3.4 Independent Events.

- Intuitive def. of independence of $E, F$:
  \[ P(E|F) = P(E), \quad P(F|E) = P(F) \] 
  \[ (*) \]
  i.e. $F$ does not affect the likelihood of $E$, and vice versa.

Ex: Flip coin 10 times.
  \[ P\{10'th\; flip=H \mid \text{first 9 flips=H}\} = \frac{1}{2} \]
  because $E, F$ independent.

- The identities in $(*)$ can be written as:
  \[ \frac{P(E \cap F)}{P(F)} = P(E), \quad \frac{P(F \cap E)}{P(E)} = P(F). \]

  \[ \text{SAME!} \]

Def Events $E, F$ are independent if \[ P(E \cap F) = P(E) \cdot P(F) \]

- Independence implies $(*)$

- Warning: independent $\neq$ mutually exclusive

Def Events $E_1, E_2, \ldots$ are independent if

\[ (*) \quad P(E_i \cap E_j) = P(E_i) \cdot P(E_j) \quad \text{for all pairs } \{i, j\}, \text{of distinct integers } i, j; \]

\[ P(E_i \cap E_j \cap E_k) = P(E_i) \cdot P(E_j) \cdot P(E_k) \quad \text{for all triples } \{i, j, k\}, \ldots \]

\[ \text{etc. . . .} \]

- Note: pairwise independence (identities $(*)$ only)
does not imply independence

Example: roll 3 dice

$E =$ \{first die = 3\}, $F =$ \{second die = 4\}, $G =$ \{sum of all three = 7\}.

Check that $E, F$ are indep, $F, G$ indep,
but $E, F, G$ are not indep.
Ex: A plane has 4 engines. It can fly if 2 or more engines are functioning.

Each engine fails independently with probability $p$.

$P$ {plane flies} $=$?

$E_i = \{\text{engine } i \text{ functions}\}$.

$P$ {plane flies} $= 1 - P$ {most one engine functions}.

$= 1 - P$ {no engine functions} $- P$ {one functions}.

$= 1 - (1-p)^4 - \frac{4p(1-p^3)}{4}$

Choosing the functioning engine.

Computing probabilities by conditioning.

Ex: (Networks). Consider the network:

Each link fails with prob. $p$ independently.

Find the prob. that $A$ and $B$ are connected.

Condition on the state of the vertical link; $V = \{\text{vertical link works}\}$.

$P(C) = P(C|V)P(V) + P(C|V^c)P(V^c)$.

1) $P(C|V) = ?$. If $V$ occurs, the network can only fail if $A$ is not connected to the vert link, or $B$ is not connected to vert. link, or both.

$P(C|V) = P + P^2 - P^4$. (by inclusion-excl.)

2) $P(C|V^c) = ?$. If $V^c$ occurs, the network looks like this:

Network works if either both top links work, or both bottom links work, or both.

$P(C|V^c) = (1-p)^3 + (1-p)^2 - (1-p)^4$.

Hence:

$P(C) = (1-2p^2+p^3)(1-p) + \left(2(1-p)^2-(1-p)^4\right)p = 1-2p^2-2p^3+5p^4-2p^5$. 
Example (Simple random walk)

A particle is placed at \( k \).

Each second, the particle moves 1 step to the left or to the right independently with prob \( \frac{1}{2} \) each.

What is the probability that the particle reaches \( n \) before reaching \( 0 \)? \( E_k \) \( \downarrow \)

Condition on the first step, \( L \) or \( R \).

\[
P(E_k) = P(E_k|L)P(L) + P(E_k|R)P(R) = P(E_{k+1}) + \frac{1}{2} + P(E_{k+1}) - \frac{1}{2} = \frac{P(E_{k+1})}{2}
\]

(Conditioned on \( L \), the game "resets" with particle at \( k+1 \) instead of \( k \))

Denoting \( P_k = P(E_k) \), we obtain

\[
\begin{cases}
P_k = \frac{1}{2}(P_{k-1} + P_{k+1}), & k = 1, ..., n-1 \\
P_0 = 0, P_n = 1
\end{cases}
\]

\( n+1 \) linear equations with \( n+1 \) unknowns. Solving (do this!) gets us

\[
P_k = \frac{k}{n}
\]

- Interpretations:

(a) Finance: \( 0 = \) bankruptcy, \( n = \) payoff, \( k = \) initial capital

\[
P(\text{payoff before bankruptcy}) = ?
\]

For this example, a biased random walk is more relevant, where \( P(R) = p, P(L) = 1 - p \) for some \( 0 < p < 1 \) (See Ross Ex 4E).

(b) Recurrence: for \( n \to \infty \), \( P_k \to 0 \) (with \( k \) fixed).

\( P_k \) with prob \( \frac{1}{2} \) ("almost surely"), the particle will visit any given site (or this case)

(If last, will return home by random walk).

26