A CHARACTERIZATION OF 2-TRIVIAL BANACH SPACES WITH UNCONDITIONAL BASIS

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Let X be a Banach space with an unconditional basis such that each operator from X into ℓ^2 is 2-absolutely summing. Then X is isomorphic either to c_0 or to ℓ^1 or to $c_0 \oplus \ell^1$.

A Banach space X is said to be 2-trivial if every linear operator, acting from the space X into a Hilbert space, is 2-absolutely summing. Let Y and Z be infinite dimensional Banach spaces. It is known that if for some p, $1 \leq p < \infty$, the space of p-absolutely summing operators $\Pi_p(Y,Z)$ coincides with the space of all linear operators $L(Y,Z)$, then the space Y is 2-trivial [1]. The properties of 2-trivial spaces have been considered in the survey [2] (instead of the term "2-trivial space" the term "Hilbert-Schmidt space" is used).

The fundamental result of the present paper is the proof of one of the conjectures regarding the structure of 2-trivial spaces, formulated in [1].

THEOREM 1. A 2-trivial Banach space with an unconditional basis is isomorphic to one of the following spaces: $c_0, \ell^1, c_0 \oplus \ell^1$.

Obviously, for the proof of Theorem 1 the basis can be assumed to be normalized and 1-unconditional.

Some definitions and notations. Everywhere in the sequel X is a 2-trivial space with a normalized 1-unconditional basis $\{e_n\}_{n=1}^{\infty}$, Q is the norm of the canonical isomorphism between the spaces $L(X,\ell^2)$ and $\Pi_2(X,\ell^2)$, while π_2 is the norm in the space $\Pi_2(X,\ell^2)$. If Y and Z are Banach spaces, then $d(Y,Z)$ is the Banach–Mazur distance between the spaces Y and Z, $S(Y)$ is the unit sphere of the space Y, $k_2(Y) \overset{def}{=} \max\{\dim V : V \subset Y, d(V, \ell^2_{\dim V}) \leq 2\}$ (see [3]).

Let $\mathcal{A}$ be some set. We say that the families of vectors $\{y_i\}_{i\in\mathcal{A}}$, $\{y_i\}_{i\in\mathcal{A}} \subset Y$, and $\{z_i\}_{i\in\mathcal{A}}$, $\{z_i\}_{i\in\mathcal{A}} \subset Z$ are C-equivalent if there exist numbers a and b such that $ab \leq C$ and for any finite subset B of the set $\mathcal{A}$ and for any collection of scalars $\{\lambda_i\}_{i\in B}$ we have the inequality

$$a^{-1}\left\| \sum\nolimits_{i\in B} \lambda_i y_i \right\| \leq \left\| \sum\nolimits_{i\in B} \lambda_i z_i \right\| \leq b \left\| \sum\nolimits_{i\in B} \lambda_i y_i \right\|.$$

By the letters I, J (possibly with indices) we denote finite subsets of the set $\mathbb{N}$, $|I|$ is the cardinality of the set I.

The smallest number C for which the sequence $\{e_i\}_{i\in I}$ is C-equivalent to the standard basis of the space $\ell^p_{|I|}$, $p=1;\infty$, is denoted by $D_p(I)$ We set

$$X_I \overset{def}{=} \operatorname{span}\{e_i : i \in I\},\ \lambda_p(I,S) \overset{def}{=} \max\{|J| : J \subset I,\ D_p(J) \leq S\}.$$

By $E(a)$ we denote the integer part of the number a.

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We start with some auxiliary statements. First of all we present a result from I. A. Komarchev's dissertation [4]. For the sake of the completeness of the presentation, we include the proof.

LEMMA 1. Let I be a finite set of natural numbers, let $\{f_i\}_{i\in I}$ be an orthonormal basis of the space $\ell^2_{|I|}$, and let $\{e_i^*\}_{i\in I}$ be a sequence in the space X_I^*, biorthogonal to the basis $\{e_i\}_{i\in I}$. Let $S_I : X_I \to \ell^2_{|I|}$ be the linear operator defined by the formula $S_I x = \Sigma_{i\in I} < x, e_i^* > f_i$. Then

1) $\mathcal{D}_1(I) \leqslant Q^2 \| S_I \|^2$;

2) $\lambda_\infty(I, 2Q^2) \geqslant \frac{\| S_I \|^2}{2Q^2}$.

Proof. Let α_i $(i \in I)$ be arbitrary real numbers and let $y_i = \sqrt{|\alpha_i|}\, e_i$. Making use of the definition of a 2-absolutely summing operator, we obtain

$$\Sigma_{i\in I} \| S_I y_i \|^2 \leqslant \pi_2^2(S_I) \sup\{ \Sigma_{i\in I} < y_i, e^* >^2 : e^* \in S(X_I^*)\} \leqslant$$
$$\leqslant \pi_2^2(S_I) \sup\{ \Sigma_{i\in I} |\alpha_i| \cdot | < e_i, e^* > | : e^* \in S(X_I^*)\}.$$

From the 2-triviality of the space X there follows that $\pi_2(S_I) \leqslant Q \| S_I \|$. Introducing this estimate into the previous inequality and taking into account that the basis $\{e_i\}_{i\in I}$ is I-unconditional, we obtain the inequality

$$\Sigma_{i\in I} |\alpha_i| = \Sigma_{i\in I} \| S_I y_i \|^2 \leqslant Q^2 \| S_I \|^2 \sup\{ \Sigma_{i\in I} |\alpha_i| \cdot | < e_i, e^* > | : e^* \in S(X_I^*)\} \leqslant Q^2 \| S_I \|^2 \cdot \| \Sigma_{i\in I} \alpha_i e_i \|. \tag{1}$$

Part 1) is proved.

We consider a point y on the sphere $S(X_I)$ such that $\| S_I y \| = \| S_I \|$. We set $\alpha_i = < y, e_i^* >$. Making use of the inequality (1), we obtain

$$\Sigma_{i\in I} | < y, e_i^* > | \leqslant Q^2 \| S_I \|^2 \cdot \| \Sigma_{i\in I} < y, e_i^* > e_i \| = Q^2 \| S_I \|^2 \cdot \| y \| = Q^2 \| S_I \|^2.$$

We denote by $\mathcal{J}$ the set of all indices i, $i \in I$ satisfying the condition $| < y, e_i^* > | \geqslant \frac{1}{2Q^2}$. We assume that $|\mathcal{J}| < \frac{\| S_I \|^2}{2}$. We have $\| S_I \|^2 = \| S_I y \|^2 = \Sigma_{i\in \mathcal{J}} < y, e_i^* >^2 + \Sigma_{i\in I \setminus \mathcal{J}} < y, e_i^* >^2 \leqslant |\mathcal{J}| + \Sigma_{i\in I\setminus \mathcal{J}} < y, e_i^* >^2$, i.e., $\| S_I \|^2 < 2\, \Sigma_{i\in I\setminus \mathcal{J}} < y, e_i^* >^2$. Thus, we have obtained the absurd inequality

$$\sum_{i\in I} | < y, e_i^* > | \leqslant Q^2 \| S_I \|^2 < 2Q^2 \sum_{i\in I\setminus \mathcal{J}} < y, e_i^* >^2 < \sum_{i\in I\setminus \mathcal{J}} | < y, e_i^* > |.$$

Consequently, $|\mathcal{J}| \geqslant \frac{\| S_I \|^2}{2}$. From the definition of the set $\mathcal{J}$ there follows that for any collection of real numbers $\{\alpha_i\}_{i\in \mathcal{J}}$ we have the inequality

$$\max_{i\in \mathcal{J}} |\alpha_i| \leqslant \| \sum_{i\in \mathcal{J}} \alpha_i e_i \| \leqslant 2Q^2 \max_{i\in \mathcal{J}} |\alpha_i| \cdot \| \sum_{i\in \mathcal{J}} | < y, e_i^* > | e_i \| \leqslant 2Q^2 \max_{i\in \mathcal{J}} |\alpha_i|. \bullet$$

Applying Lemma 1 to the operator $(S_I^*)^{-1}$, we obtain

COROLLARY 1. $\mathcal{D}_\infty(I) \leqslant Q^2 \| S_I^{-1} \|^2$; $\lambda_1(I, 2Q^2) \geqslant \frac{\| S_I^{-1} \|^2}{2Q^2}$. $\bullet$

COROLLARY 2. $\lambda_\infty(I, 2Q^2) \geqslant \frac{1}{2Q^4} \mathcal{D}_1(I)$; $\lambda_1(I, 2Q^2) \geqslant \frac{1}{2Q^4} \mathcal{D}_\infty(I)$. $\bullet$

LEMMA 2. There exists a positive number ρ such that for every n-element subset I of the set $\mathbb{N}$ we have the inequalities

1) $d(X_I, \ell_n^2) \geq \rho\sqrt{n}$,

2) $k_2(X_I) \geq \rho n$ or $k_2(X_I^*) \geq \rho n$.

Proof. We assume that $k_2(X_I^*) \leq k_2(X_I)$. By the Figiel–Lindenstraus–Milman theorem ([3], Theorem 2.9), there exist a positive number δ, a subspace Y in X_I^* of dimension $k_2(X_I^*)$ and a projection P from X_I^* onto Y such that $d(Y, \ell^2_{\dim Y}) \leq 2$ and

$$k_2(X_I)k_2(X_I^*) \geq \delta \|P\|^2 \frac{n^2}{d^2(X_I, \ell_n^2)} \quad . \tag{2}$$

Let T be an isomorphism between the spaces Y and $\ell^2_{\dim Y}$, satisfying the condition $\|T\|\cdot\|T^{-1}\| \leq 2$, let id be the identity imbedding of the space Y into X_I^*, and let $\{f_m\}_{m=1}^{\dim Y}$ be an orthonormal basis of the space $\ell^2_{\dim Y}$. From the 2-triviality of the space X^*† there follows that

$$\dim Y = \sum_{m=1}^{\dim Y} \|f_m\|^2 \leq \pi_2^2(TPidT^{-1}) \sup\{\sum_{m=1}^{\dim Y} \langle f_m, f^*\rangle^2 : \|f^*\| = 1\} \leq$$
$$\leq \pi_2^2(TP)\cdot\|T^{-1}\|^2 \leq 4Q^2\|P\|^2.$$

Introducing into (2) the obtained estimate for $k_2(X_I^*)$ we can write

$$k_2(X_I)d^2(X_I, \ell_n^2) \geq \frac{\delta}{4Q^2} n^2. \tag{3}$$

Combining the estimate (3) and the inequalities $k_2(X_I) \leq n$ and $d(X_I, \ell_n^2) \leq \sqrt{n}$, we obtain the statements 1) and 2), respectively. ●

LEMMA 3. There exists a positive number σ, such that for any finite subset I of the set $\mathbb{N}$ we have the inequality

$$\lambda_1(I, 2Q^2)\cdot\lambda_\infty(I, 2Q^2) \geq \sigma|I|.$$

Proof. Let S_I be the linear operator defined in Lemma 1. According to Lemma 1 and Corollary 1, we have the inequalities $\lambda_\infty(I, 2Q^2) \geq \frac{\|S_I\|^2}{2Q^2}$ and $\lambda_1(I, 2Q^2) \geq \frac{\|S_I^{-1}\|^2}{2Q^2}$. Applying Lemma 2, we obtain

$$\lambda_1(I, 2Q^2)\cdot\lambda_\infty(I, 2Q^2) \geq \frac{1}{4Q^4}\|S_I\|^2\cdot\|S_I^{-1}\|^2 \geq \frac{1}{4Q^4} d^2(X_I, \ell^2_{|I|}) \geq \frac{\rho^2}{4Q^2}|I|. \bullet$$

LEMMA 4. Let κ, m, n, p be natural numbers satisfying the conditions $\kappa \leq m$, $\kappa \leq n$, $p^2 \leq m$; let $I = \{1 \ldots n\}$. Let $\{I_\ell\}_{\ell=1}^m$ be a collection of subsets of the set I of power κ such that for any p mutually distinct indices ℓ_j the set $\cap_{j=1}^p I_{\ell_j}$ consists of at most p elements. Then $n > \frac{1}{2}\kappa^{1+\frac{1}{2p}}$.

Proof. Let μ be the measure on I that associates to each element of the set I a unit charge and let γ_ℓ be the characteristic function of the set I_ℓ.

†The 2-triviality of the space X is equivalent with the 2-triviality of the space X^* (see [2]).

Let $\mathcal{A} = \int_I (\sum_{\ell=1}^{m} \gamma_\ell)^{m_1} d\mu$ where $m_1 \in \mathbb{N}$. By Hölder's inequality, we have

$$(mk)^{m_1} = \Big(\int_I \big(\sum_{\ell=1}^{m} \gamma_\ell\big) d\mu\Big)^{m_1} \leq n^{m_1 - 1} \cdot \mathcal{A}\,. \tag{4}$$

On the other hand,

$$\mathcal{A} = \sum_{\alpha \in \mathcal{O}} \int_I \prod_{j=1}^{m_1} \gamma_{\alpha(j)} d\mu = \sum_{\alpha \in \mathcal{O}_1} \int_I \prod_{j=1}^{m_1} \gamma_{\alpha(j)} d\mu + \sum_{\alpha \in \mathcal{O}_2} \int_I \prod_{j=1}^{m_1} \gamma_{\alpha(j)} d\mu\,, \tag{5}$$

where $\mathcal{O}$ is the set of all mappings $\alpha:\{1\ldots m_1\} \to \{1 \ldots m\}$, $\mathcal{O}_1$ is the set of those mappings from $\mathcal{O}$ whose images have a cardinality not exceeding p, $\mathcal{O}_2 = \mathcal{O} \setminus \mathcal{O}_1$. If $\alpha \in \mathcal{O}_2$, then $\int_I \prod_{j=1}^{m_1} \gamma_{\alpha(j)}\, d\mu \leq p$. Obviously, $|\mathcal{O}| = m^{m_1}$ и $|\mathcal{O}_1| \leq \binom{m}{p} p^{m_1} \leq m^p \cdot p^{m_1}$. Introducing these inequalities into (5) and taking into account the estimate (4) we obtain

$$(mk)^{m_1} n^{1-m_1} \leq \mathcal{A} \leq k|\mathcal{O}_1| + p|\mathcal{O}_2| \leq k|\mathcal{O}_1| + p|\mathcal{O}| \leq k m^p p^{m_1} + p m^{m_1}\,.$$

We set $m_1 = 2p+1$. Then, since $p^2 \leq m$ the previous estimate gives $k^{2p+1} n^{-2p} \leq k p^{2p+1} m^{-p-1} + p \leq p(p^{2p} m^{-p} + 1) \leq 2p$. Consequently, $n \geq \left(\frac{1}{2p}\right)^{\frac{1}{2p}} \cdot k^{1+\frac{1}{2p}} > \frac{1}{2} k^{1+\frac{1}{2p}}$. ●

LEMMA 5. Let $k, n \in \mathbb{N}$ and assume that for all $i, 1 \leq i \leq n$, there are given the numbers m_i and τ_i, $m_i \in \mathbb{N}$, $\tau_i \in \mathbb{R}$, $\tau_i > 0$. Assume that the operators $T_i : \ell_k^2 \to \ell_{m_i}^\infty$ are such that $\|T_i\| \leq 1$ for all i and for any x from ℓ_k^2 there exists an index i for which we have the inequality $\|T_i x\| \geq \frac{1}{\tau_i}\|x\|$. Then $k \leq \gamma \max\{\tau_i^2 \log 2nm_i : 1 \leq i \leq n\}$ with some absolute constant γ.

The proof of the lemma is similar to the proof of Proposition 3.2 of [3]. Let $\{g_{ij}\}_{j=1}^{m_i}$ be the standard basis of the space $\ell_{m_i}^1$, $y_{ij} \overset{\text{def}}{=} T_i^* g_{ij}$. For any i and for any x from the set S_i, $S_i \overset{\text{def}}{=} \{x \in S(\ell_k^2) : \|T_i x\| \geq \frac{1}{\tau_i}\}$, we have the inequality $\max_j |\langle x, y_{ij}\rangle| = \max_j |\langle T_i x, g_{ij}\rangle| = \|T_i x\| \geq \frac{1}{\tau_i}$, from where there follows that $\min_j \min\{\|x - \frac{1}{\tau_i} y_{ij}\|^2; \|x + \frac{1}{\tau_i} y_{ij}\|^2\} \leq 1 - \tau_i^{-2}$. Thus, the points $\pm\frac{1}{\tau_i} y_{ij}$ form a $(\sqrt{1-\tau_i^{-2}})$-net for the set S_i. Therefore, $2m_i$ balls with centers at the points $\pm\frac{1}{\tau_i} y_{ij}$ and radii $1+\sqrt{1-\tau_i^{-2}}$ cover the set $[0;2]\cdot S_i \overset{\text{def}}{=} \{rx : r \in [0;2], x \in S_i\}$. If mes is the normalized Lebesgue measure on the sphere $S(\ell_k^2)$ then, due to $\bigcup_{i=1}^n S_i = S(\ell_k^2)$, for some i_0 we have the inequality $mes S_{i_0} \geq \frac{1}{n}$. Comparing the volume of the set $[0;2]\cdot S_{i_0}$ with the volume of its covering balls, we obtain the estimate $\frac{1}{n} 2^k \leq 2^k mes S_{i_0} \leq 2m_{i_0}(1+\sqrt{1-\tau_{i_0}^{-2}})^k$, which, after simple transformations, gives

$$\frac{1}{k}\log(2m_{i_0} n) \geq \log\frac{2}{1+\sqrt{1-\tau_{i_0}^{-2}}} \geq \frac{1}{\gamma\tau_{i_0}^2}\,. \quad ●$$

Now we can obtain the following statement, necessary for the proof of Theorem 1, but apparently, of interest also in its own right.

THEOREM 2. Let X_0 be a Banach space with a normalized 1-unconditional basis $\{e_{ij}\}_{i,j=1}^n$. If for every i the sequence $\{e_{ij}\}_{j=1}^n$ is $\mathcal{A}$-equivalent to the standard basis of the space ℓ_n^∞ while for every j the sequence $\{e_{ij}\}_{i=1}^n$ is $\mathcal{A}$-equivalent to the standard basis of the space ℓ_n^1, then we have the inequality $k_2(X_0) \leq c\mathcal{A}^{4/3} n^{5/3} \log n$ with some absolute constant c.

Proof. First we present the outline of the proof. We divide the sphere $S(X_0)$ into two sets S_1 and S_2. Then we extract from the sphere $S(X_0^*)$ the subsets $\mathcal{X}_1^*$ and $\mathcal{X}_2^*$ of cardinalities

m_1 and m_2, respectively, possessing the following properties:

1) $\log m_1 \le c_1 n \log n$ with some absolute constant c_1;

2) for any x from the set S_1 we have the inequality $\sup\{<x,x^*>: x^* \in \mathfrak{X}_1^*\} \ge \frac{1}{2} A^{-2/3} n^{-1/3}$;

3) $\log m_2 \le c_2 A^{4/3} n^{5/3} \log n$ with some absolute constant c_2;

4) for any x from the set S_2 we have the inequality $\sup\{<x,x^*>: x^* \in \mathfrak{X}_2^*\} \ge \frac{1}{4}$.

After the indicated objects have been constructed, we define the operators $T_i, T_i: X_0 \to \ell^\infty_{m_i}$, $i=1;2$, in the following manner: $T_i x \overset{def}{=} \{<x,x^*>\}_{x^* \in \mathfrak{X}_i^*}$. Let $\kappa = k_2(X_0)$ and let T be an imbedding of the space ℓ^2_κ in the space X_0 such that $\frac{1}{2}\|y\| \le \|Ty\| \le \|y\|$ for all y from the space ℓ^2_κ. Applying Lemma 5 to the operators $T_1 \circ T$ and $T_2 \circ T$ we obtain the assertion of the theorem. (Indeed, for each y from ℓ^2_κ the vector $Ty/\|Ty\|$ is either in S_1 or in S_2, so that, by properties 2) and 4), either $\|T_1 \circ Ty\| \ge \frac{1}{4} A^{-1/3} n^{-2/3} \|y\|$ or $\|T_2 \circ Ty\| \ge \frac{1}{8}\|y\|$. We leave to the reader to conclude the computation.)

We proceed to the construction of the sets $S_1, S_2, \mathfrak{X}_1^*, \mathfrak{X}_2^*$. We set $X_j \overset{def}{=} \operatorname{span}\{e_{ij}: 1 \le i \le n\}$ let P_j be the canonical projection of the space X_0 onto X_j; let S_1 be the subset of the sphere $S(X_0)$ consisting of those x, for which there exists an index j satisfying the condition $\|P_j x\| \ge a = A^{-2/3} n^{-1/3}$; $S_2 \overset{def}{=} S(X_0) \setminus S_1$. For every x from the set S_1 there exist an index j and a functional x^* from the sphere $S(X_j^*)$ such that $<x, P_j^* x^*> = <P_j x, x^*> = \|P_j x\| \ge a$.

By Lemma 2.4 of [3], in the set $S(X_j^*)$ there exists an $(\frac{a}{2})$-net of cardinality at most $(1+\frac{4}{a})^n$. Taking the union of these $(\frac{a}{2})$-nets, we obtain a set $\mathfrak{X}_1^*$ of cardinality at most $n(1+\frac{4}{a})^n$, such that for any x from the set S_1 there exists x^* from the set $\mathfrak{X}_1^*$, for which $<x, x^*> \ge \frac{a}{2}$. The set $\mathfrak{X}_1^*$ satisfies the conditions 1) and 2).

Assume now that $x \in S_2$. Let $\{e^*_{ij}\}_{i,j=1}^n$ be the basis in the space X_0^*, dual to the basis $\{e_{ij}\}_{i,j=1}^n$. If the functional x^* has in this basis the coordinates x^*_{ij} satisfying the condition $|x^*_{ij}| < b = \frac{1}{2} A^{-1/3} n^{-2/3}$ for all i and j, then

$$|<x,x^*>| \le \sum_{j=1}^n \|P_j x\| \cdot \Big\|\sum_{i=1}^n x^*_{ij} e^*_{ij}\Big\| \le A a \sum_{j=1}^n \|\{x^*_{ij}\}_{i=1}^n\|_{\ell^\infty_n} < A a b n = \frac{1}{2}. \tag{6}$$

Let $\mathcal{B}$ be the set of all subsets of the set $\{1,\dots,n\}$ of power $m = E(Ab^{-1})$. For $B_1 \dots B_n \in \mathcal{B}$ we set $X^*_{B_1 \dots B_n} \overset{def}{=} \operatorname{span}\{e^*_{ij}: j \in B_i, 1 \le i \le n\}$. We prove that

$$\sup\{<x,x^*>: x^* \in \bigcup_{B_1 \dots B_n \in \mathcal{B}} X^*_{B_1 \dots B_n}, \|x^*\| \le 1\} \ge \frac{1}{2}. \tag{7}$$

Let $\bar{x}^* \in S(X_0^*)$, $<x, \bar{x}^*> = 1$. We denote by b_i the set of all those indices j for which $|\bar{x}^*_{ij}| \ge b$ and we set $x^* = \sum_{i=1}^n \sum_{j \in b_i} \bar{x}^*_{ij} e^*_{ij}$.

From the inequality (6) there follows that $<x,x^*> \ge \frac{1}{2}$. For any index i we have the inequality

$$|b_i| = |\{j: |\bar{x}^*_{ij}| \ge b\}| \le b^{-1} \|\{\bar{x}^*_{ij}\}_{j=1}^n\|_{\ell^1_n} \le b^{-1} A \Big\|\sum_{j=1}^n \bar{x}^*_{ij} e^*_{ij}\Big\| \le b^{-1} A.$$

According to the previous estimate, from the set $\mathcal{B}$ we can select elements B_i such that for every index i we have the relation $b_i \subset B_i$. Then $x^* \in X^*_{B_1 \ldots B_n}$ and inequality (7) is proved.

By Lemma 2.4 [3], in each of the sets $S(X^*_{B_1\ldots B_n})$ there exists a $(\frac{1}{4})$ -net of power at most $9^{\dim X^*_{B_1\ldots B_n}} = 9^{mn}$. Combining these $(\frac{1}{4})$ -nets, we obtain a set $\mathfrak{X}^*_2$, satisfying the conditions 3) and 4). Indeed, $|\mathfrak{X}^*_2| \leq \binom{n}{m}^n \cdot 9^{mn} \leq (9n)^{mn} \leq (9n)^{2A^{4/3} n^{5/3}}$, and for every x from the set S_2 the inequality (7) leads to the estimate

$$\sup\{\langle x,x^*\rangle : x^* \in \mathfrak{X}^*_2\} \geq \sup\{\langle x,x^*\rangle : x^* \in \bigcup_{B_1\ldots B_n \in \mathcal{B}} S(X^*_{B_1\ldots B_n})\} - \frac{1}{4} \geq \frac{1}{4} \quad \bullet$$

Proof of Theorem 1. Let $p = 1;\infty$. A subset $\mathcal{J}'$ of a finite set $\mathcal{J}$ of natural numbers is said to be $(S;p)$ -maximal if $\mathcal{D}_p(\mathcal{J}') \leq S$ and $\lambda_p(\mathcal{J},S) = |\mathcal{J}'|$.

(i)We prove that there exists a positive number α, such that for any finite set of natural numbers I we have one of the inequalities: $\lambda_1(I, 2Q^2) \geq \alpha|I|$ or $\lambda_\infty(I, 2Q^2) \geq \alpha|I|$.

For the sake of brevity we denote $2Q^2$ by R.

Assume that what is asserted at (i) is not satisfied. Then for every number $\tau, \tau > 0$, there exists a finite subset I of the set $\mathbb{N}$ for which

$$\lambda_1(I,R) \leq \frac{\sigma|I|}{2\tau} \quad \text{and} \quad \lambda_\infty(I,R) \leq \frac{\sigma|I|}{2\tau} \; ^\dagger \tag{8}$$

The scheme of the subsequent operations is the following. We extract from the set I, satisfying the estimates (8), a subset $\mathcal{J}$ such that the sequence $\{e_i\}_{i\in\mathcal{J}}$ after an appropriate renumbering by pairs of indices will satisfy the assumption of Theorem 2. For large τ this subset will be so large that the conclusion of Theorem 2 will be in contradiction with the conclusion of Lemma 2.

Thus, we fix τ and suppose that for the set I the estimates (8) hold. We construct a sequence of subsets of the set I with the aid of an inductive procedure. We denote by $\mathcal{J}'_1$ some $(R;1)$ -maximal subset of the set I. By Lemma 3, we have $|\mathcal{J}'_1| \geq \frac{\sigma|I|}{\lambda_\infty(I,R)} \geq 2\tau$. Assume that the sets $\mathcal{J}'_1,\ldots,\mathcal{J}'_k$ have been already constructed. If $\lambda_1(I \setminus \bigcup_{i=1}^k \mathcal{J}'_i, R) < \tau$ then the procedure stops. Otherwise, for $\mathcal{J}'_{k+1}$ we take any $(R;1)$ -maximal subset of the set $I \setminus \bigcup_{i=1}^k \mathcal{J}'_i$.

Assume that the inductive process concludes after N steps. We set $I' = \bigcup_{k=1}^N \mathcal{J}'_k$. It is easy to see that $|I'| \geq \frac{|I|}{2} \geq \frac{\tau}{\sigma}|\mathcal{J}'_1|$ since if $|I'| < \frac{|I|}{2}$ then, by Lemma 3, $\lambda_1(I\setminus I'; R) \geq \frac{\sigma(|I|-|I'|)}{\lambda_\infty(I,R)} \geq \tau$ and the process can be continued.

We renumber the sequence $\{\mathcal{J}'_k\}_{k=1}^N$ in reverse order: $\mathcal{J}_k \overset{def}{=} \mathcal{J}'_{N+1-k}$. The sequence $\{m_k\}_{k=1}^N$, $m_k = |\mathcal{J}_k|$, is nondecreasing. We set $n \overset{def}{=} \min\{\bar{k} : 1 \leq \bar{k} \leq N, \sum_{k=1}^{\bar{k}} m_k \geq \frac{\tau}{\sigma} m_{\bar{k}}\}$, $I^2 \overset{def}{=} \bigcup_{k=1}^n \mathcal{J}_k$. The definition of the number n is correct since $\sum_{k=1}^N m_k \geq \frac{|I|}{2} \geq \frac{\tau}{\sigma}|\mathcal{J}_N| = \frac{\tau}{\sigma} m_N$.

For any number $\bar{k}, \bar{k} < n$ we have the inequality $\sum_{k=1}^{\bar{k}} m_k < \frac{\tau}{\sigma} m_{\bar{k}}$. Consequently, $m_{\bar{k}} > \frac{1}{\sigma^{-1}\tau - 1} \cdot \sum_{k=1}^{\bar{k}-1} m_k$. From here, by induction one can derive that $\sum_{k=1}^{\bar{k}} m_k \geq \tau(1-\frac{\sigma}{\tau})^{\bar{k}}$. Indeed, by

†We recall that σ is the constant from Lemma 3.

construction, $|\mathcal{J}_1| \geq \tau$, i.e., for $\bar{k}=1$ the inequality is satisfied. If it is satisfied for $\bar{k} < n-1$, then

$$\sum_{k=1}^{\bar{k}+1} m_k > \left(1 + \frac{1}{\sigma^{-1}\tau - 1}\right) \sum_{k=1}^{\bar{k}} m_k \geq \left(1 + \frac{1}{\sigma^{-1}\tau - 1}\right) \cdot \tau \left(1 - \frac{\sigma}{\tau}\right)^{1-\bar{k}} = \tau \left(1 - \frac{\sigma}{\tau}\right)^{-\bar{k}} .$$

Thus,

$$|I^2| = \sum_{k=1}^{n} m_k \geq \tau \left(1 - \frac{\sigma}{\tau}\right)^{2-n} . \tag{9}$$

From the definition of the number n there follows that $|I^2| = \sum_{k=1}^{n} m_k \geq \frac{\tau}{\sigma} m_n$. Therefore, for any subset $\mathcal{J}$ of the set I^2, satisfying the condition $|\mathcal{J}| \geq \frac{|I^2|}{2}$, we have the inequality $\lambda_\infty(\mathcal{J}, R) \geq \frac{\sigma|\mathcal{J}|}{\lambda_1(\mathcal{J}, R)} \geq \frac{\sigma|I^2|}{2m_n} \geq \frac{\tau}{2}$. In particular, from the set I^2 one can select $m = E\left(\frac{|I^2|}{\tau}\right)$ mutually disjoint subsets I_ℓ', each of cardinality $E\left(\frac{\tau}{2}\right)$, satisfying the condition $\mathcal{D}_\infty(I_\ell') \leq R$. Since $\mathcal{D}_1(\mathcal{J}_k) \leq R$ for all k, for any ℓ we have the relation $|\mathcal{J}_k \cap I_\ell'| \leq R^2$. Consequently, from each set I_ℓ one can extract a subset I_ℓ' of cardinality $K = E\left(\frac{\tau}{2R^2}\right)$ such that $|\mathcal{J}_k \cap I_\ell| \leq 1$ for all k.

We denote by $\mathcal{P}$ the projection of the set I^2 onto the set $\{1 \ldots n\}$ associating to the number i, $i \in I^2$ that k for which $i \in \mathcal{J}_k$.

We fix some natural number p, satisfying the inequality $\rho p^2 > cR^{4/3} p^{5/3} \log p$ (we recall that ρ is the constant from Lemma 2 and c is the constant from Theorem 2).

a) Let $n \leq \frac{1}{2} K^{1+\frac{1}{2p}}$. Obviously, $n \geq \sum_{k=1}^{n} \frac{m_k}{m_n} = \frac{|I^2|}{m_n} \geq \frac{\tau}{\sigma} > K$, $|I^2| = \sum_{k=1}^{m} m_k \geq \tau n$ and, consequently, $m \geq n > K$. We shall assume that τ is so large that $m \geq p^2$. Applying Lemma 4 to the sequence $\{\mathcal{P}(I_\ell)\}_{\ell=1}^{m}$ we obtain that there exists a sequence of mutually distinct indices $\{\ell_i\}_{i=1}^{p}$ such that $|\bigcap_{i=1}^{p} \mathcal{P}(I_{\ell_i})| \geq p$ This means that there exists a sequence of mutually distince indices $\{k_j\}_{j=1}^{p}$ such that for any indices i, j the intersection of the sets I_{ℓ_i} and $\mathcal{J}_{k_j}$ consists of one element. The basis vector corresponding to this element will be denoted by e_{ij}. The sequence $\{e_{ij}\}_{i,j=1}^{p}$ satisfies the assumption of Theorem 2. We set $X_0 = \operatorname{span}\{e_{ij}\}_{i,j=1}$. Making use of the estimate 2) of Lemma 2, we obtain the inequality

$$\rho p^2 \leq \max\{k_2(X_0), k_2(X_0^*)\} \leq cR^{4/3} p^{5/3} \log p ,$$

contradicting the number p.

b) Assume that now $n > \frac{1}{2} K^{1+\frac{1}{2p}}$. Then $\tau < 8R^2 n^{1-\frac{1}{2p+1}}$. We select from each set I_ℓ a subset $\tilde{I}_\ell$ of cardinality p. We call the sets $\tilde{I}_\ell$ and $\tilde{I}_s$ equivalent if $\mathcal{P}(\tilde{I}_\ell) = \mathcal{P}(\tilde{I}_s)$. If the cardinality of each equivalence class is less than p, then $m < p\binom{n}{p}$. But, according to the estimate (9), $\log m > \log \frac{|I^2|}{2\tau} > \log\left(1 - \frac{\sigma}{\tau}\right)^{2-n} - 1 > (n-2)\frac{\sigma}{\tau} - 1 > \frac{\sigma}{16R^2} n^{\frac{1}{2p+1}}$ while $\log\left[p\binom{n}{p}\right] < \log[pn^p] < (p+1)\log n$. Therefore, if τ is sufficiently large, then there exist p sets $\tilde{I}_\ell$ whose projections coincide. Reasoning in the same way as in the case a), we obtain a contradiction. Part (i) is proved.

We note that from the assertion of part (i) there follows that for $A = \left(\frac{1}{\alpha} + 1\right) R$ for any finite subset I of the set $\mathbb{N}$ we have one of the inequalities:

$$\lambda_1(I, A) \geq \frac{|I|}{2} \qquad \text{or} \qquad \lambda_\infty(I, A) \geq \frac{|I|}{2} . \tag{10}$$

(ii) We prove that for some number C, any finite set I of natural numbers can be partitioned into disjoint subsets I_1 and I_2 such that $\mathcal{D}_1(I_1)\le C$ and $\mathcal{D}_\infty(I_2)\le C$.

We prove that, as shown by Johnson's example [5], the assertion of part (ii) does not follow from (10) if 2-triviality is not assumed.

We prove statement (ii) by contradiction. We set $B=\frac{32Q^8}{\alpha}$; $n=\frac{32Q^4B}{\alpha}$; $C=B+\frac{n}{\alpha}$ [α is from statement of part (i)]. Let I be that set for which the partition indicated in part (ii) does not exist.

We construct disjoint sets $\mathcal{J}_1$ and $\mathcal{J}_2$ of cardinalities greater than n, satisfying one of the following conditions:

a) $\mathcal{J}_1$ is a ($B;1$)-maximal subset of the set $\mathcal{J}_1\cup\mathcal{J}_2$, $\mathcal{D}_1(\mathcal{J}_2)\le B$

or

b) $\mathcal{J}_1$ is a ($B;\infty$)-maximal subset of the set $\mathcal{J}_1\cup\mathcal{J}_2$, $\mathcal{D}_\infty(\mathcal{J}_2)\le B$.

It is easy to see that $|I|>\frac{n}{\alpha}$ since otherwise $\mathcal{D}_1(I)\le\frac{n}{\alpha}<C$ in spite of the fact that I has no partition of the type mentioned in (ii). We assume that $\lambda_1(I,2Q^4)\ge\alpha|I|$ (the case $\lambda_\infty(I,2Q^4)\ge\alpha|I|$ is considered in a similar manner). Let $\mathcal{J}_1'$ be a ($B;1$) -maximal subset of the set I. Then $|\mathcal{J}_1'|=\lambda_1(I,B)\ge\lambda_1(I,2Q^4)>n$. If $|I\setminus\mathcal{J}_1'|\le\frac{n}{\alpha}$ then $\mathcal{D}_1(I)\le\mathcal{D}_1(\mathcal{J}_1')+|I\setminus\mathcal{J}_1'|\le B+\frac{n}{\alpha}=C$ in spite of the manner in which I has been selected. Consequently, $|I\setminus\mathcal{J}_1'|>\frac{n}{\alpha}$. If $\lambda_1(I\setminus\mathcal{J}_1',B)>n$ then for $\mathcal{J}_2$ we take any ($B;1$)-maximal subset of the set $I\setminus\mathcal{J}_1'$. For the sets $\mathcal{J}_1=\mathcal{J}_1'$ and $\mathcal{J}_2$ condition a) is satisfied. If, however, $\lambda_1(I\setminus\mathcal{J}_1',B)\le n$ then $\lambda_1(I\setminus\mathcal{J}_1',2Q^4)\le n<\alpha|I\setminus\mathcal{J}_1'|$ so that, by part (i), we have $\lambda_\infty(I\setminus\mathcal{J}_1',B)\ge\lambda_\infty(I\setminus\mathcal{J}_1',2Q^4)\ge\alpha|I\setminus\mathcal{J}_1'|>n$. Let $\mathcal{J}_1''$ be a (B,∞)-maximal subset of the set $I\setminus\mathcal{J}_1'$. Then $|\mathcal{J}_1''|=\lambda_\infty(I\setminus\mathcal{J}_1',B)>n$. If $|I\setminus(\mathcal{J}_1'\cup\mathcal{J}_1'')|\le\frac{n}{\alpha}$ then $\mathcal{D}_\infty(I\setminus\mathcal{J}_1')\le\mathcal{D}_\infty(\mathcal{J}_1'')+|I\setminus(\mathcal{J}_1'\cup\mathcal{J}_1'')|\le B+\frac{n}{\alpha}=C$, contradicting the selection of the set I. Thus, $|I\setminus(\mathcal{J}_1'\cup\mathcal{J}_1'')|>\frac{n}{\alpha}$. Let $\lambda_1(I\setminus(\mathcal{J}_1'\cup\mathcal{J}_1''),B)>n$. We set $\mathcal{J}_1=\mathcal{J}_1'$ and let $\mathcal{J}_2$ be a ($B,1$)-maximal subset of the set $I\setminus(\mathcal{J}_1'\cup\mathcal{J}_1'')$. The sets $\mathcal{J}_1$ and $\mathcal{J}_2$ satisfy condition a). If $\lambda_1(I\setminus(\mathcal{J}_1'\cup\mathcal{J}_1''),B)\le n$, then $\lambda_\infty(I\setminus(\mathcal{J}_1'\cup\mathcal{J}_1''),B)>n$. In this case the set $\mathcal{J}_1=\mathcal{J}_1''$ and any (B,∞)-maximal subset of the set $I\setminus(\mathcal{J}_1'\cup\mathcal{J}_1'')$ satisfy condition b).

Assume that condition a) is satisfied [in the case of the validity of b), statement (ii) is established by the same arguments].

Let $\mathcal{J}$ be any subset of the set $\mathcal{J}_1\cup\mathcal{J}_2$ for which we have the relation $|\mathcal{J}|>|\mathcal{J}_1|$. By condition a) we have $\lambda_1(\mathcal{J}_1\cup\mathcal{J}_2,B)=|\mathcal{J}_1|$ and, consequently, $\mathcal{D}_1(\mathcal{J})>B$. Applying Corollary 2 of Lemma 1, we obtain the inequality

$$\lambda_\infty(\mathcal{J},2Q^4)\ge\frac{1}{2Q^4}\mathcal{D}_1(\mathcal{J})>\frac{B}{2Q^4}$$

Thus, from the set $\mathcal{J}_1\cup\mathcal{J}_2$ one can extract not less than $m=E\left(\frac{Q^4n}{B}\right)$ mutually disjoint subsets $\bar{I}_j$, of cardinality $E\left(\frac{B}{2Q^4}\right)$, such that $\mathcal{D}_\infty(\bar{I}_j)\le 2Q^4$. We extract from each set $\bar{I}_j$ a subset I_j of cardinality $N=E\left(\frac{B}{4Q^4}\right)$ contained in one of the sets $\mathcal{J}_1$ or $\mathcal{J}_2$. We renumber the sets I_j so that we should have the relations $I_1,\dots,I_\ell\subset\mathcal{J}_1$, $I_{\ell+1},\dots,I_m\subset\mathcal{J}_2$.

First we consider the case when $\ell\ge m_1=E\left(\frac{m}{2}\right)$. We set $I'=\bigcup_{j=1}^{m_1}I_j$. Then $|I'|=Nm_1>\frac{n}{16}$. Since $I'\subset\mathcal{J}$, we have the inequality $\mathcal{D}_1(I')\le B$. Consequently, $\lambda_\infty(I',2Q^4)\le 2Q^4B$. According to part (i), from the set I' we can extract a subset I'', of cardinality greater than $\frac{\alpha n}{16}$, for which $\mathcal{D}_1(I'')\leqslant 2Q^2$ (the inequality $\mathcal{D}_\infty(I'')\leqslant 2Q^2$ is not possible since $|I''|>\frac{\alpha n}{16}=2Q^2B\geqslant\lambda_\infty(I',2Q^2)$).

For all j we have the inequalities

$$\mathcal{D}_1(I_j\cap I'')\leqslant\mathcal{D}_1(I'')\leqslant 2Q^2,\quad \mathcal{D}_\infty(I_j\cap I'')\leqslant\mathcal{D}_\infty(I_j)\leqslant 2Q^2.$$

Consequently, $|I_j\cap I''|\leqslant 4Q^4$. Thus,

$$\frac{\alpha n}{16}<|I''|=\sum_{j=1}^{m_1}|I''\cap I_j|\leqslant 4Q^4m_1\leqslant 2\,\frac{Q^8 n}{B}$$

The obtained inequality contradicts the definition of the number B.

If, however, $\ell<m_1$, then $m-\ell>m_1$. We set $I'=\bigcup_{j=\ell+1}^{\ell+m_1}I_j$. Since $I'\subset\mathcal{J}_2$ the same arguments lead to a contradiction.

Part (ii) is proved.

(iii). With the aid of D. König's theorem [6] (Theorem 1, Chapter III, Sec. 5) one can show that there exists a partition of the set $\mathbb{N}$ into disjoint subsets $\mathbb{N}_1$ and $\mathbb{N}_\infty$, such that $\mathcal{D}_1(\mathbb{N}_1\cap\{1\ldots n\})\leqslant C$ and $\mathcal{D}_\infty(\mathbb{N}_\infty\cap\{1\ldots n\})\leqslant C$ for any natural number n. This means that the sequences $\{e_i\}_{i\in\mathbb{N}_1}$ and $\{e_i\}_{i\in\mathbb{N}_\infty}$ are equivalent to the standard bases of the spaces $\ell^1_{|\mathbb{N}_1|}$ and $\ell^\infty_{|\mathbb{N}_\infty|}$ (c_0, if $|\mathbb{N}_\infty|=\infty$).

Theorem 1 is completely proved. •

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