1. The model category of simplicial presheaves

1.1. The global model structure on simplicial presheaves

Let \((C, \tau)\) be a small Grothendieck site. We will denote by \(\mathbf{SPr}(C)\) the category of simplicial presheaves \(C^{\text{op}} \to \mathbf{SSet}\) on \(C\). The model structure on \(\mathbf{SSet}\) induces a natural model structure on \(\mathbf{SPr}(C)\) with objectwise fibrations and equivalences; this model structure is called \textit{global projective} model structure and denoted by \(\mathbf{SPr}(C)_{\text{glob}}\).

The category \(\mathbf{SPr}(C)_{\text{glob}}\) is a simplicial model category whose simplicial enrichment is given by

\[
\mathbf{Hom}(F, G)_n := \mathbf{Hom}(F \times \Delta[n], G).
\]

Therefore, for any \(F, G \in \mathbf{SPr}(C)_{\text{glob}}\) one has an isomorphism in \(\mathbf{Ho}(\mathbf{SSet})\)

\[
\mathbf{Map}_{\mathbf{SPr}(C)_{\text{glob}}}(F, G) \cong \mathbb{R}\mathbf{Hom}(F, G);
\]

here \(\mathbf{Map}_M\) denotes the mapping space in the model category \(M\).

Because the global projective model structure does not see the topology \(\tau\) on \(C\), we want to modify it so as to take it into account in some meaningful way. One possible answer is to replace the category \(\mathbf{SPr}(C)\) of simplicial presheaves with the category \(\mathbf{SSh}(C, \tau)\) of simplicial sheaves on \((C, \tau)\) and try to build a model structure; this is done in A. Joyal’s letter to Grothendieck.

Here we keep working with simplicial presheaves, and use a technique called the left Bousfield localization to “homotopically” invert all the coverings in the topology, considered as maps in \(\mathbf{SPr}(C)\). This gives a new model structure, called the \textit{local} model structure, whose fibrant objects have descent with respect to the topology. These two approaches are equivalent in some sense.

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Excerpt from \textit{From homotopical algebra to homotopical algebraic geometry}. 
1.2. The local model structure on simplicial sheaves. Hyperdescent. Recall ([1]) that a point in a site is just a point in the associated topos of sheaves of sets on the site, i.e., a geometric morphism $x$ from the topos $\text{Set}$ of sheaves of sets over a category with one object and one morphism, to our topos, so that we have an inverse image functor $x^* : \text{Sh}(\mathcal{C}, \tau) \to \text{Set}$ which is a left adjoint and left exact. The stalk at the point $x$ of a simplicial presheaf $F$ is the levelwise composition of the sheafification functor followed by $x^*$. A morphism of simplicial presheaves $f : F \to G$ on $\mathcal{C}$ is said to be a $\tau$-local equivalence if it induces a weak equivalence of simplicial sets $f_x : F_x \to G_x$ on the stalk for any point $x$ in the site.

For any $\tau$-covering $(U_i \to X)$, we consider the corresponding Čech nerve $N(U)_{\bullet}$ which is the simplicial object in $\mathcal{C}$ defined by

$N(U)_{n} := \coprod_{i_0 \times_{X} U_{i_1} \times_{X} \cdots \times_{X} U_{i_n}}$.

Note that there is a natural augmentation $N(U)_{\bullet} \to X$. This is only a special case of a more general kind of simplicial object in $\mathcal{C}$ augmented over $X$, called $\tau$-hypercover of $X$. A hypercover is essentially a Čech nerve in which each stage is refined by taking further covering in the given topology.

**Definition 1.1.** A simplicial presheaf $F$ has $\tau$-hyperdescent if for any object $X \in \mathcal{C}$ and any $\tau$-hypercover $U_{\bullet} \to X$ the canonical map in $\text{Ho}(\text{SSet})$

$F(X) \to \text{Holim} F(U_{\bullet}) \cong \text{Holim} \text{Hom}(U_{\bullet}, F)$

is an isomorphism.

Note that hyperdescent is really a homotopical generalization of the usual descent of sheaf property. To see this, let $F$ be a presheaf of sets considered as a constant simplicial presheaf, and $(U_i \to X)$ a $\tau$-covering. Since $\text{Holim}$ is $\lim$ for constant simplicial sets, and weak equivalences between constant simplicial sets are isomorphisms, we see that $F$ has hyperdescent with respect to the Čech nerve $N(U_{\bullet})$ if and only if $F$ has the sheaf property with respect to the covering $(U_i \to X)$.

**Theorem 1.2.** There exists a model structure $\text{SPr}_{\tau}(\mathcal{C})_{\text{loc}}$ on the category $\text{SPr}(\mathcal{C})$, called the local model structure, in which:

1. the equivalences are exactly the local equivalences;
2. the identity functors $\text{Id} : \text{SPr}_{\tau}(\mathcal{C})_{\text{glob}} \rightleftarrows \text{SPr}(\mathcal{C})_{\text{loc}} : \text{Id}$ give a Quillen adjunction; and
3. the right derived functor $\mathbb{R} \text{Id} : \text{Ho}(\text{SPr}_{\tau}(\mathcal{C})_{\text{loc}}) \to \text{Ho}(\text{SPr}(\mathcal{C})_{\text{glob}})$ is fully faithful with essential image the full subcategory on objects having $\tau$-hyperdescent.

In fact, $\text{SPr}_{\tau}(\mathcal{C})_{\text{loc}}$ is constructed as a left Bousfield localization of $\text{SPr}(\mathcal{C})_{\text{glob}}$. For a complete proof, see [2].

The objects in the category $\text{Ho}(\text{SPr}_{\tau}(\mathcal{C})_{\text{loc}})$ are called $\infty$-stacks. The left derived functor $\mathbb{L} \text{Id}$ is called the stackification functor and simply denoted by $\mathbb{a}$.

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* This definition is only correct if the site has enough points, which we assume; in the more general case, a local equivalence will be a map inducing an isomorphism on all the sheaves of homotopy groups of $F$ and $G$ for any base point.

† A morphism $f : U_{\bullet} \to X_{\bullet}$ in $\text{SPr}(\mathcal{C})$ is a hypercover if for all $n$ the canonical morphism

$U_n \to (\text{Cosk}_{n-1} U)_n \times_{(\text{Cosk}_{n-1} X)_n} X_n$

is a local epimorphism, i.e., an epimorphism under sheafification.
1.3. Hom-stacks. The category $\text{Ho}(\text{SPr}_\tau(C)_{\text{loc}})$ has a very rich structure.

First, since $\text{SPr}_\tau(C)_{\text{loc}}$ is a simplicially enriched model category with $\text{Hom}(F, G)_n := \text{Hom}(F \times \Delta[n], G)$, the category $\text{Ho}(\text{SPr}_\tau(C)_{\text{loc}})$ is enriched over $\text{Ho}(\text{SSet})$ with $\mathbb{R}\text{Hom}(F, G) = \text{Hom}(QF, RG)$, where $R$ and $Q$ are fibrant and cofibrant replacements of simplicial sets.

Moreover, it is a cartesian closed category, i.e. it has internal Hom-objects: for $F$ and $G$ in $\text{Ho}(\text{SPr}_\tau(C)_{\text{loc}})$, there is an object $\mathbb{R}\text{Hom}(F, G)$ in $\text{Ho}(\text{SPr}_\tau(C)_{\text{loc}})$, called the Hom-stack between $F$ and $G$, such that

$$\text{Hom}(F', \mathbb{R}\text{Hom}(F, G)) \cong \text{Hom}(F' \times F, G).$$

This essentially comes from the fact that $\text{SPr}(C)$ is cartesian closed with internal Hom-objects

$$\mathcal{H}om(F, G) : X \mapsto \text{Hom}_{\text{SPr}(C/X)}(F|_{C/X}, G|_{C/X}).$$

More precisely, for any $F$ and $G$ in $\text{Ho}(\text{SPr}_\tau(C)_{\text{loc}})$, the corresponding internal Hom-stack is defined as

$$\mathbb{R}\text{Hom}(F, G) := a(\text{Hom}(F, R_{\text{inj}}G)),$$

where $a = L\text{Id}$ is the stackification functor, and $R_{\text{inj}}$ is the fibrant replacement in the injective model structure on $\text{SPr}(C)$ in which equivalences and cofibrations are defined objectwise.

2. Sheaves and stacks in groupoids as truncated simplicial presheaves

2.1. Sheaves of sets as 0-truncated simplicial sets. The notion of $\tau$-covering induces a natural notion of $\tau$-local isomorphism of presheaves: a map of presheaves is a local isomorphism if it is isomorphism under sheafification. Then we may localize the category $\text{Pr}(C)$ with respect to $W_\tau$ of local isomorphisms to obtain $W^{-1}_\tau \text{Pr}(C)$, and the localization functor $\text{Pr}(C) \to W^{-1}_\tau \text{Pr}(C)$ is left exact (commutes with finite limits) and has a right adjoint. Such a localization is called a left exact localization. Moreover, $W^{-1}_\tau \text{Pr}(C)$ is naturally equivalent to the category of sheaves of sets on $(C, \tau)$, and the localization functor is identified with the sheafification functor through this equivalence.

The properties of the left Bousfield localization and Theorem 1.2 show that, if we replace the category $\text{Pr}(C)$ by $\text{SPr}(C)$, the $\tau$-local isomorphisms by the $\tau$-local equivalences, and the usual localization by the left Bousfield localization, then the identity functor induces a homotopy left exact localization $L\text{Id}$. i.e. it has a right adjoint and is homotopy left exact (commutes with homotopy fibered products).

To see usual sheaves as $\infty$-stacks, we may view any (pre)sheaf of sets as a constant simplicial presheaf. Composing with the localization to the homotopy category, this gives a functor

$$i : \text{Sh}(C, \tau) \to \text{Ho}(\text{SPr}_\tau(C)_{\text{loc}}).$$

Since any constant simplicial set has vanishing homotopy groups in dimensions $\geq 1$, the image of $i$ consists of 1-truncated simplicial presheaves.

**Proposition 2.1.** The functor $i$ is fully faithful, and its essential image consists of 1-truncated simplicial presheaves.

Therefore, the theory of sheaves of sets on the site $(C, \tau)$ is embedded in the theory of $\infty$-stacks over $(C, \tau)$.
2.2. **Stacks in groupoids as 1-truncated simplicial presheaves.** Denote by $\text{St}(\mathcal{C}, \tau)$ the category of stacks fibered in groupoids over the site $(\mathcal{C}, \tau)$; its objects are stacks $S \to \mathcal{C}$ and the morphisms between $S \to \mathcal{C}$ and $S' \to \mathcal{C}$ are the strictly commutative triangles of functors. One may also consider the 2-category $\text{St}(\mathcal{C}, \tau)$ by laxifying the commutative triangles.

We want to associate to any stack $S$ in groupoids over $(\mathcal{C}, \tau)$ a simplicial presheaf on $\mathcal{C}$. Since the rule $X \mapsto S_X$ only defines a priori a lax (or weak or pseudo) presheaf of groupoids, we need the following **strictification** (or **canonical cleavage**) construction to associate to a (pre)stack $S \to \mathcal{C}$ a genuine presheaf of groupoids, and actually of simplicial sets.

Recall that for groupoids the nerve functor is often called the **classifying space** functor and denoted by $B$.

**Proposition 2.2** (Strictification for prestacks in groupoids). Let $S \to \mathcal{C}$ be a prestack in groupoids over $(\mathcal{C}, \tau)$. The rule

$$BS : X \mapsto B\text{Hom}_{\text{Gpd}/\mathcal{C}}(X, S)$$

defines a simplicial presheaf $BS$ on $\mathcal{C}$. Here $\text{Hom}_{\text{Gpd}/\mathcal{C}}$ denotes the groupoid of morphisms between categories fibered in groupoids over $\mathcal{C}$.

Moreover, the rule $B : S \mapsto BS$ defines a functor between the category of prestacks in groupoids over $(\mathcal{C}, \tau)$ and the category $\text{SPr}(\mathcal{C})$.

Note that if $S \to \mathcal{C}$ is a stack in groupoids, then by definition it satisfies a descent condition on $(\mathcal{C}, \tau)$. Observe that $BS$ is a 1-truncated simplicial presheaf, so it satisfies the hyperdescent condition in $\text{SPr}(\mathcal{C})$. Moreover, $BS$ is a fibrant object in $\text{SPr}_\tau(\mathcal{C})_{\text{loc}}$. For the nerve of a groupoid is always a fibrant simplicial set, and therefore $BS$ is in fact objectwise, i.e. globally, fibrant.

Using the nerve functor on the Hom groupoids, one can view the 2-category $\text{St}(\mathcal{C}, \tau)$ of stacks in groupoids as a category enriched over $\text{SSet}$; we will denote it by $\text{St}(\mathcal{C}, \tau)$.

**Exercise 2.3.** Check that the functor $B$ can be enhanced to a simplicial functor $B : \text{St}(\mathcal{C}, \tau) \to \text{SPr}_\tau(\mathcal{C})_{\text{loc}}$.

By composing the enhanced $B$ with a cofibrant replacement in $\text{SPr}_\tau(\mathcal{C})_{\text{loc}}$ and using that $BS$ is always fibrant, we get a simplicial functor

$$QB : \text{St}(\mathcal{C}, \tau) \to \text{SPr}_\tau(\mathcal{C})^{cf}_{\text{loc}}.$$  

**Theorem 2.4.** The 2-truncation

$$(QB)_{\leq 2} : \left(\text{St}(\mathcal{C}, \tau)\right)_{\leq 2} \to \left(\text{SPr}_\tau(\mathcal{C})^{cf}_{\text{loc}}\right)_{\leq 2}$$

is a fully faithful morphism of 2-categories (i.e. it induces an equivalence of categories between the Hom-categories), and its essential images is the full 2-category consisting of 1-truncated simplicial presheaves.

Since the composite of $\Pi_1$ following nerve is canonically isomorphic to the identity functor on $\text{Gpd}$, the 2-truncation 2-category $\left(\text{St}(\mathcal{C}, \tau)\right)_{\leq 2}$ is canonically equivalent to the 2-category of stacks in groupoids.

**Corollary 2.5.** The 1-truncation of $QB$ defines a fully faithfult functor

$$(1\text{-Iso})^{-1}\text{St}(\mathcal{C}, \tau) \to \text{Ho}(\text{SPr}_\tau(\mathcal{C})_{\text{loc}})$$
whose essential image is the fully subcategory of 1-truncated simplicial presheaves.

Therefore, the theory of stacks in groupoids is embedded in the theory of ∞-stacks over the same site.

References
