NOTES ON PERVERSE SHEAVES AND INTERSECTION COHOMOLOGY

CONTENTS

1. Constructible derived category of sheaves 1
   1.1. Derived category of sheaves 1
   1.2. Constructible derived category 2
   1.3. Verdier duality 4
   1.4. Six-functor formalism 6
2. Perverse sheaves 7
   2.1. Truncation structures 7
   2.2. Recollement 10
Appendix A. Whitney stratification 11
References 11

Unless specified otherwise, by “manifolds” and “varieties” we shall always mean, respectively, complex manifolds and complex algebraic varieties. Consequently, “dimension” (of a manifold or variety) always refers to the complex dimension. We also fix, once and for all, a commutative Noetherian ring $k$ of finite global dimension as our ring of “coefficients”; the reader is welcome to take $k = \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \mathbb{F}_q$, and so on. Unless specified otherwise, by “sheaves” and “complexes”, we shall always mean sheaves of $k$-modules and complexes of such sheaves, respectively.

1. Constructible derived category of sheaves

1.1. Derived category of sheaves. Let $X$ be a variety with analytic topology. Since the category $\text{Sh}(X, k) = \text{Sh}(X)$ of sheaves on $X$ is an abelian category, we may consider the category $\text{Ch}(X)$ of (cochain) complexes of sheaves, the homotopy category $K(X)$ of $\text{Ch}(X)$ obtained as a quotient by chain homotopies, and the derived category of sheaves $D(X)$ on $X$, obtained by formally inverting quasi-isomorphisms. Since $\text{Ch}(X)$ has enough injective objects (given by the Godement resolution), $D(X)$ is equivalent to the full subcategory of $K(X)$ on injective complexes. It is well-known that $D(X)$ is not an abelian category, but instead a triangulated category. These notions have variants obtained by considering the bounded below, bounded above, or bounded complexes, and we are particularly interested in the bounded derived category $D^b(X)$.

If a functor $F$ from $\text{Ch}(X)$ to an abelian category is additive, then $F$ preserves chain homotopies, so its right derived functor $RF$ exists, and could be computed by injective resolutions.

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following Geordie Williamson’s notes [6].
For example, we obtain the derived internal hom functor
\[ R\mathcal{H}om : D^b(X)^{\text{op}} \times D^b(X) \to D^b(X) \]
by injectively resolving the second (covariant) argument of the internal sheaf hom functor
\[ \mathcal{H}om : \text{Ch}^b(X)^{\text{op}} \times \text{Ch}^b(X) \to \text{Ch}^b(X). \]
For any complex \( \mathcal{F}^\bullet \), the functor
\[ \mathcal{H}om(\mathcal{F}, -) : \text{Ch}^b(X) \to \text{Ch}^b(X) \]
has a left adjoint
\[ - \otimes \mathcal{F} : \text{Ch}^b(X) \to \text{Ch}^b(X). \]
To obtain the derived tensor-hom adjunction, we need the tensor product to be \emph{left-derived}. Although left-derived functors may not exist for general additive functors on \( \text{Ch}(X) \) due to lack of projective objects, the left derived functor \(- \otimes^L -\) of tensor product nonetheless exists, and is computed by \emph{flat} resolutions.

Before moving on, we also fix some notations and conventions. For each \( k \in \mathbb{Z} \) there is a \emph{shift} endofunctor \([k]\) on \( D^b(X) \) (and similarly for \( K^b(X), \text{Ch}^b(X), \) and other variants) defined by
\[ (\mathcal{F}[k])^n := \mathcal{F}^{n+k}. \]
The \( i \)th \emph{cohomology sheaf} \( \mathcal{H}^i(\mathcal{F}) \) of a complex \( \mathcal{F}^\bullet \) is defined as the sheafification of the presheaf
\[ U \mapsto H^i(\mathcal{F}(U)). \]
Here, we use the identification \( \text{Ch}(X) \cong \text{Sh}(X, \text{Ch}(k)) \), so \( \mathcal{F} \) is regarded as a sheaf of complex of \( k \)-modules.

1.2. \textbf{Constructible derived category.} Before stating the Verdier duality, which is a local generalization of the Poincaré duality, we introduce a notion which serves as the finiteness condition for the duality theory to hold. One sees already that some finiteness condition is necessary in the case of Poincaré duality at a singleton space. It reduces to the natural morphism from a \( k \)-module \( V \) into its double dual \((V^*)^*\), which is an isomorphism if, for example, \( M \) is finitely generated and projective. In general context of category of sheaves, constructibility is one such finiteness condition.

\textbf{Definition.} A \emph{local system} (of \( k \)-modules) is a locally constant and locally finitely generated sheaf.

The category \( \text{Loc}(X) = \text{Loc}(X, k) \) of local systems on \( X \) is an abelian subcategory of \( \text{Sh}(X) \).

\textbf{Theorem 1.} For \( X \) connected, there is an equivalence of categories
\[ \begin{array}{ccc}
\text{Loc}(X) & \cong & k[\pi_1(X)]\text{-Mod}_f \\
\downarrow & & \downarrow \\
\text{Sh}(X) & \cong & k[\pi_1(X)]\text{-Mod}
\end{array} \]
where \( k\pi_1(X) \) denotes the group \( k \)-algebra on \( \pi_1(X) \), and \( \text{Mod} \) (resp. \( \text{Mod}_f \)) denotes the category of (resp. finitely generated) left modules. The equivalence is induced by the stalk functor, with the \( \pi_1(X) \)-action by monodromy.
More generally, for $X$ not necessarily connected, the equivalence above remains valid if we replace $\pi_1(X)$ by the fundamental groupoid on $X$, regarded as a category, and the categories of modules by the functor categories from $\pi_1(X)$ to $k\text{-Mod}$ and to $k\text{-Mod}_f$.

Proof. See for instance [4, Thm. 2.5.14].

In particular, a local system $L$ on a simply connected space $X$ is canonically isomorphic to the constant sheaf $\underline{L}_{|x}$ with values in the stalk $L_x$ for any $x \in X$.

**Example 2.** Consider smooth varieties $X = Y = \mathbb{C}^*$, and proper map $f : X \to Y : z \mapsto z^n$; we are interested in the sheaves $R^i f_* \mathbb{C}_X$, which is equivalently the sheafification of the presheaf

$$U \mapsto H^i(f^{-1}(U), \mathbb{C}).$$

For any $y \in Y$, we could choose a small enough open disk $U \ni y$ so that $f^{-1}(D) = D_1 \sqcup \cdots \sqcup D_n$ is a disjoint union of open disks $D_i \cong D$, which are contractible. It then follows that $R^i f_* \mathbb{C}_X$ is, when restricted to $D$, the constant sheaf with values in $k^n$, and hence is a local system.

In light of Theorem [4] this local system is characterized by the representation of $\pi_1(Y) \cong \mathbb{Z}$ on the free $k$-module $k^n$, or equivalently on the set $\{1, \ldots, n\}$, corresponding to an $n$-cycle in the symmetric group $S_n$. In particular, this is not a constant sheaf.

For general smooth and proper morphism $f$ between smooth varieties, it is a locally trivial fibration by Ehresmann’s fibration lemma, so it follows similarly that $R^i f_* \mathbb{C}_X$ is again a local system.

**Definition.** Fix a stratification $X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$. A sheaf $\mathcal{F}$ on $X$ is $\Lambda$-constructible if there exists the restriction $\mathcal{F}|_{X_\lambda}$ of $\mathcal{F}$ on each stratum is a local system. $\mathcal{F}$ is constructible if it is $\Lambda$-constructible for some stratification $\Lambda$.

**Example 3.** Again let $X = Y = \mathbb{C}$ and $f : X \to Y : z \mapsto z^n$, and consider the sheaves $R^i f_* \mathbb{C}_X$. We have seen that, when restricted to $\mathbb{C}^* \subseteq Y$, $R^i f_* \mathbb{C}_X$ is a local system with stalks $k^n$ and monodromy action $\sigma \in S_n$. Take any small open disk $D$ around $0 \in Y$, then $f^{-1}(D)$ is connected and also contractible, so the presheaf $U \mapsto H^i(f^{-1}(U), \mathbb{C})$ takes value $k$ on $D$. It follows that $R^i f_* \mathbb{C}_X$ has the same stalk at 0 as this presheaf, which then must be $k$. Moreover, for small disks $D' \subset D$, $0 \notin D'$ and $0 \in D$, the restriction map

$$k \cong \Gamma (R^i f_* \mathbb{C}_X, D) \longrightarrow \Gamma (R^i f_* \mathbb{C}_X, D') \cong k^n$$

is the inclusion of the invariants $k \cong (k^n)_\sigma \cong \text{span}\{(1, \ldots, 1)\} \subseteq k^n$.

In particular, $R^i f_* \mathbb{C}_X$ is a constructible sheaf on $Y$ with respect to the stratification $Y = \mathbb{C}^* \sqcup \{0\}$.

In general, a constructible sheaf on a smooth curve $C$ is determined by the following data:

1. a stratification $C = U \sqcup \bigsqcup_{i=1}^n \{x_i\}$;
2. a local system $\mathcal{L}$ on $U$;
3. for each $i$, a $k$-module $M_i$, a generator $\phi_i$ of $\pi_1(C, x_i)$, and a map $M_i \to \mathcal{L}_{|x_i}$, where $x_i \in U$ is a point near $x_i$.

**Definition.** A complex $\mathcal{F}^\bullet$ of sheaves is (resp. $\Lambda$-)constructible if all of its cohomology sheaves $\mathcal{H}^i(\mathcal{F})$ is. The full subcategory of $D^b(X)$ on (resp. $\Lambda$-)constructible is denoted $D^b_c(X)$ (resp. $D^b_\Lambda(X)$).
1.3. Verdier duality. Given a map \( f : X \to Y \) of varieties, the direct image functor \( f_* \) is defined as usual, while the proper direct image functor \( f! \) is defined by taking sections with proper support. They are both additive, hence right-derivable, and we denote their right derived functors by
\[
Rf_* : \mathcal{D}^b(X) \to \mathcal{D}^b(Y) \quad \text{and} \quad Rf! : \mathcal{D}^b(X) \to \mathcal{D}^b(Y).
\]
Take \( Y = \ast \) to be a singleton and \( f = p_X \) the unique projection, then \( f_* \) and \( f! \) are the global section functor and global section with compact support functor.

**Definition.** For any \( i \in \mathbb{Z} \) and \( F \in \mathcal{D}^b(X) \), the \( i \)-th cohomology and the \( i \)-th cohomology with compact support on \( X \) with coefficient \( F \) are
\[
H^i(X; F) := R^i p_{X!} F \quad \text{and} \quad H^i_c(X; F) := R^i p_{X!}^c F,
\]
respectively.

It is well-known that, for a constant sheaf \( M_X := p_{X!}^* M \), where \( M \) is a \( k \)-module regarded as a sheaf on a point, there are isomorphisms
\[
H^i(X; M) \cong H^i(X; M_X) \cong R^i p_{X*}^* p_X^* M, \quad H^i_c(X; M) \cong H^i_c(X; M_X) \cong R^i p_{X!}^c p_X^* M,
\]
where the left-hand sides denote singular cohomology and singular cohomology with compact support, respectively.

On the level of complexes, \( f_* \) has a left adjoint \( f^* \), the inverse image functor, defined as usual. In particular, \( f_* \) is left exact but not right exact;

The inverse image functor preserves stalks and hence quasi-isomorphisms, so it is an exact functor, and it descends to a functor between the derived categories, which we also denote by \( f^* \).

**Theorem 4.** There are natural isomorphisms of functors
\[
(1) \quad Rf_* R\text{Hom} (f^* , -) \cong R\text{Hom} (-, Rf_*( -)).
\]
In particular, by taking global sections, we conclude that \( f^* \) is left adjoint to \( Rf_* \).

Furthermore, \( Rf_* \) is also right adjoint to \( f^* \).

**Proposition 5** (?). Both \( Rf_* \) and \( f^* \) preserve constructibility.

On the other hand, \( f! \) has no left adjoints.

**Example 6.** It could be checked that
\[
f! \left( \prod_{i=1}^\infty k_X \right) \cong \bigoplus_{i=1}^\infty (f_i k_X),
\]
thus it does not preserve products, and cannot be a right adjoint.

In general, \( f! \) is does not have a right adjoint either. For if it did, it would be right exact, hence has no higher right derived functors, and in particular any higher cohomology with compact support vanishes; this is absurd. Nonetheless, in some cases \( f! \) indeed has a right adjoint.

**Example 7.** Let \( h : Z \hookrightarrow X \) be the inclusion of a locally closed subset. For a sheaf \( \mathcal{F} \in \text{Sh}(X) \), define the presheaf \( \mathcal{F}^Z \) of sections of \( \mathcal{F} \) with support in \( Z \) by
\[
\mathcal{F}^Z : U \mapsto \{ s \in \mathcal{F}(U) | \text{supp } s \subseteq Z \};
\]
this presheaf is in fact a sheaf. Moreover, we define a functor $h^! : \text{Sh}(X) \to \text{Sh}(Z) : \mathcal{F} \mapsto h^*\mathcal{F}^Z$, and extend it to complex.

**Proposition 8** ([1 Prop. II.6.6]). The functor $h^!$ is right adjoint to $h_!$.

However, passing to the constructible derived category, $Rf_!$ indeed has a right adjoint.

**Theorem 9** (Verdier duality; cf. [5]). There exists a functor $f^! : D^b_c(Y) \to D^b_c(X)$, called the twisted inverse image functor, such that the functors
\[
(2) \quad Rf_* R\mathcal{H}om (\cdot, f^! (-)) \cong R\mathcal{H}om (Rf_! (-), \cdot)
\]
are isomorphic. In particular, by taking global sections, we conclude that $f^!$ is right adjoint to $Rf_!$.

Moreover, we have $(gf)^! \cong f^! g^!$.

**Theorem 10** (Proper base change). For any pullback square
\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow{g'} & & \downarrow{g} \\
Y' & \xrightarrow{g} & Y
\end{array}
\]
there is an isomorphism of functors
\[
f^* \circ Rg_! \cong Rg^! \circ f^{**}.
\]

**Proposition 11** (??). The tensor product and internal hom functors both preserve constructibility.

**Theorem 12** (Projection formula; cf. [2 Prop. 1.4.1]). There is an isomorphism of functors
\[
(3) \quad (Rf_! (-)) \otimes^L (\cdot) \cong Rf_! \left( (\cdot) \otimes^L f^*(-) \right).
\]

**Corollary 13.** There exists an isomorphism of functors
\[
(4) \quad R\mathcal{H}om \left( f^*(-), f^! (-) \right) \cong f^! R\mathcal{H}om (\cdot, \cdot).
\]

**Proof.** There is a chain of isomorphisms
\[
\begin{align*}
\text{Hom} \left( \cdot, f^! R\mathcal{H}om (\cdot, \cdot) \right) & \cong \text{Hom} \left( Rf_! (-), R\mathcal{H}om (\cdot, \cdot) \right) \\
& \cong \text{Hom} \left( Rf_! (-) \otimes^L (\cdot), \cdot \right) \\
& \cong \text{Hom} \left( Rf_! \left( (\cdot) \otimes^L f^*(-) \right), \cdot \right) \\
& \cong \text{Hom} \left( (\cdot) \otimes^b f^*(-), f^! (-) \right) \\
& \cong \text{Hom} \left( \cdot, R\mathcal{H}om \left( f^*(-), f^! (-) \right) \right),
\end{align*}
\]
where the first and the fourth isomorphisms are from the adjunction $Rf_! \dashv f^!$, the second and the last are from the derived tensor-hom adjunction, and the third is the projection formula. Now the desired isomorphism follows from Yoneda lemma. \hfill \Box

**Definition.** The **dualizing complex** is defined as $\omega_X := p_X^! k$, where $k$ is regarded as a sheaf on a point. The **Verdier dual** functor by
\[
\mathbb{D}_X := R\mathcal{H}om (\cdot, \omega_X) : D^b_c(X)^{\text{op}} \to D^b_c(X).
\]

In particular, $\omega_X \cong \mathbb{D}_X \mathbb{D}_X$. If $X = \ast$, then $\mathbb{D}_X$ is simply $R\text{Hom} (\cdot, k)$.

**Proposition 14** (??). The Verdier dual $\mathbb{D}$ is an involution, that is, $\mathbb{D}^2 = \text{Id}$.
Corollary 15. There are isomorphisms of functors
\[ \mathcal{D}_Y Rf_! \cong Rf_* \mathcal{D}_X \quad \text{and} \quad \mathcal{D}_X f^! \cong f^* \mathcal{D}_Y. \]

Proof. Substituting \( \omega_Y \) into the second argument of both sides of (2) and (4), respectively, and observe that \( \omega_X = f^! \omega_Y \) by functoriality in \( f \).

Proposition 16 ([2] Prop. 3.3.6). If \( X \) is smooth (and automatically orientable) of dimension \( n \), then \( \omega_X \cong \underline{k}_X[2n] \).

Corollary 17. In the context of the projection formula, there is also an isomorphism of functors
\[ f^! \circ Rg_* \cong Rg'_* \circ f^! . \]

Proof. This follows from the projection formula, along with the observations that the Verdier dual \( \mathcal{D} \) is an involution and intertwines \( ! \) and \( * \).

Corollary 18. For any complex \( \mathcal{F} \) on a manifold \( X \) of dimension \( n \), we have
\[ \mathcal{D} \mathcal{F} \cong \mathcal{F}^\vee [2n], \]
where \( (-)^\vee \) is the (ordinary) dual functor \( R\text{Hom}(-, \underline{k}_X) \).

Proof. Compute
\[ \mathcal{D} \mathcal{F} := R\text{Hom}(\mathcal{F}, \omega_X) \cong R\text{Hom}(\mathcal{F}, \underline{k}_X[2n]) \cong R\text{Hom}(\mathcal{F}, \underline{k}_X)[2n] =: \mathcal{F}^\vee [2n]. \]

Corollary 19 (Poincaré duality). If \( k \) is a field, there is an isomorphism
\[ H^*(X; k)^\vee \cong H^{2n-*}(X; k). \]

Proof. There is a chain of isomorphisms of functors
\[ \mathcal{D} Rp_X \underline{k}_X \cong Rp_X \mathcal{D} \underline{k}_X \cong Rp_X \omega_X \cong Rp_X \underline{k}_X[2n]. \]
Note that \( k \) being a field is an injective \( k \)-module, so \( \mathcal{D} = R\text{Hom}(-, k) \) is the same as the dual functor \( (-)^\vee \). Therefore, taking the \( (-i) \)-th cohomology of the two complexes gives an isomorphism
\[ H^i(X; \underline{k}_X)^\vee \cong H^{2n-i}(X; \underline{k}_X). \]

Definition. For any \( i \in \mathbb{Z} \) and \( \mathcal{F} \in D^b(X) \), the \( i \)-th homology and \( i \)-th Borel–Moore homology on \( X \) with coefficient \( \mathcal{F} \) are
\[ H_i(X; \mathcal{F}) := R^{-i} p_X \mathcal{D} \mathcal{F} \quad \text{and} \quad H_i^!(X; \mathcal{F}) := R^{-i} p_X \mathcal{D} \mathcal{F}, \]
respectively.

They are defined so that homology is Verdier dual to cohomology with compact support, and Borel–Moore homology is dual to cohomology, just as the classical case. In fact, there are isomorphisms
\[ H^i(X; M) \cong H^{-i}(X; \mathcal{D}_M X) \cong R^{-i} p_X p_X^! M^\vee, \]
\[ H_i^!(X; M) \cong H^{-i}_!(X; \mathcal{D}_M X) \cong R^{-i} p_X p_X^! M^\vee. \]

Since we will be mostly working in the derived category, it is convenient to drop the direction of derivations from the notations, and simply write
\[ f_* := Rf_*, \quad f^! := Rf^!, \quad \text{Hom}(-, -) := R\text{Hom}(-, -), \quad - \otimes - := - \otimes^L -. \]

1.4. Six-functor formalism. In this section we collect properties of the six functors on the constructible derived category.
• Adjunctions: there are three adjoint pairs: \( f^* \dashv f_* \), \( f_! \dashv f^! \), and \(- \otimes F \dashv \text{Hom}(F, -)\).

• Proper maps and open embeddings: There is a natural transformation \( f_! \rightarrow f^* \), which is an isomorphism if \( f \) is proper, for instance if \( f = i : Z \hookrightarrow X \) is a closed embedding. For an open embedding \( j : U \hookrightarrow X \), we have \( j^! = j^* \). In particular, in this case we have adjunctions

\[
i^* \dashv i_* = i^* - i^!, \quad j_! \dashv j^! = j_! - j^!
\]

• Open-closed triangles: Given an open-closed decomposition \( X = U \sqcup Z \), we have functorial exact triangles

\[
i^! i^! \rightarrow \text{Id} \rightarrow j_* j^* \quad [1]
\]

\[
\rightarrow j_! j^! \rightarrow \text{Id} \rightarrow i_* i^* \quad [1]
\]

Proof. Exactness can be tested on stalks. \( \square \)

**Corollary 20.** There are long exact sequences in Borel–Moore homology and in cohomology with compact support:

\[
\cdots \rightarrow H^i_{n+1}(U) \rightarrow H^i_n(Z) \rightarrow H^i_n(X) \rightarrow H^i_{n-1}(U) \rightarrow \cdots
\]

\[
\cdots \rightarrow H^m_{n-1}(Z) \rightarrow H^m_n(U) \rightarrow H^m_n(X) \rightarrow H^m_{n+1}(Z) \rightarrow \cdots
\]

Proof. The homology sequence is obtained by specializing the first triangle to the dualizing sheaf \( \omega_X \) and taking \( R^{-n}p_{X*} \), while the cohomology sequence is obtained by specializing the second triangle to the constant sheaf \( k_X \) and taking \( R^n p_{X!} \). \( \square \)

In general, for a locally closed embedding \( h : W \hookrightarrow X \), we write \((-)W! := h_! h^*\). For \( h = j \) open, this is equal to \( j_! j^! \), while for \( h = i \) closed, this is equal to \( i_* i^* \). Thus the second open-closed triangle can be rewritten into

\[
\mathcal{F}_{W!} \rightarrow \mathcal{F} \rightarrow \mathcal{F}_{Z!} \quad [1]
\]

for any complex \( \mathcal{F} \).

• Grand octahedron: Let \( Z \subseteq Y \subseteq X \) be a filtration by closed sets, and let \( U = X - Y \), \( V = Y - Z \), and \( W = U \cup V = X - Z \). For any complex \( \mathcal{F} \), we have an octahedron

\[
\begin{array}{ccc}
\mathcal{F}_{Z!} & \xrightarrow{[1]} & \mathcal{F}_{V!} \\
[1] & + & + \\
\mathcal{F}_{U!} & \xleftarrow{[1]} & \mathcal{F}
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{F}_{Z!} & \xleftarrow{[1]} & \mathcal{F}_{V!} \\
+ & + & + \\
\mathcal{F}_{U!} & \xrightarrow{[1]} & \mathcal{F}
\end{array}
\]

where the triangles with “+” are exact, and those without commute.

2. **Perverse sheaves**

2.1. **Truncation structures.** Any abelian category \( \mathcal{A} \) embeds into \( \text{Ch}(\mathcal{A}) \) by sending each object \( A \) into the complex concentrated in degree 0. The composition \( \mathcal{A} \rightarrow D(\mathcal{A}) \) is fully faithful, hence exhibits \( \mathcal{A} \) as an abelian subcategory of the triangulated category \( D(\mathcal{A}) \). This is abstracted as follows:
Definition. Let $\mathcal{D}$ be a triangulated category. $\mathcal{C} \subseteq \mathcal{D}$ is an admissible abelian subcategory if

1. $\mathcal{C}$ is full;
2. $\text{Hom}(\mathcal{C}, \mathcal{C}[i]) = 0$ for $i < 0$;
3. any short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{C}$ extends to an exact triangle

$$A \to B \to C \to [1].$$

Clearly $\mathcal{A} \subseteq D(\mathcal{A})$ is an admissible abelian subcategory.

On the other hand, $\mathcal{A}$ may be recovered from $D(\mathcal{A})$ by taking (the skeleton of) the full subcategory on complexes whose cohomology concentrates in degree 0. To see this, we define two truncation functors

$$\tau_{\leq p}^n M := \begin{cases} M^n & n < p \\ \text{Ker} d^n & n = p \\ 0 & n > p, \end{cases}$$

and

$$\tau_{\geq p}^n M := \begin{cases} 0 & n < p \\ \text{Coker} d^{p-1} & n = p \\ M^n & n > p. \end{cases}$$

Then clearly we have natural transformations $\tau_{\leq p} \to \text{Id}$ and $\text{Id} \to \tau_{\geq p}$, and also note that

$$H^n(\tau_{\leq p} M) \cong \begin{cases} H^n(M) & n \leq p \\ 0 & n > p, \end{cases}$$

and

$$H^n(\tau_{\geq p} M) \cong \begin{cases} 0 & n \leq p \\ H^n(M) & n > p. \end{cases}$$

Therefore, if $H^n(M) = 0$ for $n \neq 0$, then the span

$$M \sim \tau_{\geq 0} M \sim \tau_{\leq 0} \tau_{\geq 0} M \cong H^0(M)$$

is a quasi-isomorphism, i.e. an isomorphism in $D(\mathcal{A})$. In light of this construction, we may define $D^{\leq 0}(\mathcal{A})$ and $D^{\geq 0}(\mathcal{A})$ to be the full subcategories on complexes fixed by $\tau_{\leq 0}$ and $\tau_{\geq 0}$, respectively, then we have an equivalence of categories

$$\mathcal{A} \sim D^{\leq 0}(\mathcal{A}) \cap D^{\geq 0}(\mathcal{A}).$$

This motivates the following abstraction.

Definition. A truncation structure, or “$t$-structure”, on a triangulated category $\mathcal{D}$ is a pair of full subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$; we set $\mathcal{D}^{<i} := \mathcal{D}^{\leq 0}[-i]$ and $\mathcal{D}^{>i} := \mathcal{D}^{\geq 0}[-i]$ (we shall also use, whenever convenient, the notations $\mathcal{D}^{<i} := \mathcal{D}^{\leq i-1}$ and $\mathcal{D}^{>i} := \mathcal{D}^{\leq i+1}$). These categories are subject to the following conditions:

1. $\text{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$.
2. $\mathcal{D}^{\leq -1} \subseteq \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1} \subseteq \mathcal{D}^{\geq 0}$.
3. For any $M \in \mathcal{D}$, there exists an exact triangle

$$M^{\leq 0} \to M \to M^{\geq 1} \to [1]$$

with $M^{\leq 0} \in \mathcal{D}^{\leq 0}$ and $M^{\geq 1} \in \mathcal{D}^{\geq 1}$.

The intersection $\mathcal{C} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the core (or heart) of the $t$-structure.

It follows immediately that, for $p < q$, we have

- $\mathcal{D}^{\leq p} \subseteq \mathcal{D}^{\leq q}$,
- $\mathcal{D}^{\geq q} \subseteq \mathcal{D}^{\geq p}$, and
- $\text{Hom}(\mathcal{D}^{\leq p}, \mathcal{D}^{\geq q}) = 0$. 


Clearly the full subcategories $D^\leq 0(\mathcal{A})$ and $D^\geq 0(\mathcal{A})$ form a t-structure on $D(\mathcal{A})$, and the core of this t-structure is $\mathcal{A}$.

**Remark 21.** In fact, $\mathcal{D}^p \cong (\mathcal{D}^\leq p)^\perp$ and similarly $\mathcal{D}^p \cong (\mathcal{D}^\geq p)^\perp$.

To obtain the truncation functors associated to a t-structure, we need the following lemma.

**Lemma 22 (Uniqueness of triangle).** In the diagram in a triangulated category

$$
\begin{array}{cccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \ [1] \\
\downarrow f & & \downarrow g & & \downarrow h \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' \ [1]
\end{array}
$$

where the rows are exact triangles, the following are equivalent:

1. there exists a morphism $(f, g, h)$ between the exact triangles;
2. there exists $f$ making the first square commute;
3. there exists $h$ making the second square commute;
4. $v' gu = 0$.

If in addition $\text{Hom}(X[1], Z') \cong \text{Hom}(X, Z'[-1]) = 0$, then all existence above are unique.

**Proof.** Clearly (1) contains (2) and (3), either of which implies (4). Moreover, (2) or (3) implies (1) by axiom of triangulated category. Thus it remains to show that (4) implies (2) and (3). To this end, consider the long exact sequences

$$
\cdots \rightarrow \text{Hom}(X, Z'[−1]) \rightarrow \text{Hom}(X, X') \xrightarrow{w' o} \text{Hom}(X, Y') \rightarrow \text{Hom}(X, Z') \rightarrow \cdots
$$

and

$$
\cdots \rightarrow \text{Hom}(X[1], Z') \rightarrow \text{Hom}(Z, Z') \xrightarrow{v' o} \text{Hom}(Y, Z') \rightarrow \text{Hom}(X, Z') \rightarrow \cdots
$$

whereas the uniqueness statements follow if $\text{Hom}(X[1], Z') = 0$. $\square$

**Proposition 23.** Suppose $(\mathcal{D}^\leq 0, \mathcal{D}^\geq 0)$ is a t-structure on a triangulated category $\mathcal{D}$. Then the inclusions $\mathcal{D}^\leq p \hookrightarrow \mathcal{D}$ and $\mathcal{D}^\geq p \hookrightarrow \mathcal{D}$ have, respectively, right and left adjoints

$$
\tau^\leq p : \mathcal{D} \rightarrow \mathcal{D}^\leq p \\
\tau^\geq p : \mathcal{D} \rightarrow \mathcal{D}^\geq p,
$$

called the truncation functors. Moreover, the truncation functors satisfy

$$
\tau^\leq p(M[q]) \cong (\tau^\leq p+q M) [p] \\
\text{and} \quad \tau^\geq p(M[q]) \cong (\tau^\geq p+q M) [q].
$$

**Proof.** Since $\text{Hom}(M^\leq 0[1], M^\geq 1) = 0$, we apply the uniqueness lemma above to conclude that the functors

$$
\tau^\leq 0 : \mathcal{D} \rightarrow \mathcal{D}^\leq 0 \\
\tau^\geq 0 : \mathcal{D} \rightarrow \mathcal{D}^\geq 0
$$

are well-defined. To see the adjunction, for $N^\leq 0 \in \mathcal{D}^\leq 0$ and $N^\geq 1 \in \mathcal{D}^\geq 1$, apply the uniqueness lemma again to the pairs of exact triangles

$$
\begin{array}{ccc}
N^\leq 0 & \xrightarrow{\text{id}} & N^\leq 0 \rightarrow 0 \ [1] \\
\downarrow & & \downarrow 0 \\
M^\leq 0 & \rightarrow & M \rightarrow M^\geq 1 \ [1]
\end{array}
$$
and

\[ \begin{array}{c}
M \leq 0 \rightarrow M \rightarrow M \geq 1 [1] \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \rightarrow N \geq 1 \rightarrow N \geq 1 [1]
\end{array} \]

Furthermore, define

\[ \tau \leq p M := (\tau \leq 0 (M[p])) [-p] \quad \text{and} \quad \tau \geq p M := (\tau \geq 1 (M[p - 1])) [-p]; \]

since \([1]\) is an auto-equivalence, the desired adjunction statements follows from those of \(\tau \leq 0\) and \(\tau \geq 1\). The properties follow directly from the definitions. □

**Theorem 24.** The core \(\mathcal{C}\) of a t-structure \((\mathcal{D} \leq 0, \mathcal{D} \geq 0)\) on a triangulated category \(\mathcal{D}\) is an admissible abelian subcategory. Moreover, the functors

\[ H^p : \mathcal{D} \rightarrow \mathcal{C} \]

\[ M \mapsto \tau \leq 0 \tau \geq 0 (M[p]) \]

are cohomological, namely, takes exact triangles to short exact sequences.

**Proof.** Given a morphism \(f : A \rightarrow B\) in \(\mathcal{D}\), extend it to an exact triangle \(A \xrightarrow{f} B \rightarrow C \overset{[1]}{\rightarrow}\). Then cokernel and kernel are defined by

\[ \text{Coker } f := \tau \geq 0 C \quad \text{and} \quad \text{Ker } f := \tau \leq 0 (C[-1]). \]

For the proof that \(\mathcal{C}\) is abelian, see [2, Prop. 10.1.11].

We prove that \(\mathcal{C}\) is admissible. \(\mathcal{C}\) is full since both \(\mathcal{D} \leq 0\) and \(\mathcal{D} \geq 0\) are. For \(i < 0\), since \(\mathcal{C}[i] \subseteq \mathcal{D} \leq 0 \mathcal{D} \geq -1\), we have

\[ \text{Hom}(\mathcal{C}, \mathcal{C}[i]) = \text{Hom}(\mathcal{C}[-i], \mathcal{C}) \subseteq \text{Hom}(\mathcal{D} \leq 0, \mathcal{D} \geq 0) = 0. \]

Given a short exact sequence \(0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0\) in \(\mathcal{C}\), we have from above that

\(A \cong \text{Ker } g \cong \tau \leq 0 (A'[\geq 1])\) \quad \text{and} \quad \(C \cong \text{Coker } f \cong \tau \geq 0 C').\]

where \(A'\) and \(C'\) extend \(g\) and \(f\) into exact triangles

\[ B \xrightarrow{g} C \rightarrow A' \overset{[1]}{\rightarrow} \quad \text{and} \quad A \xrightarrow{f} B \rightarrow C' \overset{[1]}{\rightarrow}, \]

in \(\mathcal{D}\). Applying the uniqueness lemma to the diagram

\[ \begin{array}{c}
A \rightarrow B \rightarrow C' \overset{[1]}{\rightarrow} \\
\downarrow \quad \downarrow \quad \downarrow \\
A'[-1] \rightarrow B \rightarrow C \overset{[1]}{\rightarrow}
\end{array} \]

and observing \(\text{Hom}(A[1], C) = 0\) since \(A[1] \in \mathcal{D} \leq -1\) while \(C \in \mathcal{D} \geq 0\), we conclude that the dotted arrows exist uniquely to make the diagram a morphism between exact triangles. To construct the inverse morphism, we use the uniqueness lemma again by checking

\[ \text{Hom}(A'[-1], C') = 0. \]

Note that since \(\mathcal{D} \leq p\) and \(\mathcal{D} \geq p\) are both closed in extension, we have \(A', C' \in \mathcal{D} \leq 0 \cap \mathcal{D} \geq -1\), so \(A'[-1] \in \mathcal{D} \leq 1 \cap \mathcal{D} \geq 0\).

For that \(H^0\) and thus \(H^p\) are cohomological, see [2, Prop. 10.1.12]. □

2.2. **Recollement.** This section is devoted to proving gluing t-structures in the special case which is referred to as the “recollement” situation. Specifically, consider the open-closed
decomposition of a space $X = Z \sqcup U$, with the embeddings denoted by $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$. Then the embeddings induces a sequence of functors

$$D_Z \xrightarrow{i_*} D \xrightarrow{j^*} D_U.$$ 

it has the following properties:

1. There are adjunctions

   $$i^* \dashv i_* = i_! \dashv i^!,$$

   $$j_! \dashv j^! = j^* \dashv j_*.$$

2. **Appendix A. Whitney stratification**

   **References**


