frac{\partial F_1}{\partial q}(q, Q, t), t) = 0 \tag{15}$$

$$\frac{\partial F_2}{\partial t}(q, P, t) + H(q, \frac{\partial F_2}{\partial q}(q, P, t), t) = 0 \tag{16}$$

$$\frac{\partial F_3}{\partial t}(p, Q, t) + H(-\frac{\partial F_3}{\partial p}(p, Q, t), p, t) = 0 \tag{17}$$

$$\frac{\partial F_4}{\partial t}(p, P, t) + H(-\frac{\partial F_4}{\partial p}(p, P, t), p, t) = 0 \tag{18}$$

Hence, by choosing (Q, P) as initial conditions, the Hamilton-Jacobi equation and F_i describe the phase flow as a boundary value problem. Given initial and final positions Q and q, we use equation 15 to solve for F_1, and subsequently Eqns. 5 and 6 to find the corresponding initial and final momenta. In the same way, if q and P are given, equation 16 together with 7 and 8 allow us to solve for the corresponding initial position and final momentum. Hence, each generating function solves a specific two-point boundary value problem.

To solve the Hamilton-Jacobi equation for the generating functions for the phase flow transformation, we need to specify initial or boundary values. At $t = 0$, the current position and momentum are the same as the initial position and momentum, therefore, the phase flow induces the identity transformation. Such a transformation can be described by F_2 and F_3 but not by F_1 and F_4[6,9,10]. To understand why F_1 and F_4 cannot describe the identity transformation, we must realize that knowledge of Q and $q = Q$ is not sufficient to uniquely determine P and $p = P$. Thus, the boundary value problem which consists of finding the initial and final momenta given the initial and final positions is ill-posed at $t = 0$, and F_1 and F_4 are not well-defined functions at initial time. We say that they develop singularities at the initial time. To bypass this problem we first solve F_2 or F_3 and then use the Legendre transformation to compute the values of F_1 or F_4 at some later epoch in order to integrate the Hamilton-Jacobi equation (Eqns. (15) and (18)). The singularity in F_1 and F_4 arises from a non-uniqueness of the boundary value problem, i.e., there are an infinite number of solutions that satisfy $q = Q$ at $t = t_0$.

Finally, we mention that, in general, generating functions develop singularities at specific times. It can be proven, however, that they cannot all be singular at the same time[8](pp267)[†]. More details on singularities can be found in[8,13–15]. Global singularities again correspond to boundary value problems that have a multiple or infinite number of solutions. In general, computation of an F_i cannot be continued through a singularity, although the F_i are well-defined on either side of the singularity. This is the situation that occurs, for example, in solving Lambert's problem over a $180°$ transfer, there are an infinite number of transfers that satisfy the boundary value problem all occurring on different orbit planes.

In this previous section, we have presented theoretical results on Hamilton-Jacobi theory and have emphasized how generating functions can be used to solve a two-point boundary value problem. In the following, we focus on the study of periodic orbits. First, we pose the problem of finding periodic orbits as a two-point boundary value problem. Second, using generating functions we solve this problem by deriving both necessary and necessary and sufficient conditions for the solution.

Study of periodic orbits

Formulation as a two-point boundary value problem

The aim of this section is to transform the search for periodic orbits into the resolution of a boundary value problem that can be handled with the theory outlined above.

For a periodic orbit of period T, both position and momentum take the same values at t and at $t + kT, \quad k \in \mathbb{Z}$. In terms of initial conditions, this reads:

$$q(T) = Q \tag{19}$$

$$p(T) = P \tag{20}$$

For a dynamical system with n degrees of freedom Eqns. 19 and 20 can be viewed as $2n$ equations of $2n + 1$ variables, the initial conditions (Q, P) and the period T. To solve such a problem, for each trial (Q, P, T) one needs to integrate the equation of motion and check if the $2n$ equations are verified, and if not try again. On the other hand, Eqns. 19 and 20 can also be viewed as a two-point boundary value problem. Suppose the initial momentum P and the position at time T, q, are given, then Eqns. 19 and 20 define $2n$ equations with $2n + 1$ variables, the initial position Q, the momentum at time T, p, and the time period T. Solutions to these equations characterize all periodic orbits. Indeed, if (Q, p, T) satisfies equations 19 and 20, then the trajectory whose initial conditions are $(Q, P = p)$ is periodic with period T. In the next section we will see that such a formulation yields a simpler set of equations when combined with the Hamilton-Jacobi theory. However, whatever formulation we choose, Eqns. 19 and 20 define an under-determined system ($2n$ equations and $2n+1$ variables) and thus we need to take at least one variable as a known parameter. Depending on the variables we set, the problem we solve has different physical interpretations. In this paper we focus on two which are of particular interest in astrodynamics:

[†]Actually, one can define 2^n generating functions, Arnold proved that among them, one at least must be well-defined. In practice, among the four generating functions we defined, one is always non-singular.

1. *Search in time domain*: Given a point in phase space, find all periodic orbits going through this point. In this case, Eqns. 19 and 20 reduce to an over-determined system of $2n$ equations with only one variable T.
2. *Search in phase space*: Find all points which belong to a periodic orbit of given period T. Eqns. 19 and 20 reduce to a system of $2n$ equations with $2n$ unknowns (q, p).

The use of generating functions

We have seen that finding periodic orbits is equivalent to solving a two-point boundary value problem, and that boundary value problems can be handled using canonical transformations, generating functions and the Hamilton-Jacobi equation. Let us assemble these two topics in order to build a new approach to the study of periodic orbits.

Generating function of the second kind

The generating function F_2 allows us to solve a two-point boundary value problem for which the initial momentum and the position at time T are given. The solution to this problem is then found using Eqns. 7 and 8.

$$p = \frac{\partial F_2}{\partial q}(q, P, T) \tag{21}$$

$$Q = \frac{\partial F_2}{\partial P}(q, P, T) \tag{22}$$

On the other hand, the boundary value problem is defined by equations 19 and 20. Combining these four equations yields:

$$\begin{aligned} P &= p(T) \\ &= \frac{\partial F_2}{\partial q}(q, P, T) \end{aligned} \tag{23}$$

$$\begin{aligned} q(T) &= Q \\ &= \frac{\partial F_2}{\partial P}(q, P, T) \end{aligned} \tag{24}$$

Eqns. 23 and 24 are $2n$ equations with $2n + 1$ variables. As pointed out before, this system of equations is under-determined, hence we will focus on one of the two following problems:

1. *Search in time domain*:

   Eqns. 23 and 24 reduce to an over-determined system of $2n$ equations with only one variable T. The $2n$ equations can be viewed as a vector equation

   $$X_2(T) = \begin{pmatrix} P - \frac{\partial F_2}{\partial q}(q, P, T) \\ q - \frac{\partial F_2}{\partial P}(q, P, T) \end{pmatrix} = 0 \tag{25}$$

   Taking the norm of X_2 yields:

   $$\sqrt{(P - \frac{\partial F_2}{\partial q}(q, P, T))^2 + (q - \frac{\partial F_2}{\partial P}(q, P, T))^2} = 0 \tag{26}$$

   Eq. 26 is an equation of one variable only, T. Suppose we have knowledge of F_2, then we can solve Eq. 26 graphically or using Newton iteration. However, the solution of equation 26 depends on the values of $(q = Q, P)$, that is, on position at time T and initial momentum. In general, for an arbitrary choice of (q, P), there is no periodic orbit going through the point $(Q = q, P)$, and thus Eq. 26 does not have any solution. On the other hand, by the Poincaré conjecture, we know that for almost every point in any n-dimensional subspace of the phase space there exists at least one periodic orbit going through this point. This leads us to consider another problem: Given either a point in position space or a point in the momentum space, find all periodic orbits which go through this point. Eqns. 23 and 24 reduce to a system of $2n$ equations with $n+1$ variables, (q, T) or (P, T) which can be solved using Newton iteration[‡] for instance.

2. *Search in phase space*:

   Eqns. 23 and 24 reduce to a system of $2n$ equations with $2n$ variables, (q, P). Supposing we have knowledge of F_2, then again we can use Newton iteration to solve these equations.

Generating function of the third kind

The generating function F_3 allows us to solve a two-point boundary value problem for which momentum at time T and initial position are given. This problem is very similar to the one solved by F_2 and hence yields a similar set of equations (the role played by q and p are inverted). Combining Eqns. 9 and 10 with Eqns. 19 and 20 yields:

$$\begin{aligned} p(T) &= P \\ &= -\frac{\partial F_3}{\partial Q}(p, Q, T) \quad (27) \\ Q &= q(T) \\ &= -\frac{\partial F_3}{\partial p}(p, Q, T) \quad (28) \end{aligned}$$

Eqns. 27 and 28 are $2n$ equations with $2n+1$ variables, (p, Q, T). We again focus on the two following problems:

1. *Search in time domain*:

   Eqns. 27 and 28 reduce to an over-determined system of $2n$ equations with only one variable T. The $2n$ equations can be viewed as a vector equation

$$X_3(T) = \begin{pmatrix} p + \frac{\partial F_3}{\partial Q}(p, Q, T) \\ Q + \frac{\partial F_3}{\partial p}(p, Q, T) \end{pmatrix} = 0 \quad (29)$$

   Taking the norm of X_3 yields:

$$\sqrt{(p + \frac{\partial F_3}{\partial Q}(p, Q, T))^2 + (Q + \frac{\partial F_3}{\partial P}(p, Q, T))^2} = 0 \quad (30)$$

   Again we need to consider Eq. 30 as an equation of $n+1$ variables, (p, T) or (Q, T) that can be solved using Newton iteration.

2. *Search in phase space*:

   Eqns. 27 and 28 reduce to a system of $2n$ equations with $2n$ variables, (p, Q). If we have knowledge of F_3, then we can use Newton iteration to solve these equations.

Generating function of the first kind

Generating functions of second and third kinds provide $2n$ equations whose solutions characterize periodic orbits (depending on the problem we consider, we find either the period of all periodic orbits going through a given point or points which belong to periodic orbits of given period). In this

[‡]There exist more sophisticated methods to find the solution to a set of equations with better convergence properties than Newton's iteration. We are either using the function *FindRoot* (based on Newton's method and the secant method) of *Mathematica*© or the Fortran code *DNSQE* based on a modification of the Powell hybrid method.

section we show that by using the generating functions F_1 or F_4, we are able to reduce the search for periodic orbits to only n equations with at most n variables. However, instead of finding necessary and sufficient conditions, such as Eqns. 23 and 24, we only find necessary conditions, i.e., solutions to the equations we develop may not correspond to a periodic orbit.

The generating function F_1 allows us to solve a two-point boundary value problem for which initial position and position at time T are given. The solution to this problem is found using Eqns. 5 and 6.

$$p = \frac{\partial F_1}{\partial q}(q, Q, T) \tag{31}$$

$$P = -\frac{\partial F_1}{\partial Q}(q, Q, T) \tag{32}$$

On the other hand, the boundary value problem is defined by equations 19 and 20. Hence, combining these four equations yields:

$$P = -\frac{\partial F_1}{\partial Q}(q = Q, Q, T) \tag{33}$$

$$p(T) = \frac{\partial F_1}{\partial q}(q = Q, Q, T) \tag{34}$$

That is, since $p(T) = P$:

$$\frac{\partial F_1}{\partial q}(q = Q, Q, T) + \frac{\partial F_1}{\partial Q}(q = Q, Q, T) = 0 \tag{35}$$

Eq. 35 defines n equations with $n+1$ variables, (Q, T). Whereas solutions to Eqns. 33 and 34 characterize periodic orbits, i.e., Eqns. 33 and 34 are necessary and sufficient conditions to solve our problem, Eq. 35 is only a necessary condition in that we may find solutions to equation 35 which are not solutions to Eqns. 33 and 34. We will see an example of this when studying periodic orbits in Hill's problem. Hence, by using F_1 we only need to solve n equations instead of $2n$, but we only get necessary conditions; we need to verify, *a posteriori*, if the solutions found characterize periodic orbits. Further, as pointed out before, Eq. 35 defines an under-determined system, and hence we will focus on one of the two following problems:

1. *Search in time domain*: Given a point in the position space, find all periodic orbits going through this point and their associated momentum. Eq. 35 defines n equations of a single variable T. Taking the norm of the left hand side yields:

$$\left\| \frac{\partial F_1}{\partial q}(q = Q, Q, T) + \frac{\partial F_1}{\partial Q}(q = Q, Q, T) \right\| = 0 \tag{36}$$

   Eq. 36 is a single equation of one variable that can be solved graphically. To find the corresponding momentum, we can use either Eq. 7 or Eq. 8:

$$P = -\frac{\partial F_1}{\partial Q}(q = Q, Q, T) \tag{37}$$

   Note that this problem is different from the previous one since we are looking for periodic orbits going through a point in position space, not in phase space.

2. *Search in position space*: Find all periodic orbits of a given period. Eq. 35 reduces to a system of n equations with n unknowns, Q. For dynamical systems with n degrees of freedom the solution may be represented on a n-dimensional plot. In practice, solving this problem graphically is efficient only for systems with at most 2 degrees of freedom[§], as we will see later.

[§]We use the function *ImplicitPlot* of *Mathematica*© which takes two equations with two variables and plots the set of solution to these equations

For Hamiltonian systems with more than 2 degrees of freedom, we use Newton's iteration or an equivalent method. When a solution to Eq. 35 is obtained, then we use Eq. 5 or 6 to find the corresponding momentum:

$$P = -\frac{\partial F_1}{\partial Q}(q = Q, Q, T) \tag{38}$$

Generating function of the fourth kind

As with F_2 and F_3, F_1 and F_4 are very similar. F_4 allows us to solve a two-point boundary value problem for which initial momentum and momentum at time T are given. The solution to this problem is then found using Eqns. 11 and 12.

$$q = -\frac{\partial F_4}{\partial p}(p, P, T) \tag{39}$$

$$Q = \frac{\partial F_4}{\partial P}(p, P, T) \tag{40}$$

On the other hand, the boundary value problem is defined by equations 19 and 20. Hence, combining these four equations yield:

$$\frac{\partial F_4}{\partial p}(p = P, P, T) + \frac{\partial F_4}{\partial P}(p = P, P, T) = 0 \tag{41}$$

Eq. 41 defines n equations with $n+1$ variables, (Q, T). These equations are necessary conditions but not sufficient. Further, Eq. 41 defines an under-determined system of equations, and hence we focus on one of the two following problems again:

1. *Search in time domain*: Given a point in the momentum space, find all periodic orbits going through this point and the associated position. Eq. 41 defines n equations of a single variable T. Taking the norm of the left hand side yields:

$$\|\frac{\partial F_4}{\partial p}(p = P, P, T) + \frac{\partial F_4}{\partial P}(p = P, P, T)\| = 0 \tag{42}$$

   As for F_1, Eq. 42 is a single equation with one variable that can be solved graphically. To find the corresponding position, we can use either Eq. 11 or Eq. 12:

$$Q = \frac{\partial F_4}{\partial P}(p = P, P, T) \tag{43}$$

2. *Search in momentum space*:

   Eq. 41 reduces to a system of n equations with n unknowns, p. We use Eq. 11 or Eq. 12 to find the corresponding momentum:

$$Q = \frac{\partial F_4}{\partial P}(p = P, P, T) \tag{44}$$

Using F_2 and F_3, necessary and sufficient conditions for existence of periodic orbits have been derived. They consist of $2n$ equations with $2n+1$ variables. On the other hand, using F_1 and F_4 we found n equations with $n+1$ variables that define necessary but not sufficient conditions for existence of periodic orbits. Moreover, solving these equations does not provide all the required information, and we need Eqns. 5 and 6 (or Eqns. 11 and 12) to find the corresponding momentum (position).

Examples

Using generating functions we are able to derive conditions for finding periodic orbits which are numerically tractable in theory. In practice, since we need to have knowledge of the generating functions to solve these conditions, a closed form solution of the Hamilton-Jacobi equation is required, which may be hard to find. An algorithm developed by Guibout and Scheeres[6] finds closed form solutions to the Hamilton-Jacobi equation, however this algorithm only converges for a certain class of problem and thus we can apply the theory developed above only to these problems. In the next sections, we summarize the specifics of this algorithm and then use it to find periodic orbits in the three-body problem.

Properties of the algorithm

We briefly discuss the properties of the algorithm we use, as these provide the necessary background for the following. For more details on the algorithm, we refer the reader to[6].

The algorithm developed by Guibout and Scheeres[6] solves the Hamilton-Jacobi equation for the generating functions describing relative motion of a particle with respect to another one whose trajectory (called the "nominal" trajectory) is known. Let $(\Delta q, \Delta p)$ be the relative position and relative momentum of the two particles. Then, the dynamics of the relative motion is Hamiltonian with the Hamiltonian function H_h given by the Taylor series expansion:

$$H_h = \sum_{\substack{i,j=0 \\ i+j\geq 2}}^{n} \frac{\Delta q^i \Delta p^j}{i!j!} \frac{\partial^{i+j} H}{\partial q^i \partial p^j}(q_0, p_0, t) \tag{45}$$

where (q_0, p_0) is the position of the particle in phase space on the nominal trajectory. The coefficients of the Taylor series $\frac{1}{i!j!}\frac{\partial^{i+j} H}{\partial q^i \partial p^j}(q_0, p_0, t)$ are time varying quantities and are easily evaluated for any Hamiltonian function with a defined trajectory.

Let us truncate the series H_h in order to only keep finitely many terms. Suppose N terms are kept, then we say that we describe the relative motion using an approximation of order N. Clearly, the greater N is, the better the approximation of the nonlinear motion of a particle about the nominal trajectory will be. The procedure of the algorithm is summarized as follows:

- Approximate the Hamiltonian describing the relative motion by truncation of high order terms.
- Assume the generating function is a polynomial in its spatial variables with time-dependent coefficients (i.e., a truncated Taylor series expansion about the nominal trajectory).
- Transform the partial differential Hamilton-Jacobi equation into a system of ordinary differential equations for the coefficients of the generating function by balancing terms of the same order in the Taylor series expansion.
- Solve the system of ordinary differential equations.

Solutions to the Hamilton-Jacobi equations are functions of either $(\Delta q, \Delta Q, t)$, $(\Delta q, \Delta P, t)$, $(\Delta p, \Delta Q, t)$ or $(\Delta p, \Delta P, t)$, i.e., they describe the relative motion. To recover the displaced trajectory we use the knowledge of the nominal trajectory together with equations 5-12. Moreover, the solution found is valid only locally in the spatial domain (in the vicinity of the nominal trajectory) and globally in the time domain.

Let us now illustrate the theory developed above with some examples. First we analyze how our conditions transform for linear systems, and then look at some more sophisticated problems such as periodic orbits in the Hill's three-body problem and in the restricted three-body problem.

Linear systems analysis

We consider a linear Hamiltonian system to be a system whose Hamiltonian function is quadratic in position and momentum without any linear term¶, i.e.,

$$H(q,p,t) = \frac{1}{2}X^T \begin{pmatrix} H_{qq}(t) & H_{qp}(t) \\ H_{pq}(t) & H_{pp}(t) \end{pmatrix} X \tag{46}$$

where $H_{qp} = H_{pq}^T$, H_{qq} and H_{pp} are symmetric and $X = \begin{pmatrix} q \\ p \end{pmatrix}$.

For linear systems, the generating function F_i is also quadratic in its spatial variables without any linear term[6], i.e,

$$F_i(Y_i,t) = \frac{1}{2}Y_i^T \begin{pmatrix} F_{11}^i(t) & F_{12}^i(t) \\ F_{21}^i(t) & F_{22}^i(t) \end{pmatrix} Y_i \tag{47}$$

where $F_{12}^i = {F_{21}^i}^T$, F_{11}^i and F_{22}^i are symmetric and $Y_1 = \begin{pmatrix} q \\ Q \end{pmatrix}$, $Y_2 = \begin{pmatrix} q \\ P \end{pmatrix}$, $Y_3 = \begin{pmatrix} p \\ Q \end{pmatrix}$, $Y_4 = \begin{pmatrix} p \\ P \end{pmatrix}$.

Then, Eqns. 35 and 41 transform into:

$$(F_{11}^1(t) + F_{21}^1(t) + F_{12}^1(t) + F_{22}^1(t))q = 0 \tag{48}$$

$$(F_{11}^4(t) + F_{21}^4(t) + F_{12}^4(t) + F_{22}^4(t))p = 0 \tag{49}$$

and Eqns. 23, 24 and 27, 28 read:

$$\begin{pmatrix} F_{11}^2 & F_{12}^2 - I \\ F_{21}^2 - I & F_{22}^2 \end{pmatrix} Y_2 = 0 \tag{50}$$

and

$$\begin{pmatrix} F_{11}^3 & F_{12}^3 - I \\ F_{21}^3 - I & F_{22}^3 \end{pmatrix} Y_3 = 0 \tag{51}$$

Eqns. 48, 49, 50 and 51 are linear matrix equations that can be solved using linear algebra tools. These equations have a trivial solutions $Y_i = 0$ which is not unique if the matrices on the left-hand side are not full rank, i.e., if their determinant vanishes.

As an example, consider the following problem: Find the periods of all periodic orbits going through a given point in position space using linearized equations of the dynamics about the Libration point L_2 in the Hill's problem (more details on Hill's problem are given in the appendix).

In Figure 1, we have plotted the determinant of $(F_{11}^1(T) + F_{21}^1(T) + F_{12}^1(T) + F_{22}^1(T))$ and in Fig. 2 the determinant of $\begin{pmatrix} F_{11}^2 & F_{12}^2 - I \\ F_{21}^2 - I & F_{22}^2 \end{pmatrix}$. Using F_1, we find ‖ that the determinant of $(F_{11}^1(T) + F_{21}^1(T) + F_{12}^1(T) + F_{22}^1(T))$ vanishes at $T^1 = 3.0330101$ and that the rank of this matrix at T^1 is 0. These results are in agreement with linear systems theory: any point in the position space belongs to a periodic orbit of period T^1. Furthermore, T^1 is a very good approximation to the true period $T^* = 3.0330193236451116$ of the linear motion, the error being less than $3 \cdot 10^{-4}\%$.

The use of F_2 provides similar results. We find that at $T^2 = 3.03301387$ the matrix $\begin{pmatrix} F_{11}^2 & F_{12}^2 - I \\ F_{21}^2 - I & F_{22}^2 \end{pmatrix}$ has rank 2. This means that there exists a periodic orbit of period T^2 going through any given point which belongs to a 2-dimensional subspace of the phase space. In particular, every point of the position space belongs to a periodic orbit of period T^2.

¶If the Hamiltonian function has linear terms, we can remove them by changing coordinates.

‖using Newton iteration

Nonlinear systems

In this section, we apply our method to some non-linear systems. As mentioned above, using F_1 or F_4 allows us to reduce the search for periodic orbits to solving n equations only, the counterpart being that we need to verify, *a posteriori*, if the solutions found are periodic orbits. For dynamical systems with 2 degrees of freedom this can be done graphically, for that reason we have chosen to illustrate our method using F_1 and not F_2 or F_3, which would have lead to more computations (solving $2n$ equations) and which would not have allowed graphical resolution. Moreover, due to the limitations of our algorithm, we limit our study to periodic orbits that go through points in the phase space which belong to the domain where our solution to the Hamilton-Jacobi equation is well-defined.

The first example we address is the study of periodic orbits around the Libration point L_2 in Hill's three-body problem. First, we solve the Hamilton-Jacobi equation for the generating function F_1 up to order 5, that is we use an approximation of order 5 of the dynamics. For the present study, such an approximation is sufficient since generating functions predict the non-linear dynamics with an error of 10^{-5} in normalized units in the spatial domain of interest[**] (i.e., an approximation of the dynamics of order 5 predicts the position of a particle within 10^{-5} units , about $70km$ for the Sun-Earth-spacecraft system, if the particle is within 0.01 units, about $21,000km$, of the Libration point).

We will be considering the following two problems:

1. *Search in time domain*: Given a point in the spatial domain, $Q = (0.01, 0)$, find the periods of all periodic orbits[††] going through this point and the corresponding momentum. To solve such a problem, we use equation 36:

$$\|\frac{\partial F_1}{\partial q}(q = Q, Q, T) + \frac{\partial F_1}{\partial Q}(q = Q, Q, T)\| = 0 \tag{52}$$

   where $Q = (0.01, 0)$. In figure 3 we have plotted the left-hand side of Eq. 52 as a function of time. We observe that there may exist a periodic orbit of period $T = 3.03353$ going through $Q = (0.01, 0)$. In order to check if we have found a periodic orbit, we use Eq. 6 to obtain the corresponding initial momentum and we generate the trajectory using Hamilton's equation. We find the initial momentum to be $P = (0, -0.0573157)$ which generates a periodic orbit of the correct period.

2. *Search in position space*: Another problem is to find all periodic orbits of a given period $T = 3.0345$. To solve this problem we use Eq. 35 which defines 2 equations with 2 unknowns that can be solved graphically. In Figure 4 we plot solutions to each of these 2 equations and then superimpose them to find their intersection. The intersection is the set of solutions to Eq. 35, that is, the set of points that may belong to periodic orbits of period T. We observe that the intersection is composed of a circle and two points whose coordinates are $(q_x, q_y) = (-0.0603795, \pm 0.187281)$. We check if the circle is a periodic orbit and if the two points are equilibrium points and find that the circle is a periodic orbit of period T, whereas the points are not equilibrium points.

   By plotting the intersection for different periods T, we generate a map of a family of periodic orbits around the Libration point. In Figure 5 we represent the solutions to Eq. 35 for $t = 3.033 + 0.0005n \quad, n \in \{1 \cdots 10\}$. For $t = 3.033$, the intersection only contains the origin, which is why there are only 9 periodic orbits shown around the origin.

The second problem we address is the search for periodic orbits in the vicinity of a given periodic orbit (see Figure 6) of period $T^* = 3.568576$ going through the point $(1.2, 0)$ in the circular restricted

[**] Actually the spatial domain in which the solution we find is valid is closely connected to the time domain of study. For every time, there exists a domain of validity which shrinks as we approach a singularity in time. In the following, we will study periodic orbits around the Libration point L_2, whose periods are about 3. F_1 is singular at 3.1938, and therefore the domain of validity is fairly small.

[††] We cannot find all the periodic orbits in general because some of them may have large momentum, so that the point in the phase space does not belong to the domain where our solution is valid.

three-body problem with $\mu = 3.03590999999 \cdot 10^{-6}$. Now we solve the generating functions up to order 4 and look for periodic orbits. We emphasize the following two problems:

1. *Search in time domain*: Given a point in the position space $(1.21, 0.01)$, find the periods of all periodic orbits going through this point. In figure 7 we plot the left hand side of Eq. 36. We notice the existence of two periodic orbits with respective period T and $2T$ where $T = 3.5727$, these are candidate periods for periodic orbits. We use Eq. 6 to compute the associated initial momentum and verify that there are periodic orbits going through the point $(1.21, 0.01)$ with that period.

2. *Search in position space*: Find all periodic orbits of a given period, $T = 3.6$. To solve this problem we use Eq. 35, which defines two equations with two variables. In figure 8 we have plotted the set of solutions to each of these equations and their superposition. The intersection of the two sets of solutions represent the set of solutions to Eq. 35 and is an arc of circle. In the domain of convergence of our generating function we find a set of intersections that defines a periodic orbit. We verify using Eqns. 6 that the intersection corresponds to a periodic orbit (see figure 9). We note that our method does not recover the entire periodic orbit, because it does not lie in the domain of convergence. Indeed, we should have found an almost circular trajectory close to the nominal one, that is of radius larger than 1.2 that corresponds to relative position as large as 2.4. Nonetheless, the set of intersections we find is enough to recover the whole trajectory using Eq. 6 and Hamilton's equations.

Conclusions

This paper derives a novel application of Hamilton-Jacobi theory to find periodic orbits. We pose the problem of finding periodic orbits as a two-point boundary value problem that can be solved using generating functions, canonical transformations and the Hamilton-Jacobi equation. Through a few examples, we show that the method proposed is numerically tractable once the generating functions are known, and that it can capture the nonlinear dynamics of relative motion. In future work, we plan to develop an algorithm that solves the Hamilton-Jacobi equation in three dimensions in order to recover Halo orbits and bifurcations. Finally, if we use the relative position of a particle with respect to the nominal trajectory instead of its "true" position, then we will be able to find periodic orbits relative to the nominal trajectory. Such a modification of our method should have applications to formation flight and also to the study of stability of periodic orbits.

Appendix I: The circular restricted three-body problem and the Hill's problem

The circular restricted three-body problem is a three-body problem where the second body is in circular orbit about the first one and the third body has negligable mass[16]. The coordinate system is centered at the center of mass of the two bodies with mass and the Hamiltonian function describing the dynamics of the third body is:

$$H(q_x, q_y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + p_x q_y - q_x p_y - \frac{1-\mu}{\sqrt{(q_x+\mu)^2 + q_y^2}} - \frac{\mu}{\sqrt{(q_x - 1 + \mu)^2 + q_y^2}} \tag{53}$$

where $q_x = x$, $q_y = y$, $p_x = \dot{x} - y$ and $p_y = \dot{y} + x$. Equations of motion for the third body can be found using Eqns. 1 and 2:

$$\ddot{x} - 2\dot{y} = x - (1-\mu)\frac{x+\mu}{((q_x+\mu)^2 + q_y^2)^{3/2}} - \mu\frac{x - 1 + \mu}{((q_x - 1 + \mu)^2 + q_y^2)^{3/2}} \tag{54}$$

$$\ddot{y} + 2\dot{x} = y - (1-\mu)\frac{y}{((q_x+\mu)^2 + q_y^2)^{3/2}} - \mu\frac{y}{((q_x - 1 + \mu)^2 + q_y^2)^{3/2}} \tag{55}$$

There are five equilibrium points for this system that are called Libration points. L_2 is the one whose coordinates are $(1.0100701985928262, 0)$ for a value of $\mu = 3.03590999999 \cdot 10^{-6}$.

If the first body has a larger mass than the second one we can expand the equations of motion about $\mu = 0$. Then, shifting the coordinate system so that its center is the second body yields Hill's formulation of the three-body problem. The equations are motion are:

$$\ddot{x} - 2\dot{y} = -\frac{x}{r^3} + 3x \tag{56}$$

$$\ddot{y} + 2\dot{x} = -\frac{y}{r^3} \tag{57}$$

$$\tag{58}$$

where $r^2 = x^2 + y^2$.

The Lagrangian then reads:

$$L(q, \dot{q}, t) = \frac{1}{2}(\dot{q}_x^2 + \dot{q}_y^2) + \frac{1}{\sqrt{q_x^2 + q_y^2}} + \frac{3}{2}q_x^2 - (\dot{q}_x q_y - \dot{q}_y q_x) \tag{59}$$

Hence,

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{q}_x} \\ &= \dot{q}_x - q_y \end{aligned} \tag{60}$$

$$\begin{aligned} p_y &= \frac{\partial L}{\partial \dot{q}_y} \\ &= \dot{q}_y + q_x \end{aligned} \tag{61}$$

From Eqns. 59, 60 and 61 we obtain the Hamiltonian function H:

$$\begin{aligned} H(q,p) &= p_x\dot{q}_x + p_y\dot{q}_y - L \\ &= \frac{1}{2}(p_x^2 + p_y^2) + (q_y p_x - q_x p_y) - \frac{1}{\sqrt{q_x^2 + q_y^2}} + \frac{1}{2}(q_y^2 - 2q_x^2) \end{aligned} \tag{62}$$

There are two equilibrium points for this system that are called libration points. Their coordinates are $L_1(-\left(\frac{1}{3}\right)^{1/3}, 0)$ and $L_2(\left(\frac{1}{3}\right)^{1/3}, 0)$

References

1. Henri Poincaré. *Les Méthodes Nouvelles de la Mécanique Céleste*, volume 1,2,3. Gauthier-Villars, Paris; reprinted by Dover, New York (1957), 1892,1893,1899.

2. R. S. MacKay. Some aspects of the dynamics and numerics of hamiltonian systems. In *The dynamics of numerics and the numerics of dynamics*, pages 137–193. Oxford Univ. Press, 1990.

3. Michel Hénon. *Generating families in the restricted three-body problem.* Springer-Verlag, 1997.

4. Pierre-Louis Lions. *Generalized solutions of Hamilton-Jacobi equations.* Pitman, 1982.

5. Jerry Markman and I. Norman Katz. An iterative algorithm for solving hamilton-jacobi type equations. *SIAM Journal on Scientific Computing*, 22(1):312–329, 2000.

6. Vincent M. Guibout and Daniel J. Scheeres. Formation flight with generating functions: Solving the relative boundary value problem. In *AIAA/AAS Astrodynamics Specialist Conference and Exhibit.* AIAA, 2002.

7. Ralph Abraham and Jerrold E. Marsden. *Foundations of mechanics.* W. A. Benjamin, 2nd edition, 1978.

8. Vladimir I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer-Verlag, 2nd edition, 1988.

9. Herbert Goldstein. *Classical Mechanics*. Addison-Wesley, 1965.

10. Donald T Greenwood. *Classical Dynamics*. Prentice-Hall, 1977.

11. Cornelius Lanczos. *The variational principles of mechanics*. University of Toronto Press, 4th edition, 1977.

12. Jerrold E. Marsden and Tudor S. Ratiu. *Introduction to mechanics and symmetry : a basic exposition of classical mechanical systems*. Springer-Verlag, 2nd edition, 1998.

13. Vladimir I. Arnold. *Catastrophe Theory*. Springer-Verlag, 3rd, revised and expanded edition, 1992.

14. Juergen Ehlers and Ezra T. Newman. The theory of caustics and wavefront singularities with physical applications. *Jounral of Mathematical Physics A*, 41(6):3344–3378, 2000.

15. Lars Hörmander. *Seminar on singularities of solutions of linear partial differential equations*. Princeton University Press, 1979.

16. Vladimir I. Arnold, V. V. Kozlov, and A. I. Neishtadt. *Mathematical Aspects of Classical and Celestial Mechanics, Dynamical Systems III*. Springer-Verlag, 1988.

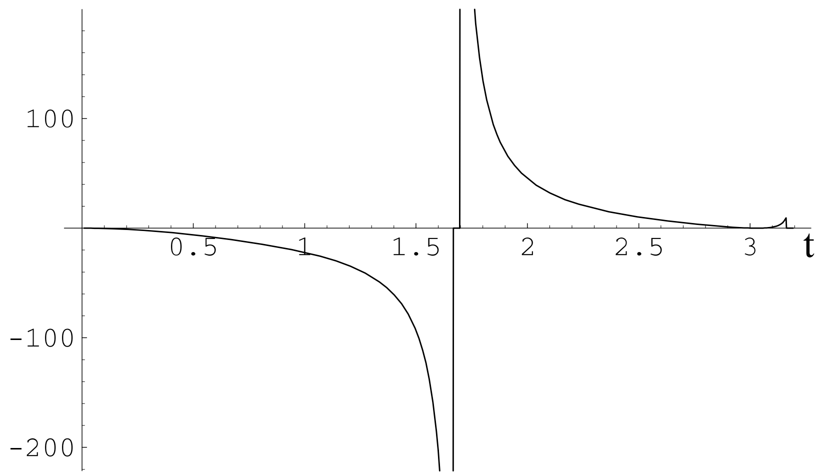


Figure 1: Determinant of the matrix defined in Eq. 48

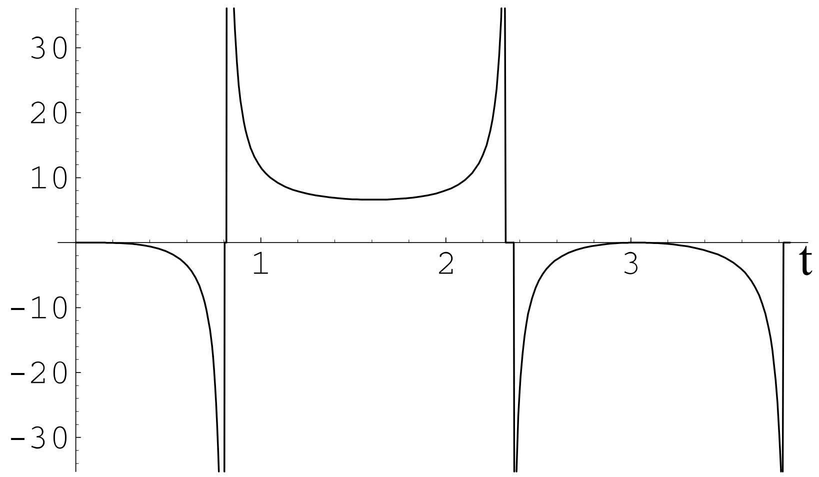


Figure 2: Determinant of the matrix defined in Eq. 50

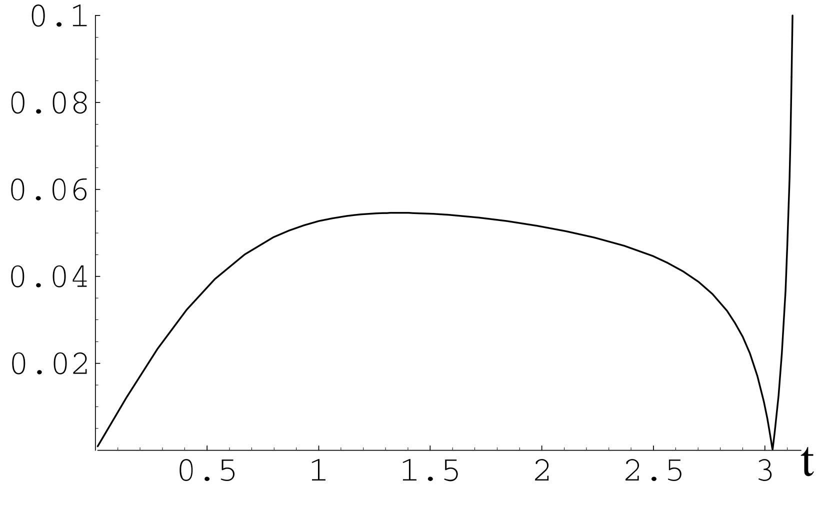
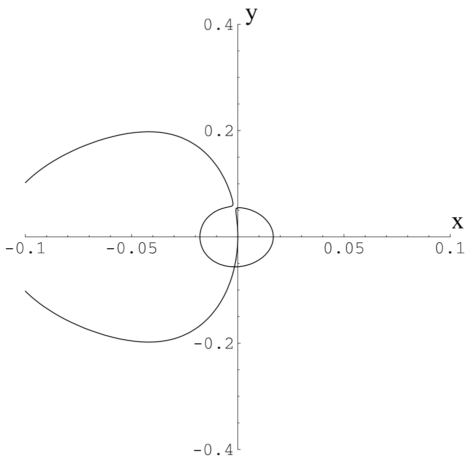
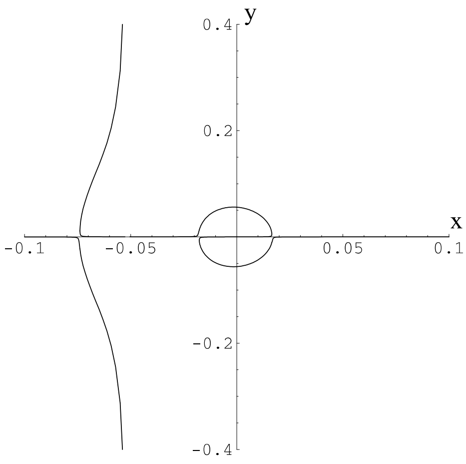


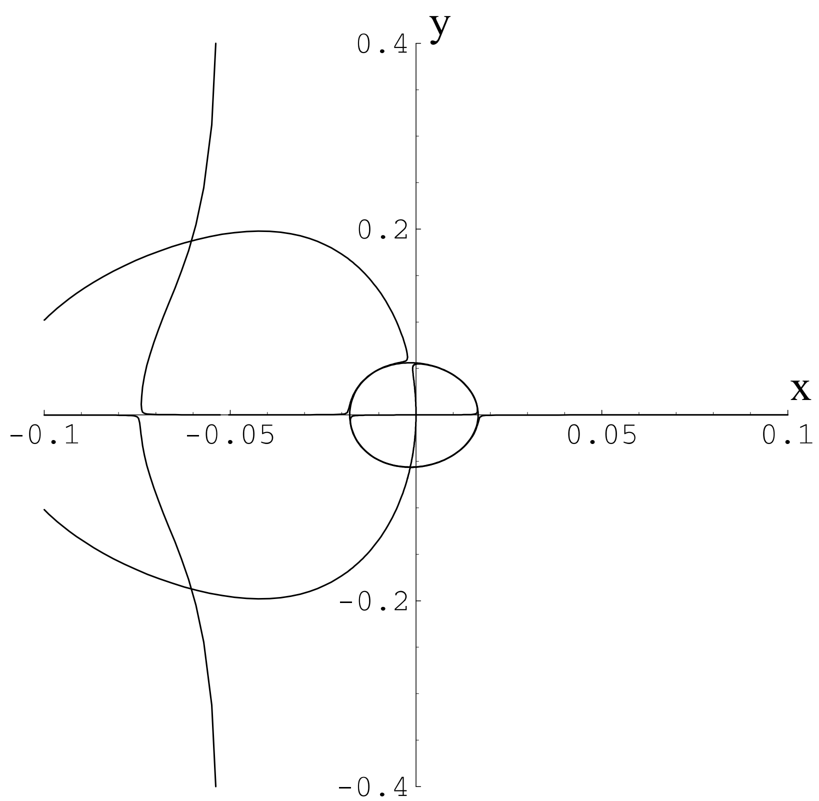
Figure 3: Plot of $\|\frac{\partial F_1}{\partial q}(q=Q,Q,T)+\frac{\partial F_1}{\partial Q}(q=Q,Q,T)\|$ where $Q=(0.01,0)$



(a) Plot of the solution to the first equation defined by Eq. 36

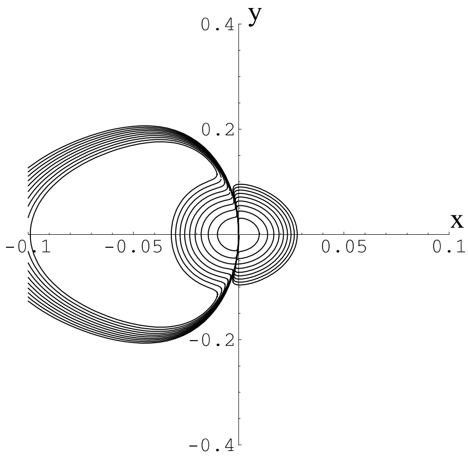


(b) Plot of the solution to the second equation defined by Eq. 36

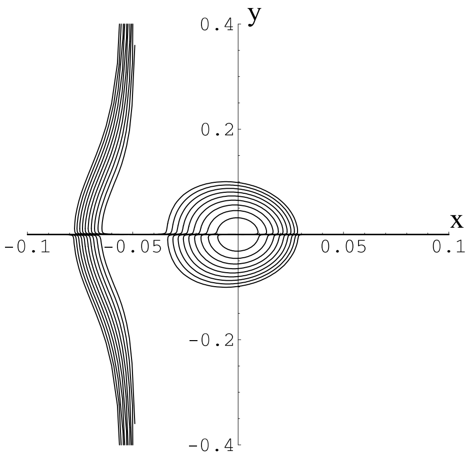


(c) Superposition of the two sets of solutions

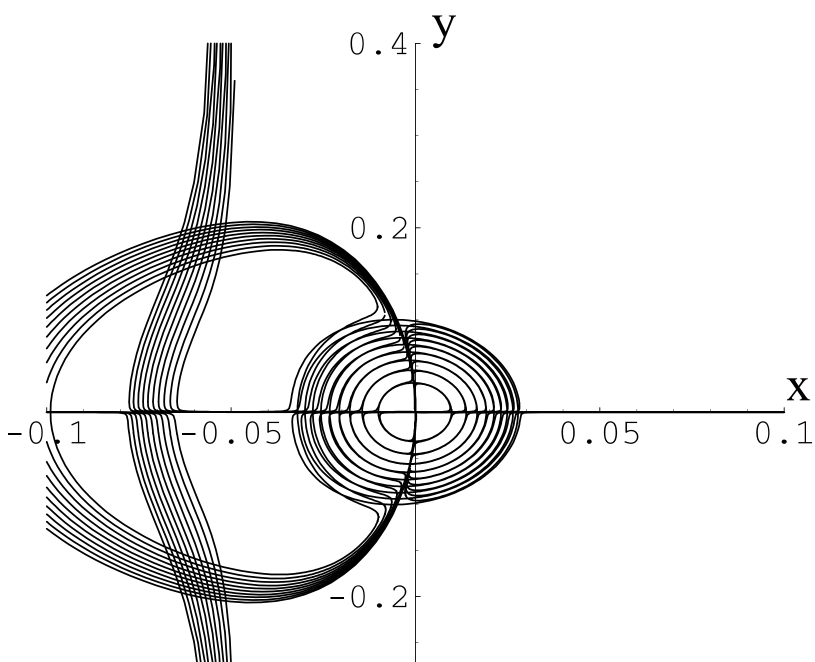
Figure 4: Periodic orbits for the nonlinear motion about a Libration point



(a) Plot of the solution to the first equation defined by Eq. 36 for $t = 3.033 + 0.0005n \quad n \in \{1 \cdots 10\}$



(b) Plot of the solution to the second equation defined by Eq. 36 for $t = 3.033 + 0.0005n \quad n \in \{1 \cdots 10\}$



(c) Superposition of the two sets of solutions for $t = 3.033 + 0.0005n \quad n \in \{1 \cdots 10\}$

Figure 5: Periodic orbits for the nonlinear motion about a Libration point

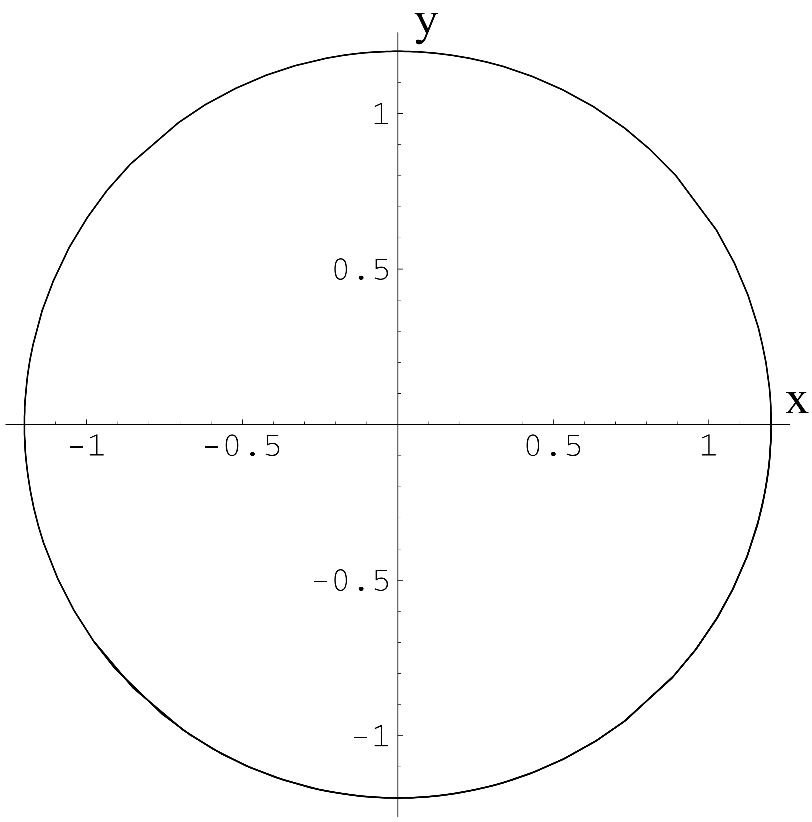


Figure 6: Nominal trajectory of a periodic orbit in the restricted three-body problem with period $T = 3.568576$

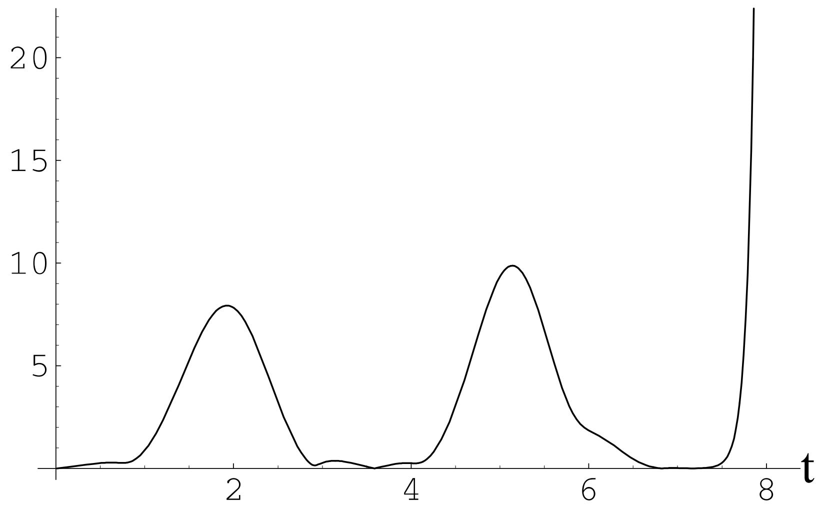
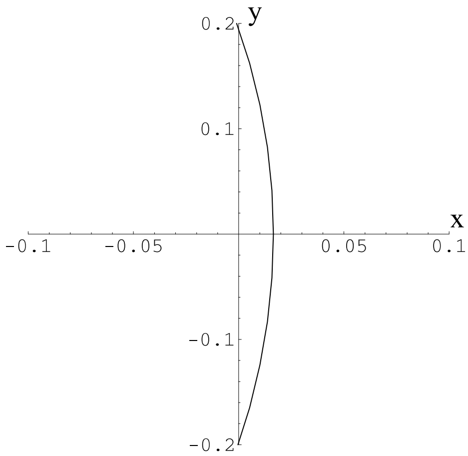
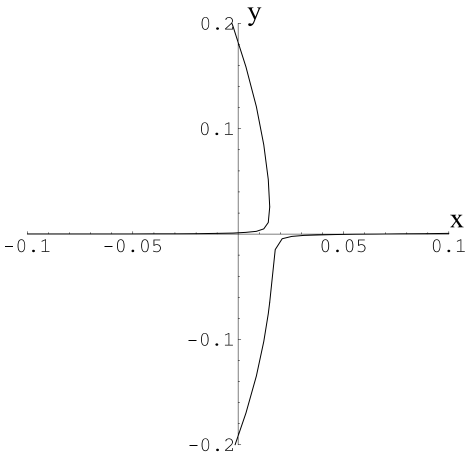


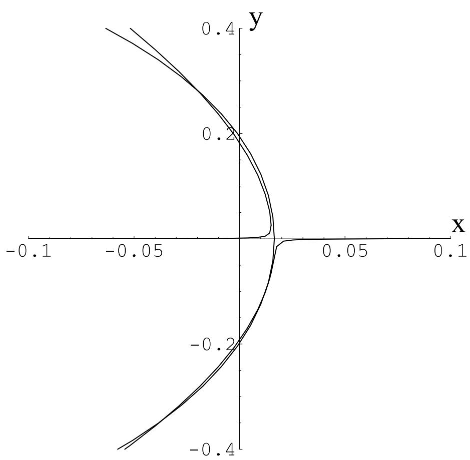
Figure 7: Periodic orbits in the vicinity of a circular orbit in the restricted three-body problem



(a) Set of solutions to the first equation defined by Eq. 36

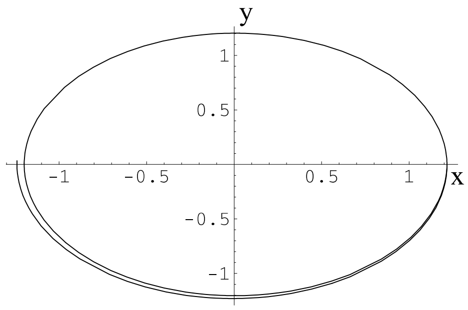


(b) Set of solutions to the first equation defined by Eq. 36

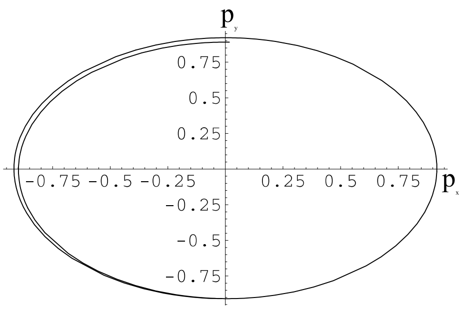


(c) Set of solutions to Eq. 36

Figure 8: Periodic orbits in the vicinity of a circular orbit in the restricted three-body problem



(a) Position space



(b) Momentum space

Figure 9: Periodic orbit of period 3.6 in the restricted three-body problem obtained using generating functions