AAS03-519

THIRD BODY DRIVEN *vs.* ONE IMPULSE PLANE CHANGES

B.F. Villac* and D.J. Scheeres†

Third body driven plane changes, i.e., plane change maneuvers that uses 3^{rd} body forces to reduce the cost of the maneuver, are compared to one impulse plane changes, the only classical approach still applicable in environments strongly perturbed by a third body. It is shown in particular that, while having inherent restrictions, the realizable range of third body driven plane changes covers a wide range of initial conditions and is closely related to the existence of $\pm 180^o$ plane changes. The question of the optimal method for performing a given plane change with given initial conditions and the maximal cost of such transfers are addressed semi-analytically to help in the design of plane change maneuvers in third body perturbed environments. Mars and Callisto orbiter exmaples have been chosen to illustrate the theory.

INTRODUCTION

The classical approach to orbit transfers is to assume an underlying two body model (2BP) and investigate the different ways of transferring from one conic orbit to another[1]. Between two successive impulsive burns, the orbit is characterized by a set of constant values, the orbital elements, which allows us to decide, most often analytically, about the optimality of a given transfer. However, when perturbations occur, this classical approach breaks down due to more complex dynamics that can result in large orbital element changes from one periapsis passage to the next[2]. Placing a spacecraft on a transfer trajectory prescribed by a two body model may result in an impact with the primary before reaching the next periapsis passage and the ΔV estimates thus obtained do not necessarily yield useful information. However, transfers such as WSB ballistic capture[3], or insertions into halo orbits[4], using invariant manifold theory shows that these strong dynamics can be used effectively in control purposes, reducing, in some case drastically, the total cost of the transfers.

*Corresponding author. Ph.D student, Department of Aerospace engineering, The University of Michigan, Ann Arbor, Michigan. E-mail: bvillac@umich.edu

†Associate professor, Department of Aerospace Engineering, The University of Michigan, Ann Arbor Michigan. E-mail: scheeres@umich.edu

In a recent paper[5], third body perturbations have been shown to be possibly used to perform plane change maneuvers at reduced costs over the classical approaches. The resulting new class of plane change maneuvers, third body driven plane changes, are similar to the classic bi-elliptic plane change maneuvers with the apoapsis maneuvers replaced by the cost free action of the 3^{rd} body perturbations (see Figure 1).

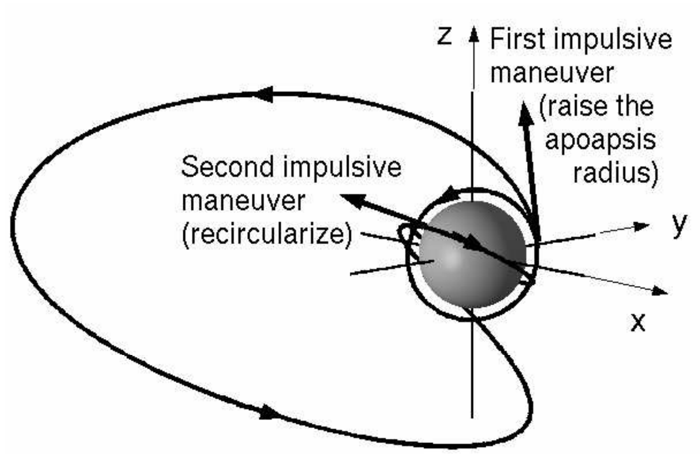

Figure 1: Third body driven plane change

While inherently restricted by the underlying dynamics (modeled by Hill's problem), these transfers have been shown to be optimal over the one impulse plane changes (the only classical maneuver still applicable in these perturbed environments) over a large range of initial conditions[6]. This paper reviews and extends these results while applying the theory to the case of Mars and Callisto orbiters ($\sim 200\ km$ altitude) .

In particular, the allowable range of third body driven plane changes is computed and related to the existence of $\pm 180^o$ plane changes. The question of optimality of transfers over one impulse maneuvers is answered semi-analytically and an upper bound on the costs of these transfers is also presented.

THIRD BODY DRIVEN PLANE CHANGES

The plane change problem considered consist of finding a trajectory that transfers a spacecraft from an initial circular orbit of given radius r_0 and inclination i to a final circular orbit with same radius r_0 but with a different inclination, $i + \Delta i$.

Classically, the methods used for performing the maneuvers consists of one impulse, bi-elliptic, parabolic or general three impulse maneuvers[1]. Among these methods, only the one impulse maneuvers, which consist of rotating the velocity vector of the initial circular orbit at one of the nodes, is still applicable in strongly perturbed environments. Indeed, the remaining methods involve a transfer trajectory with a large apoapsis radius ("infinity" for plane changes larger than 60^o) so that the third body perturbations, which increase linearly with distance from the primary, induce large changes in the orbital elements over the transfer (and possibly cancel the effect of the classical apoapsis burn) and the two body assumption does not hold anymore. This section reviews the model used to represent the third body perturbed orbital environment and the basic idea and results about third body driven plane changes.

Hill's problem

The Hill problem is an approximate model derived from the three body model that represents the motion of two close, gravitationally interacting, small masses, as perturbed by a larger, distant body. This model is well suited to represent the motion of an orbiter around a primary (planet or planetary satellite) as perturbed by a massive body (Sun or giant planet, respectively). As compared to the more widely used circular restricted three body problem, Hill's problem has the advantage of a simple and parameterless set of equations with the presence of symmetries that simplify the numerical computations[7].

Hill's problem is formulated in a rotating frame with angular velocity N (corresponding to the mean motion of the primary around the disturbing body in the case of the orbiter problem), as shown in Figure 2.

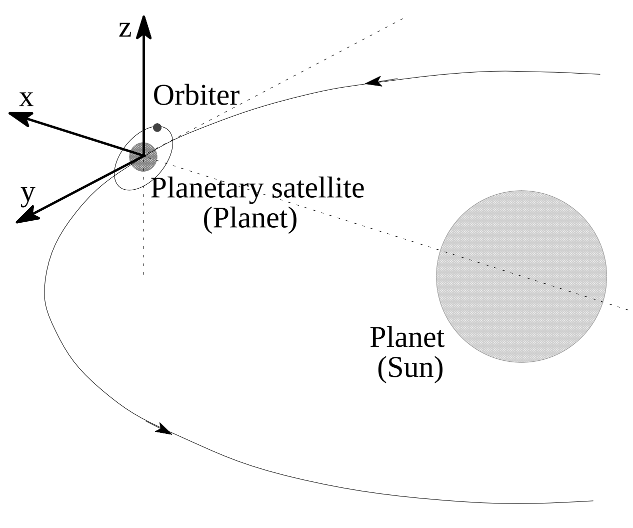

Figure 2: Geometry of Hill's problem in the case of an orbiter

Denoting μ the gravitational parameter of the primary, one can define the length and time scales, l and τ respectively, as:

$$l = \left(\frac{\mu}{N^2}\right)^{1/3} \quad \text{and} \quad \tau = \frac{1}{N}$$

so that the equations of motion are represented in parameterless form as:

$$\begin{cases} \ddot{x} - 2\dot{y} &= -\frac{x}{r^3} + 3x \\ \ddot{y} + 2\dot{x} &= -\frac{y}{r^3} \\ \ddot{z} &= -\frac{z}{r^3} - z \end{cases} \quad (1)$$

In the sequel, all the computations will be performed in this normalized setting (and may thus be applied to a range of physical system by a simple scaling). However, some dimensional results will be given for the cases of Mars and Callisto orbiters to illustrate the theory. Table 1 summarizes the numerical values used for these primaries. Notably, an initial $\sim 200\ km$ altitude orbiter has a normalized radius of 0.00333 in the case of Mars and 0.355 in the case of Callisto*. These examples are representative of two different realm of dynamics and are separated by an order of magnitude in normalized radii.

Table 1: Numerical parameters for Mars and Callisto

Primary	μ $(km^3.s^{-2})$	N $(rad.s^{-1})$	l (km)	τ (hr)	Radius (km)	Normalized radius	ΔV scale $(m.s^{-1})$
Mars	42832	1.058e-7	1084027.7	2623.82	3397	0.00313	114.7
Callisto	7171	4.357e-6	72283.6	63.75	2400	0.0332	314.9

Note also that Hill's equations consist of a Kepler problem in a rotating frame with an additional linear term representing the third body perturbation. Thus, when close to the primary, the perturbation tends to zero as compared to the Keplerian term and the low altitude dynamics can be represented for several orbits by a Keplerian motion in a rotating frame. Averaging theory indicates that this approximation is valid for a normalized radius less than $\simeq 0.2$ (see Refs. 7 and 8). Above this radius, the dynamics become strongly non-Keplerian and at $x = \pm(1/3)^{1/3} \simeq 0.69$, the perturbing and Keplerian terms cancel each other, resulting in the equilibrium solutions that correspond to the libration points, L_1 and L_2.

A first integral of the equations of motion, known as the Jacobi constant, is expressed in the inertial space as:

$$J = \frac{1}{2}V^2 - \frac{1}{r} - G\cos i + \frac{1}{2}r^2 - \frac{3}{2}x^2 \quad (2)$$

where r denotes the magnitude of the position vector, V represents the inertial velocity, G is the magnitude of the angular momentum and i is the inclination. This expression will be used to derive estimates of the cost of the third body driven plane changes in section 3.

*More precisely, an initial radius of 0.00333 corresponds to a $212.8\ km$ altitude Mars orbiter and the initial radius of 0.0355 corresponds to a $166\ km$ altitude Callisto orbiter.

Finally from (1), it can easily be checked that the equatorial plane is invariant under the flow and that the symmetries with respect to the origin (both in the spatial and planar case) leave the equations invariant. That is, the dynamics is invariant with respect to shifts of $m\pi$, $m \in \mathbb{Z}$ in longitude of ascending node, Ω, and argument of periapsis, ω. Thus, numerical investigations need only be performed on the "reduced" torus-space $[0, 180]^2$.

Third body driven plane changes

Since for low altitudes, quasi-circular motion exists and can be approximated by a Keplerian motion in a rotating frame (i.e., with a precession of the nodes) for several spacecraft orbits, the plane change problem in a third body perturbed environment still makes sense. However, one should use Hill's problem to represents the dynamics of the transfer trajectories. In particular, different choices of ω and Ω of the transfer trajectory (i.e. the orientation of this trajectory with respect to the disturbing body) leads to completely different transfers. Third body driven plane change consists of placing the spacecraft on a transfer trajectory whose initial orientation has been chosen so that the plane change is performed without any apoapsis burn, thus resembling the classical bi-elliptic maneuvers with the apoapsis maneuver suppressed by the use of the third body perturbations.

More precisely, when the spacecraft moves along the initial, low altitude, circular orbit, the mean anomaly, M, and the longitude of the ascending node, Ω, are linear functions of time and can be represented as a line wrapping in the (M, Ω)-torus space (all the other orbital elements remaining fixed during the motion by our approximation of circular motion), as illustrated in Figure 3.

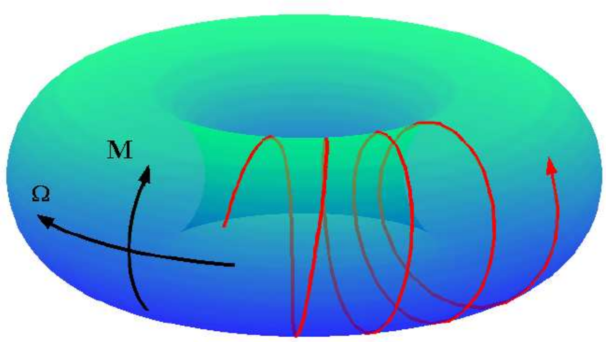

Figure 3: Motion of a spacecraft on a circular orbit as viewed in the (M, Ω)-torus space

Note that, since the slope is almost certainly irrational, any value of M and Ω can be approximated to any degree of accuracy by waiting for a sufficiently long time. In practice, any value of M and Ω can be approximated to within $\sim 1^o$ after a few spacecraft orbits. For example, in the case of a Mars orbiter, Ω varies by 0.04^o during a spacecraft orbit ($\sim 200\ km$ orbiter) so that any value of (M, Ω) can be reached to within 0.02^o accuracy.

Similarly, in the case of the Callisto orbiter, any value of (M, Ω) can be approximated to within 1.23^o.

A tangential impulsive maneuver at any point of this initial circular orbit will place the spacecraft at the periapsis of a transfer trajectory, the orbital elements of which are determined using the following correspondences:

$$\begin{array}{ccc} \underline{\text{Initial, circular orbit}} & & \underline{\text{Transfer trajectory}} \\ r_0 & \longrightarrow & r_p \\ \Delta V & \longrightarrow & r_{a(\Delta V)} \\ i & \longrightarrow & i \\ \Omega = -Nt & \longrightarrow & \Omega \\ M = nt & \longrightarrow & \omega \end{array} \tag{3}$$

where $n = \sqrt{1/r_0}$ represents the mean motion of the spacecraft on the initial circular orbit. In particular, ω and Ω are directly related to the value of M and Ω of the initial circular orbit, i.e., time. Thus, from the previous discussion, any value of (ω, M) of the transfer trajectory can be targeted by timing this first maneuver.

Then, the key point is the existence of some values of (ω, Ω) that result in a zero change in periapsis radius at the next periapsis passage of the transfer trajectory, while having a non-zero change in inclination (and, in fact, this change can be rather large, as we will see below). Thus, choosing such values for ω and Ω, one only needs perform a recircularization burn at the next periapsis to complete a plane change maneuver.

Mathematically, the choice of ω and Ω amounts to solving the following non-convex optimization problem:

$$\max_{\omega, \Omega} \quad \Delta i_{(\omega,\Omega)} \quad \text{subject to} \quad \Delta r_{p_{(\omega,\Omega)}} = 0 \tag{4}$$

where the notation Δ represents a change between successive periapsis passages[†]. This procedure can be visualized by computing the changes Δr_p and Δi for each value of ω and Ω and plotting the intersection of the curves $\Delta r_p(\omega, \Omega) = 0$ with the level curves of Δi. This is shown in Figure 4 where the spacecraft orbit in its initial circular orbit has been also represented (owing to the correspondences (3)).

Finally, we note that Figure 1, used to illustrate the geometry of the third body driven plane changes, corresponds indeed to an actual transfer computed with this method.

Dynamical restrictions

The previous method allows us to show that 3^{rd} body driven plane changes are less expensive than one impulse maneuvers for some large enough plane change values. However, the

[†]That is, $\Delta x = x(\text{final peripsis}) - x(\text{initial periapsis})$.

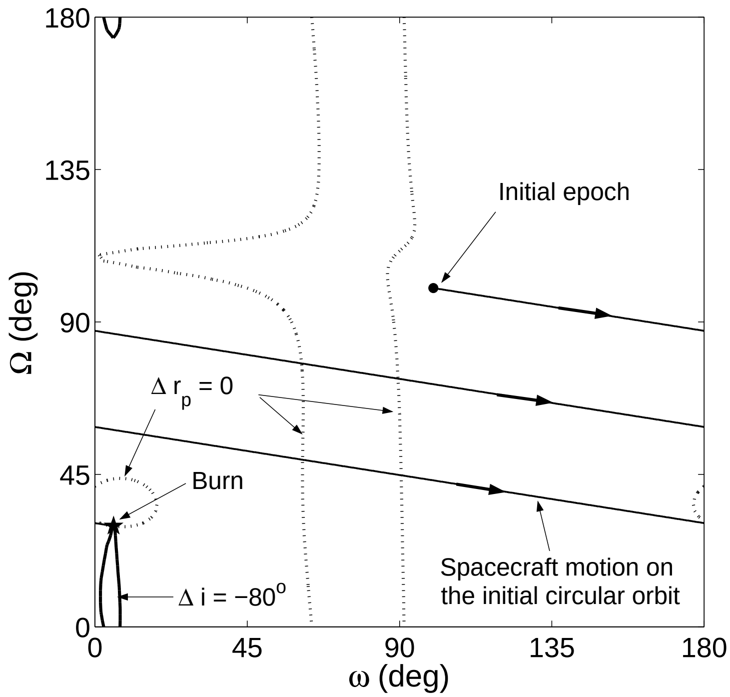

Figure 4: A geometric view of third body driven plane changes

invariance of the equatorial plane in Hill's problem restricts the range of 3^{rd} body driven plane changes realizable. Transfers from a non-equatorial to an equatorial trajectory is not possible using this method and one impulse maneuvers must be used to perform such transfers.

Besides the limitations on the extrema values of the solution of (4), this restriction manifests itself also in the topology of this optimization problem. More precisely, the extrema of plane change may be located on two disconnected components of the level curves $\Delta r_p(\omega, \Omega) = 0$ and not all the plane change values in between these two extrema may be possible. Figure 5 shows a situation where there is several disconnected zero level curves of Δr_p, but the global Δi extrema are located on a single component. Thus, in this case, all the value of plane change between -80.9^o and $+79.2^o$ are possible. Near zero inclination, these global extrema are on different components and a gap in the values of realizable plane change exists.

Therefore, numerically, one has to compute the extrema of plane change on each connected component of $\Delta r_p(\omega, \Omega) = 0$ to determine the range of 3^{rd} body driven plane changes. These computations have been performed for different values of apoapsis radius in the case of Mars and Callisto, as shown in Figure 6.

We can see from these graphs that the 3^{rd} body driven plane change range increases

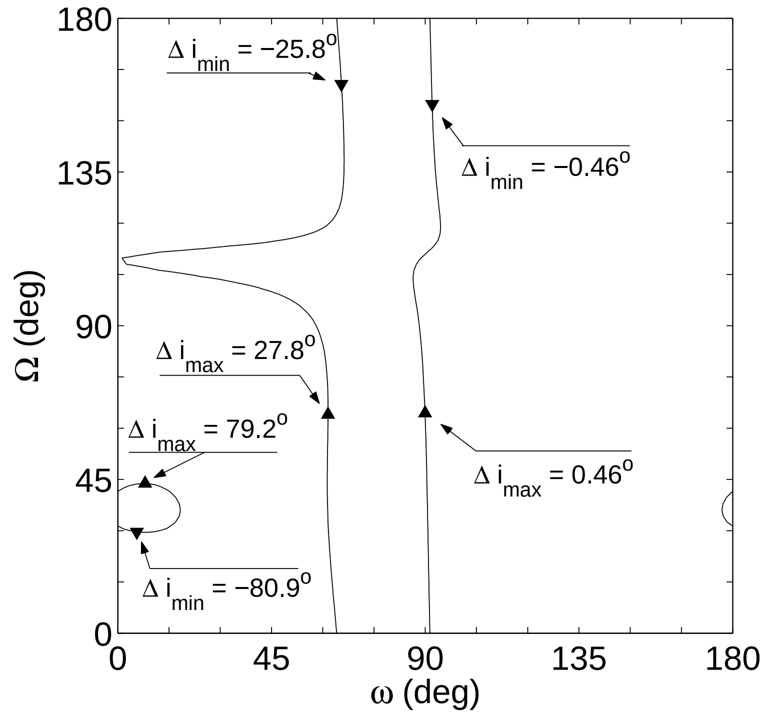

Figure 5: $\Delta r_p = 0$ **line for** $r_p = 0.003$**,** $r_a = 0.5$ **and** $i = 90^o$

significantly with r_a until the $\pm 180^o$ plane changes appear to be possible. In this case, the range still increases with r_a but very slowly.

Also, we can observe that negative values of plane change are larger than positive values (globally) for any given r_a. In particular, while -180^o plane change are possible for planetary satellites (here Callisto), $+180^o$ plane changes are not necessarily possible for reasonable values of r_a, showing that the potential advantages of third body driven plane changes should be more apparent for negative plane change values.

These results indicate that the maximal range of 3^{rd} body driven plane changes is well approximated by the range obtained when $\pm 180^o$ plane change are realizable. These "change of direction" transfers only involve the planar dynamics and their domain of existence can be computed as a function of the initial conditions r_p and r_a. The results of these computations are shown in Figure 7.

We can note that these transfers correspond to nearly zero angular momentum trajectories when $r_p \to 0$ and even if their possibility appears as a bifurcation in the planar case, Figure 6 shows that these transfers follows from the increasing variations of the range of 3^{rd} body driven plane changes with apoapsis radius.

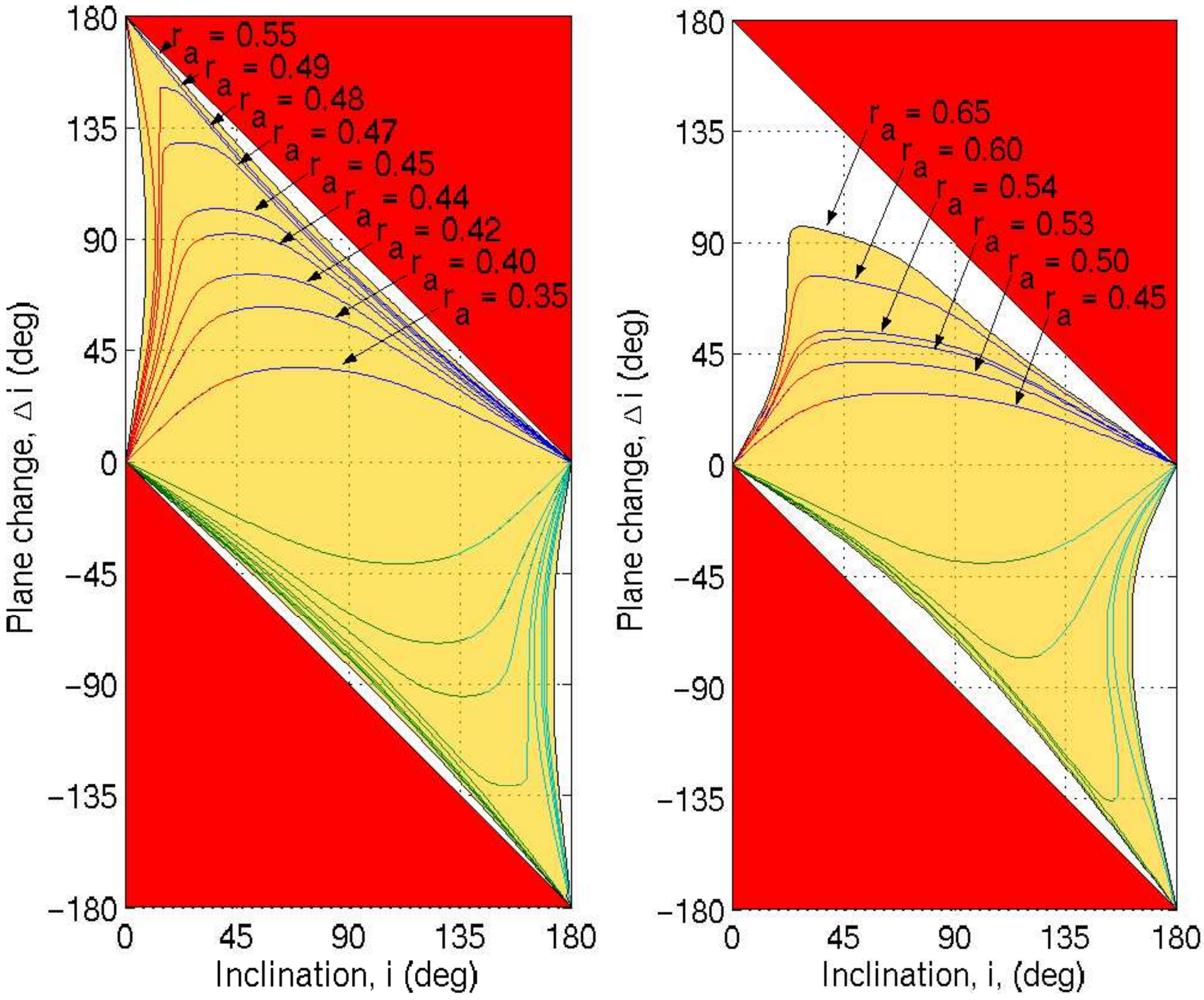

Figure 6: Range of possible third body driven plane changes in the case of the Mars (left) and Callisto (right) orbiters.

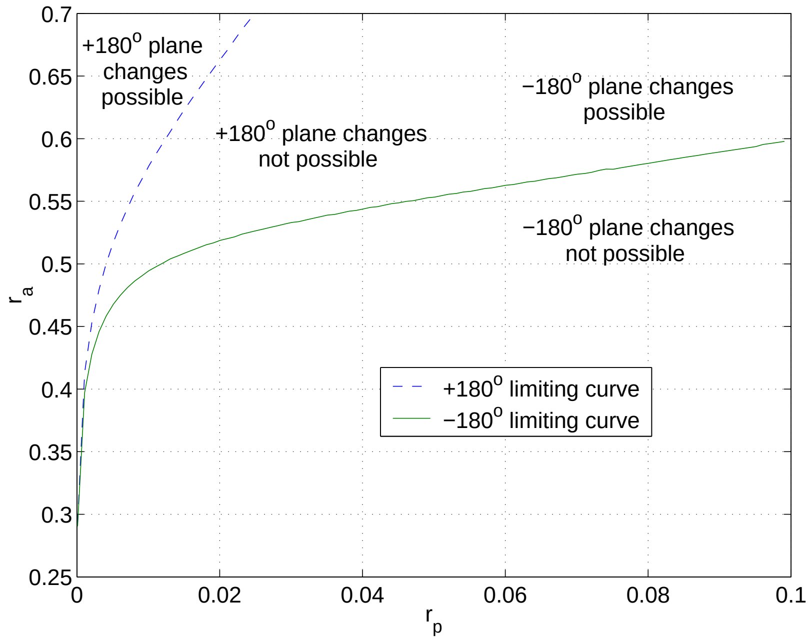

Figure 7: Limits of existence for $\pm 180^o$ plane changes

OPTIMAL PLANE CHANGE STRATEGY

The previous discussion shows that for equatorial or near equatorial orbits, one impulse maneuvers must be used to perform plane change maneuvers. However, in the range of third body driven plane change, one impulse maneuvers may be much more expensive than the third body driven transfers. This section makes explicit the relation between these strategies by deriving analytical estimates of the cost of the third body driven plane changes.

Estimate of the cost of third body driven plane changes

We have seen in the previous section that Hill's problem has a first integral of motion, J. This integral depends on the position and speed of the spacecraft, and the main contribution due to the position in the case of low altitudes depends on the magnitude of the radius vector, r. Thus, in the case of third body driven plane change maneuvers where both impulsive burns are made at a given, low altitude, the Jacobi constant can be used to obtain a sharp estimate of the cost of the maneuver. More precisely, since J is constant along the transfer trajectory, its difference in value between the two successive periapsis passages (initial and final) of the transfer trajectory is zero. Hence, using expression (2) and denoting the initial and final periapsis by the subscripts 1 and 2 respectively, one obtains:

$$\Delta J = J_2 - J_1 = 0 = \frac{1}{2}(V_2^2 - V_1^2) - r_p V_2 \cos(i + \Delta i) + r_p V_1 \cos(i) - \frac{3}{2}(x_2^2 - x_1^2) \quad (5)$$

The term $c = (3/2)(x_2^2 - x_1^2)$ can be bounded independently of ω and Ω:

$$-\frac{3}{2} r_p^2 \cos^2(i) \leq c \leq \frac{3}{2} r_p^2 \cos^2(i + \Delta i)$$

showing its smallness for low initial and final altitudes.

Now, (5) represents a quadratic from in V_2 and can be solved explicitly:

$$V_2 = r_p \cos(i + \Delta i) + \sqrt{r_p^2 \cos^2(i + \Delta i) + V_1^2 - 2 r_p V_1 \cos(i) + 2c}$$

so that, upon substituting the bound on c, one obtains a bound on V_2:

$$V_2^- \leq V_2 \leq V_2^+ \tag{6}$$

where $\begin{cases} V_2^- = r_p \cos(i + \Delta i) + \sqrt{r_p^2 \cos^2(i + \Delta i) - 3 r_p^2 \cos^2(i) + V_1^2 - 2 r_p V_1 \cos(i)} \\ V_2^+ = r_p \cos(i + \Delta i) + \sqrt{4 r_p^2 \cos^2(i + \Delta i) + V_1^2 - 2 r_p V_1 \cos(i)} \end{cases}$

Note that V_1 corresponds to a tangential impulse used to raise the apoapsis of the transfer trajectory and only depends on r_p and r_a:

$$V_1 = \alpha V_{lc} \quad ; \quad \alpha = \sqrt{\frac{2 r_a / r_p}{1 + r_a / r_p}}$$

where $V_{lc} = 1/r$ represents the local circular speed of the spacecraft.

Thus, the bound (6) on V_2 only depends on r_p, r_a, i and Δi and since the cost of the third body driven plane changes is expressed as:

$$\Delta V = V_1 + V_2 - 2 V_{lc}$$

one obtains a bound on this cost, independent of ω and Ω:

$$\Delta V^- \leq \Delta V \leq \Delta V^+ \tag{7}$$

where $\Delta V^{\pm} = V_1 + V_2^{\pm} - 2 V_{lc}$.

Note also that, the error made when using this bound (i.e., $\Delta V^+ - \Delta V^-$) is smaller than $2\sqrt{3/2} r_p$, so that the estimate $\tilde{\Delta V} = (\Delta V^- + \Delta V^+)/2$ has an accuracy of $\pm\sqrt{3/2} r_p$. For low altitude orbiters, this may be sufficient for mission preliminary analysis. For example, in the case of the $200\ km$ altitude Mars orbiter, this bound is accurate to within $\pm 0.47\ m.s^{-1}$ and in the case of the Callisto orbiter, the accuracy is $\pm 13.7\ m.s^{-1}$ (both cases represent less than 1% error on the actual cost).

This result does not give a complete picture of the problem since we don't know a priori what value of plane change Δi is realizable for a given apoapsis radius. However, one can use the previous numerical computations of the ranges to obtain an upper bound on the cost of these transfers.

More precisely, we note that the dependence on i and Δi in the above cost estimate is small and can be bounded:

$$\Delta V \leq V_{lc} \left\{ \alpha - 2 + r_p^{3/2} + \sqrt{(\alpha + r_p^{3/2})^2 + 3r_p^{3/2}} \right\} \tag{8}$$

Now, since the maximal range of third body driven plane changes is approximated by the range obtained at the limit of existence of $\pm 180^o$ plane changes, one can evaluate (8) along the curves shown in Figure 7 to obtain a maximal cost estimate. This is shown on Figure 8.

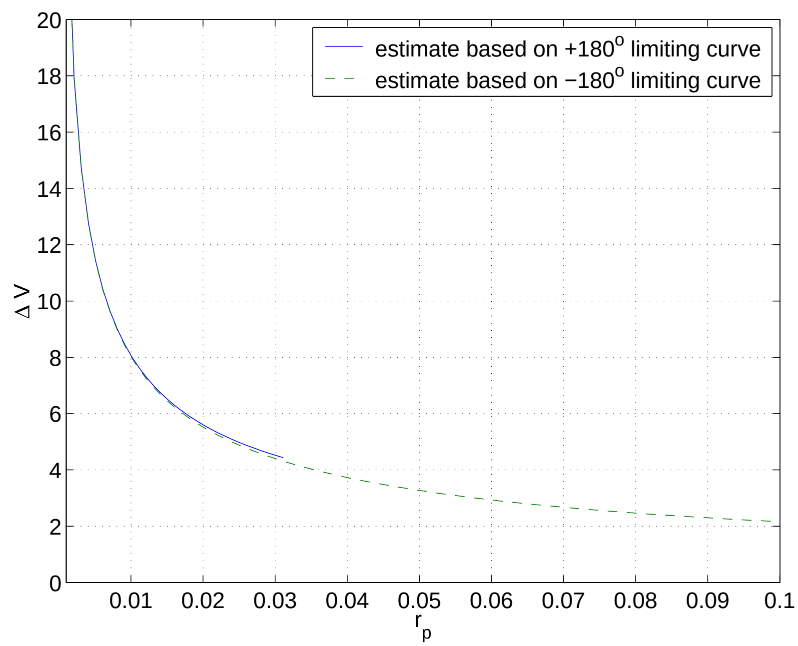

Figure 8: Maximal cost of third body driven plane changes as a function of initial periapsis radius

In the case of the Mars orbiter, this cost estimate is on the order of $1629.6\, m.s^{-1}$ (based on the existence of $+180^o$ plane change limit, i.e., $r_a \simeq 0.49$), while in the case of the Callisto orbiter, this estimate is only $1258.6\, m.s^{-1}$ (based on the existence of -180^o plane change limit, $r_a \simeq 0.539$).

Note that the difference, already observed in section 2, between the case of planetary and planetary satellite orbiters is also apparent here. In the case of a planetary orbiter, this upper estimate is very close to the cost of a parabolic transfer, thus seemingly, not improving on the two body analysis. These results show, however, that the implementation of the plane change must be performed more carefully than indicated by the two body problem analysis, i.e., the timing of the maneuver does matter.

In the case of planetary satellite orbiters, the use of the third body forces is even more apparent, especially for the negative values of plane change. In the case of Callisto, the savings (as compared to the parabolic estimates) are on the order of $126\ m.s^{-1}$. These savings increase with the magnitude of the normalized radius of the planetary satellite. Even though these savings are significant, these results still indicate that, similarly as in the two body problem analysis, large plane change maneuvers are expensive.

Limit of optimality

Using the above cost estimate, an approximation to the limit of optimality between one impulse and third body driven plane changes can be derived. Indeed, since the cost of one impulse maneuvers is known:

$$\Delta V_0 = 2 V_{lc} \sin\left(\frac{|\Delta i|}{2}\right) \tag{9}$$

and the range of 3^{rd} body driven plane changes increases with r_a, we see that 3^{rd} body driven plane changes will be cheaper than one impulse maneuvers, at least when:

$$\Delta V^+(r_{a_{max}}) \leq \Delta V_0$$

where $r_{a_{max}}$ is obtained from the limits shown in Figure 7. As shown previously, one can bound the small dependence on i and Δi in ΔV^+ to obtain an explicit upper bound on the optimality limit:

$$|\Delta i| \geq 2 \arcsin\left\{\frac{\alpha}{2} - 1 + \frac{r_p^{3/2}}{2} + \frac{1}{2}\sqrt{(\alpha + r_p^{3/2})^2 + 3 r_p^{3/2}}\right\} \tag{10}$$

Evaluating this bound on the limiting curves for the existence of $\pm 180^o$ plane changes, one obtains the upper bound on the limit of optimality as shown on Figure 9. That is, for plane change values above the values shown, third body driven plane change are cheaper than one impulse maneuvers, when realizable. This bound is illustrated on the range plots for the cases of the Mars and Callisto orbiters, as shown on Figure 10. Third body driven plane changes are cheaper than one impulse plane change in the range of realizable transfers and outside the shaded region.

As for the cost estimate, this result is very close to the limit of optimality between one impulse and parabolic plane changes in the case of planetary orbiters (very small value of r_p, while being significantly smaller for the case of planetary satellite orbiters (larger r_p). In fact, since for small values of r_p, the dependence on r_a is small, the bound obtained is very accurate. For example, in the case of the Mars orbiter, it can be checked that for $r_a = 0.35$, one impulse plane changes are always optimal and since the variation of (10) with r_a (for larger values of r_a) is less than 0.5^o, we deduce that the bound obtained is

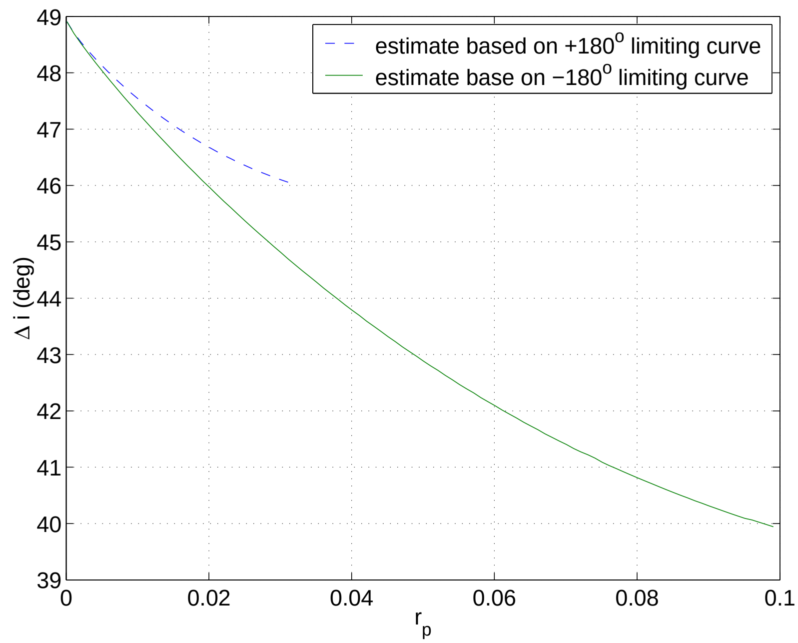

Figure 9: Optimality limit between one impulseand 3^{rd} body driven plane changes

accurate to within this value.

When r_p is larger, the bound obtained is a priori not so accurate and a numerical correction may be necessary. Based upon the previous estimates, one can derive a quick algorithm to compute this limit[6].

Instead of comparing the ΔV's at fixed Δi, one can compare the Δi's realized for a fixed, ΔV. This is possible because of the 1-1 relation between Δi and ΔV_0 on one hand, and the same monotonic dependence of Δi and ΔV on r_a for third body driven plane changes, on the other hand.

Given an initial value Δi_0, we have ΔV_0 given by equation (9). Then we can solve for r_a in $\Delta V^{\pm}(r_a) = \Delta V_0$, yielding two solutions $r_a^{\pm}$:

$$r_a^{\pm} = \frac{(\alpha^{\pm})^2 r_p}{2 - (\alpha^{\pm})^2} \quad ; \quad \alpha^{\pm} = \frac{A^2 - B^{\pm}}{2V_{lc}(A - r_p \cos(i))} \tag{11}$$

$$\text{where} \quad \begin{cases} A = \Delta V_0 - r_p \cos(i + \Delta i) + 2V_{lc} \\ B^+ = 4r_p^2 \cos^2(i + \Delta i) \;\; ; \;\; B^- = r_p^2 \cos^2(i + \Delta i) - 3r_p^2 \cos^2(i) \end{cases}$$

Now, problem (4) can be solved numerically with $r_a^{\pm}$, resulting in $\Delta i^{\pm}$ to be compared with Δi_0.

Note that when $\Delta i^+ \leq \Delta i_0 \leq \Delta i^-$, one needs to solve exactly for ΔV to decide of the optimality by comparing with ΔV_0. In practice, this case covers only a very thin strip

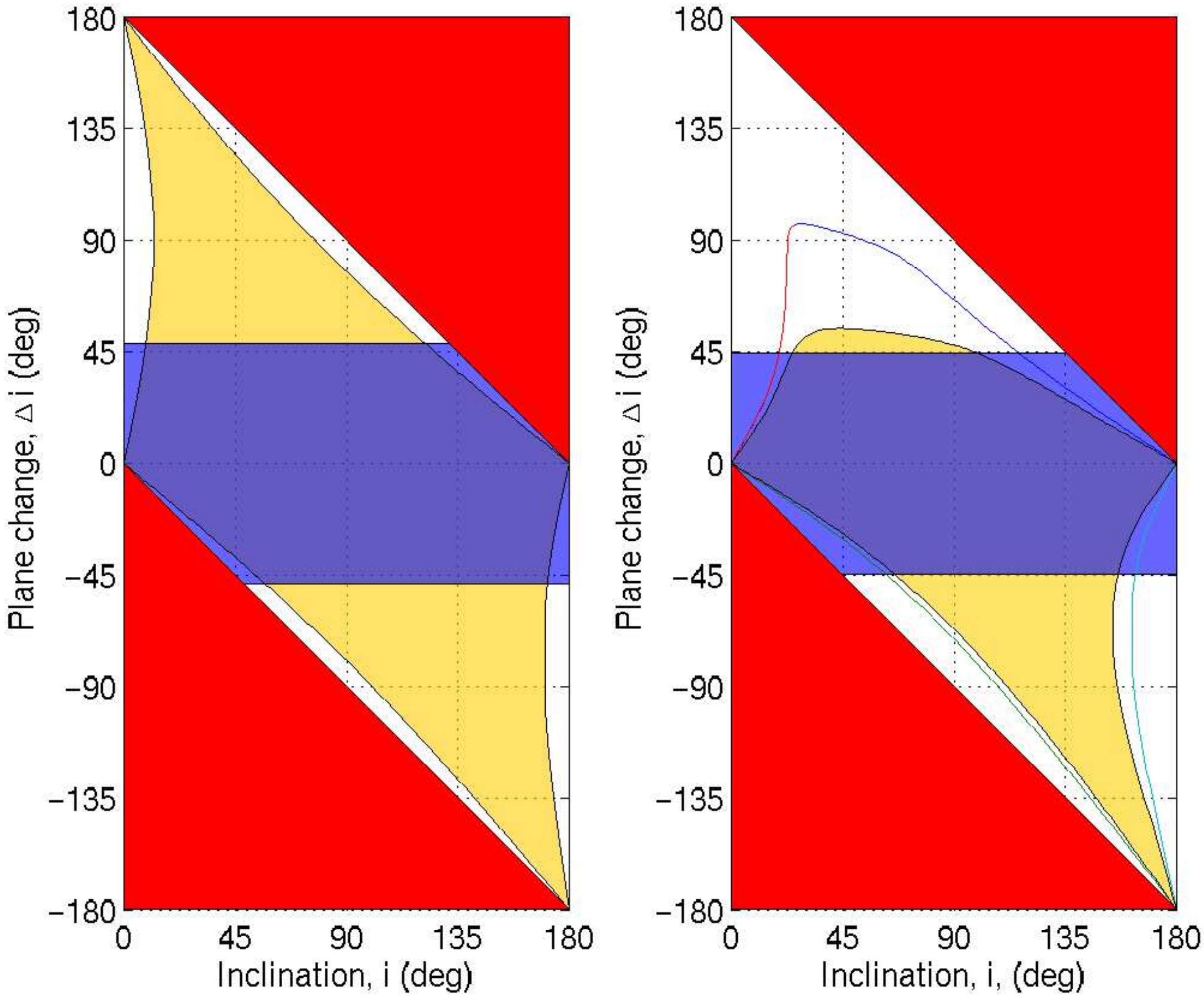

Figure 10: Range of third body driven plane change with optimality limit in the case of the Mars (left) and Callisto (right) orbiters.

of initial conditions and generally, the above algorithm requires only solving problem (4) once or twice per initial conditions to settle the question.

Algorithm[‡]**:**

- Given r_p, i and Δi_0, solve for ΔV_0 using (9).
- Solve for $r_a^\pm$ using the relation $\Delta V^\pm(r_a) = \Delta V_0$.
  (Note that $r_a^+ \leq r_a^-$).
- Solve the optimization problem (4) with r_p, $r_a^\pm$, i to obtain $\Delta i^\pm$.
  (Note that $\Delta i^+ \leq \Delta i^-$).
- compare:
  - If $(\Delta i_0 \geq \Delta i^-)$ Then one impulse transfers are optimal
  - If $(\Delta i_0 \leq \Delta i^+)$ Then tidally driven plane changes are optimal
  - If $(\Delta i^+ \leq \Delta i_0 \leq \Delta i^-)$ Then use a dichotomy on r_a to solve exactly for ΔV

Applying this scheme to the case of Callisto, one can check that the analytical limit (10) is, at most, 1.5^o above the numerically computed values.

CONCLUSION

The optimality of third body driven plane changes has been investigated. In particular, an analytical estimate of the limit of optimality has been derived, showing that third body driven plane changes are indeed less expensive than one impulse maneuvers for a large domain of initial conditions and plane changes larger than $\simeq 40^o$. Limitations of third body driven plane changes have been found because of the restrictions imposed by the natural dynamics. It seems that these limitations could be overcome by using artificial propulsion systems in conjunction with the use of the cost free,third body forces, resulting in further optimality criteria. In particular, the general two impulse case, that is the addition of small plane changes at each impulsive burn of the 3^{rd} body driven plane change maneuvers considered in this article should suffice for this purpose and should be analyzable with the results presented (the only difficulty being the numerical computation of $\partial \Delta V / \partial r_a$, which results in lengthy computational times). Note also that multiple revolution plane changes should be possible, extending the realizable range of third body driven plane changes. The criteria for optimality derived in this paper would apply directly to this situation. However, the practical computation of such transfer becomes more difficult as one has to restrict the domain of initial conditions (in ω and Ω) to avoid impact with the satellite (planet) before

[‡] formulated here for $\Delta i_0 \geq 0$. A similar algorithm can be written for $\Delta i_0 \leq 0$

the desired periapsis is reached. Future research needs to be performed to find an answer to such possibilities.

REFERENCES

1. V.A.Chobotov, *Orbital Mechanics*, 2^{nd} edition, AIAA Education Series, 1996.

2. B.F.Villac, D.J.Scheeres , L.A.D'Amario and M.D.Guman, "The effect of tidal forces in orbit transfers", *AAS/AIAA Spaceflight Mechanics Meeting*, Santa Barbara, CA, 2001.

3. E.Belbruno, J.Miller, "Sun-Perturbed Earth-to-Moon Transfers with Ballistic Capture", *Journal of Guidance, Control and Dynamics*, vol. 16, pp. 770-775.

4. M.W.Lo, B.G.Williams, W.E.Bollman, D.S.Han, Y.S.Hahn, J.L.Bell, E.A.Hirst, R.A.Corwin. P.E.Hong, K.C.Howell, B.Barden and R.Wilson, "Genesis Mission Design", *Journal of the Astronautical Sciences*, vol. 49, No.1, pp. 169-184.

5. B.F.Villac and D.J.Scheeres, "A new class of optimal plane change maneuvers", *Journal of Guidance, Control and Dynamics*, in press.

6. B.F.Villac and D.J.Scheeres, "Optimal plane changes using 3^{rd} body forces", *New Trends in Astrodynamics and Applications - An international conference*, College Park, Maryland, January 2003.

7. B.F.Villac, *Dynamics in the Hill problem with Applications to spacecraft maneuvers*, Ph.D. Dissertation, The University of Michigan, 2003.

8. D.J.Scheeres, M.D.Guman, B.F.Villac, "Stability analysis of planetary satellite orbiters: Application to the Europa Orbiter", *Journal of Guidance, Control and Dynamics*, vol. 24, No. 4, July-August, 2001.

9. V.I. Arnol'd, editor, *Dynamical Systems III*, Springer-Verlag, 1988.