MAT-127A-2  FINAL EXAM  March 23, 2004

2 hours, 7 problems, closed books, closed notes, no calculators

1. [9 points] Give lim inf$(s_n)$ and lim sup$(s_n)$ for the following sequences $(s_n)$ (no proofs are needed).

   (a) [3 points]  $(-1)^n(1 - \frac{1}{n})$

   (b) [3 points]  $\sin(\frac{\pi n}{3})$

   (c) [3 points]  $(-1)^n n - n$

2. [12 points] Assume that $\lim_{x \to 1^+} f(x) = 0$ and that $f(x) > 0$ for all $x \in \mathbb{R}$. Find $\lim_{x \to 1^+} \frac{1}{f(x)}$ and prove your assertion.
3. [18 points] Let

\[
f(x) = \begin{cases} 
-x & \text{for } x < 1 \\
x & \text{for } x \geq 1.
\end{cases}
\]

Prove or disprove the following statements:
(a) \( f \) is uniformly continuous on \((-\infty, 1)\)
(b) \( f \) is uniformly continuous on \((1, \infty)\)
(c) \( f \) is uniformly continuous on \((1, \infty) \cup (-\infty, 1)\).
4. [9 points] Prove that $x^3 = \cos x$ for some $x$ in $(0, \frac{\pi}{2})$.

5. [12 points] Prove that the function $f(x) = \ln x - \frac{x-1}{e-1}$ attains its maximum on $(1, e)$. 
6. [20 points] Let $f$ be a function such that

$$|f(x)| \leq x^2 \quad \text{for all } x \in [-1, 1].$$

Prove that $f$ is differentiable at $0$ and $f'(0) = 0$. 


7. [20 points] Let \( (s_n) \) and \( (t_n) \) be two sequences such that \( (s_n) \) is unbounded and \( (t_n) \) is convergent (to a real number).

(a) Prove that the sequence \( (s_n + t_n) \) is unbounded.

(b) Will the result in (a) remain always true if \( (t_n) \) diverges to \(+\infty\) (instead of being convergent)? (Prove this or find an example showing that this fails. Just a correct guess will not earn any credit)