**Name:** **Date:**

# Problem Set # 6

***Instructions:*** *Numerical answers require units and appropriate numbers of significant digits.* ***Remember to show your work!***

**C-1** (12 points) In the 1800's it was recognized that temperatures in mines and caves increased with depth. Listed below are some measurements taken in a rather deep mine. Assume steady-state equilibrium:

a. Graph the temperature versus depth using a computer (EXCEL spreadsheet or another programming software).

b. Using a computer "tend line" (or least square) function, find the best-fit straight line through your points and determine the thermal gradient from the slope.

c. Determine the average thermal conductivity from the values given.

d. Use the results from (b) and (c) to determine the heat flux through the surface in this region.

e. How many milliWatts per square meter are there in 1 HFU

f. Convert the result from (d) into $\dfrac{\text{calories}}{\text{cm}^2 \times \text{s}}$ and also convert it into HFU.

| Depth (km) | Temperature (°C) | Conductivity (W/m°C) |
|---|---|---|
| 0.0 | 12.0 | 2.46 |
| 0.5 | 24.0 | 2.5 |
| 1.0 | 37.0 | 2.47 |
| 1.5 | 49.0 | 2.50 |
| 2.0 | 62.5 | 2.45 |
| 2.5 | 75.0 | 2.49 |
| 3.0 | 86.5 | 2.61 |
| 3.5 | 100.0 | 2.59 |
| 4.0 | 111.0 | 2.57 |

**Recall:**

1 Watt = 1 Joule/s , 1 Calorie = 4.2 Joules

1 Heat Flow Unit (HFU) = $10^{-6}$ calories/cm$^2$s

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**1.** (10 points) Let's use the heat equation to find an expression relating temperature to depth within the Earth. Assuming a steady-state equilibrium (e.g. no change with respect to time), $\partial T/\partial t = 0$, so the heat equation becomes,

$$k\frac{d^2T}{dy^2} + \rho H = 0 \tag{1}$$

Given that we know the heat flux ($q = -q_s$) through the Earth's surface and the temperature ($T_s$) at the surface as well,

a. Integrate the heat equation with respect to $y$ to determine the heat flow as a function of depth. Recall that you will need to use the surface heat flux to solve for the constant of integration.

b. Integrate the result from (a) to determine the relation of temperature with depth inside the Earth to show that,

$$T(y) = T_s + \frac{q_s}{k}y - \frac{\rho H}{2k}y^2 \tag{2}$$

Again, you will need to use the surface temperature to evaluate the constant of integration.

**C-2** (16 points) Use Eq. (2) from problem 2b, and the values below to

a. Determine the temperature profile of the upper 210 km within the Earth. To do this, use the spreadsheet which you will get from

`http://www.earth.northwestern.edu/~amir/202/PS7_2_C.xls`.

Calculate the temperature values under "geothermal profile" and plot the profile *on the same graph* in the spreadsheet.

$$q_s = 70 \text{ mW/m}^2 \qquad \rho = 3.0 \text{ g/cm}^3 \qquad H = 10^{-10} \text{ W/kg}$$

$$k = 4.0 \text{ W/m.°C} \qquad T_s = 10 \text{ °C}$$

b. Also shown on the graph are the liquidus (the line separating the liquid phase [or magma] of a rock from the partially molten phase) and the solidus (the line separating the solid phase of the rock from the partially molten phase), for a peridotite

   i. Determine the depth where the mantle starts to melt.

   ii. Does this depth correspond with any other observational attribute of the mantle?

   iii. At what depth is the mantle entirely liquid?

   iv. What observations tell us that the mantle does not heat up to such temperatures?

   v. How is such melting prevented by the mantle?

c. Using the parameters listed in part (a), find the temperature in the crust at 10, 15, and 25 km by varying your assumption about the heat generation in crustal rocks:

   i. Assume $H = 0$; no heat is generated in the rock

   ii. Assume $H = 10^{-9}$ W/kg

**2.** (12 points) For a planet with radius a and surface temperature zero, temperature as a function of radius is given by,

$$T(r) = \frac{\rho H(a^2 - r^2)}{6k} \tag{3}$$

where $\rho$ is density, $H$ is heat production, and $k$ is thermal conductivity.

a. Show that this solution satisfies the heat equation for a spherical planet

$$\frac{k}{r^2}\frac{d}{dr}\{r^2\frac{dT}{dr}\} + \rho H = 0 \tag{4}$$

b. Assuming that $k = 4.0$ W/m°C, $H = 10^{-10}$ W/kg, $\rho = 5.5$ g/cm$^3$ find the temperature at the centers of planets with radii 20 km, 200 km, 2000 km. (Be careful with units!)

c. How large must such a planet be before it melts in the center, assuming melting occurs at 1000°C ?

**3.** (6 points) From the heat equation, we could find that the time $t$ needed for heat to conduct a distance $y$ is given by $\sqrt{\kappa t}$, where $\kappa$ is the thermal diffusivity. Assume $\kappa = 10^{-6}$ $m^2/s$. Find the conductive cooling time (in years) for material of thickness

a. 6 meter

b. 6 kilometer

c. 6371 kilometers

The last time is approximately the time needed for the earth to cool down by conduction alone. How does it compare to the age of the earth?

**4.** (5 points) Estimate the total energy in ergs per year flowing from the Earth's interior, assuming a surface heat flow of 1 HFU (1 HFU = 4.184 $\times 10^{-2}$ W/$m^2$). Compare this to the total annual energy per year from solar radiation, assuming an incident solar flux at the Earths orbit of 1.36 $\times 10^3$ W/$m^2$.

**5.** (4 points) Determine the Rayleigh number as a function of $\Delta$T for a solid mantle. Use the following physical parameters,

$$\alpha = 2.0 \times 10^{-5}\ {}^\circ\mathrm{C}^{-1} \qquad g = 10\ \mathrm{m/s^2} \qquad \rho = 3.4 \times 10^3\ \mathrm{kg/m^3}$$

$$\kappa = 10^{-6}\ \mathrm{m^2/s} \qquad \eta = 10^{21}\ \mathrm{kg/ms}$$

Under what conditions would you expect the solid rock in the mantle to convect?