S. Arlinghaus, to appear, Structural Models in Geography.

An application of graph theory to group relationships in history
(source: Williams: Finite Mathematics)

The term transition is often used to describe the return of a system to a balanced situation following a period in which it has been out of balance. One classical transition is the demographer's "demographic transition (Notestein, 1945; Thompson, 1929, 1944). Typically this transition describes a condition of high vital (birth and death) rates, followed by a drop in the death rate during a period in which the corresponding drop in the birth rate lags behind, followed by a drop in the birth rate and the realignment of a low birth rate with a low death rate (Bogue, 1969). The transition is from high vital rates to low vital rates; if the intermediate stage does not lead to eventual low vital rates, then there is no transition.

As we have seen, one need not confine the idea of transition to demography; it extends naturally to a variety of real-world realms (Borchert, 1967; Drake, 1992). For any system to be in some sort of functional balance, the inputs and outputs must be fairly close to each other in number: if the inputs dominate, the system explodes. If the outputs dominate, the system withers. Abstractly, a transition within a system occurs when the input/output level starts in balance, at a high level, experiences a drop in outputs so that the inputs dominate for a period of time, and then returns to a balanced state by a corresponding drop in the input level so that once again the input/output level is in balance. Symmetry promotes systemic stability.

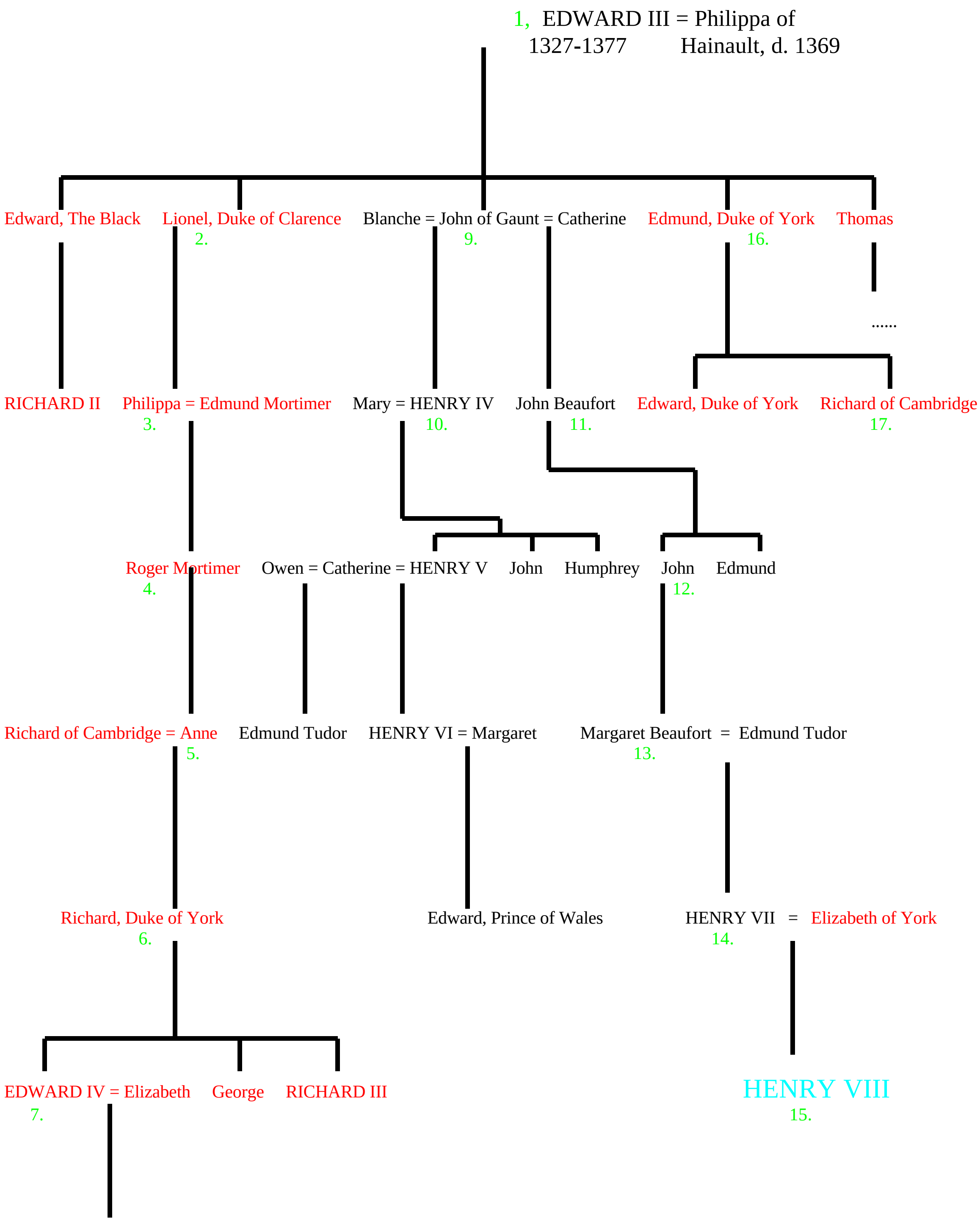
The transition is from the high level of input/output to a low level of input/output. Because the rates generally do not drop evenly, one curve has more area under it than does the other, signifying a period of "boom." What happens during the transition, in the boom time, is critical in determining whether or not the transition is completed and it is in this intermediate stage that so many complexities often arise. Drake (1992) notes this situation in a variety of contexts: from forestry, to education, to environmental toxicity, to a host of others. We consider it here in an historical context: in succession to the British throne.

In the peiord of British history from William the Conqueror (about 1066) to Richard the II (about 1399), the pattern of hereditary succession to the British throne was clear. When Henry IV overthrew Richard II in 1399, the Wars of the Roses, involving issues between the House of York and the House of Lancaster, concerning succession to the British throone, were the result. In 1485, when Henry VII of the House of Lancaster married Elizabeth of the House of York, the pattern of succession once again became clear. An historical transition was achieved, but, in the "boom" time from 1399 to 1485, what was the pattern of the dispute? Structural models, or graphs, offer a way to resolve historical complexity (Luce and Perry, 1968; Williams, 1979).

Generally speaking, a graph is a collection of nodes together with edges linking pairs of nodes. There may be more than one component to a graph. Some graphs are trees. Some graphs have directed edges indicating the direction of flow (digraphs). The subject of graph theory is a very broad one with numerous applications. The classical text in graph theory is by Frank Harary, entitled Graph Theory, published in 1969.

A partial genealogical table of the family relationships, for both the Houses of Lancaster an York, is shown below. This genealogical table can be made into a digraph. Let the relationship linking people be "is the father/mother of." Let each person represent a node. Thus, if person P is related to person Q, draw an edge from P to Q, with the direction pointing from P to Q. Thus, Q is adjacent within the structural model from P (Harary, Norman, and Cartwright, 1965).

We can code this sort of adjacency in a binary matrix: the entry from P to Q described above would be a 1; the entry from Q to P would be a 0. Using this idea of adjacency, based on "is the father/mother of," the structural model of the genealogical table can be expressed as an adjacency matrix focusing only on the set of 17 people noted in the family tree. In this case, adjacency is based on certain key relationships gleaned from historical evidence.



Elizabeth = HENRY VII   EDWARD V   Richard of York
8.

HENRY VIII--15.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1. Edward III		1							1							1	
2. Lionel, Duke of Clarence			1														
3. Philippa				1													
4. Roger Mortimer					1												
5. Anne						1											
6. Richard, Duke of York							1										
7. Edward IV								1									
8. Elizabeth															1		
9. John of Gaunt, Duke of Lancaster										1	1						
10. Henry IV																	
11.John Beaufort, Earl of Somerset												1					
12. John, Duke of Somerset													1				
13. Margaret Beaufort														1			
14. Henry VII															1		
15. Henry VIII																	
16. Edmund, Duke of York																	1
17. Richard, Earl of Cambridge						1											

The matrix A

$$
\begin{matrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}
$$

The matrix A^2

0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	1
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0

A^3

0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0

A^4

0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0

A^5

0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

A^6

0	0	0	0	0	0	1	0	0	0	0	0	0	0	2	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Note that there are three 1s in the first row of the matrix A since Edward III was the father of Lionel, John of Gaunt, and Edmund. Taking powers of the adjacency matrix, A, counts the number of paths through the genealogical hierarchy. The power A^2 counts the number of paths of length two--grandparent/grandchildren relationships. In A^2 there are 1s in the third, tenth, eleventh, and seventeenth columns, indicating that Edward III was the grandfather of Philippa, Henry IV, John Beaufort, and Richard. These observations tally with the family tree. Generally, the matrix A is the "parent" matrix; the matrix A^2 is the grandparent matrix; the matrix A^3 is the great-grandparent matrix, and so forth. To find the previous generation of a given individual, focus on the columns. Thus, Richard Duke of York's (number 6) ancestors can be read directly from each matrix by looking down column 6.

The fifth power of the adjacency matrix, A^5, has a value of 1 in the (1,14) position of the matrix (first row, Fourteenth column), showing that there is one path of length five between Edward III and Henry VIII; that
Henry VIII was descended directly from Edward III through five Lancaster generations. Because a value of 1 is recorded in this position for the first time in the fifth power matrix, we know that it is exactly five (and not fewer) generations for the descent.

The fifth power of the adjacency matrix shows an entry of 1 in the (1,8) position reflecting the fact that Elizabeth is descended over five York generations from Edward III.

The sixth power matrix has an entry of 2 in the (1,15) entry--for
Henry VIII, the son of Henry VII of Lancaster and Elizabeth of York. Both Elizabeth and Henry are descended from Richard III over the same number of generations:
a 1 from each line
of descent. Thus, Henry VIII, the son of Henry VII and Elizabeth combined the claim of both the Houses of Lancaster and York to the British throne. Historical complexity, that can occur in the "boom" time within a transition--between symmetric periods of stability, is resolved easily using adjacency matrices of structural models.

(Note that Richard, Duke of York is descended over three generations from the male York side, and five generations from the female York side: in A^3 and A^5 there are entries in the first row in the sixth column.)

Reference
:
R. Duncan Luce and Albert D. Perry, "A Method of Matrix Analysis of Group Structure." Readings in Mathematical Social Science, ed. Paul F. Lazarsfeld and Neil W. Henry Cambridge, MIT Press, 1968.