

**International Society of Soil Mechanics
and Geotechnical Engineering**

Prediction and Simulation Methods in Geomechanics

State-of-the-art report on selected topics by TC-34

Technical Committee 34 of ISSMGE

October 2005

Limit analysis in geotechnical engineering

Radoslaw L. Michalowski

Department of Civil and Environmental Engineering, University of Michigan, Ann Arbor, U.S.A.

ABSTRACT: The background of limit analysis is given first, and various aspects, considered in the last decades, are described. These include: application of limit analysis for soils governed by the non-associative flow rule, solving problems with frictional boundaries, including pore water pressure in limit analysis calculations, application to materials with non-linear yield conditions, seismic effects, and application to reinforced soils. These aspects are presented briefly with the exception of the pore water consideration, where, in addition to a short description, some analytical background is given. 3D analyses are discussed. Because the static approach of limit analysis is hindered by the difficulties in finding admissible stress fields, the major part of the report focuses on the kinematic approach. The static approach, however, can be used conveniently if the numerical approach is used, and this approach is mentioned in the last section.

1 INTRODUCTION

Although limit analysis is a well-established method in structural engineering, its application in geotechnical engineering has not been accepted as widely. The first contributions to geotechnical limit analysis (Drucker and Prager 1952) did not lead to immediate implementation in geotechnical engineering. Several developments in the last two decades made limit analysis more attractive as an engineering tool. These include a practical approach to accounting for the nonassociativity of plastic deformation (flow), calculations with nonlinear yield conditions, accounting for presence of pore water pressure, and application of limit analysis to reinforced soil. In addition, development of numerical limit analysis methods now makes it possible to more efficiently address geotechnical problems with nonhomogeneous soils and complicated boundary conditions. These contributions are summarized in this report.

Emphasis is placed on the contributions to the kinematic approach of limit analysis, since the static approach is used rarely because of the difficulties in finding admissible stress fields. However, the numerical approach is mentioned in the last section, and this approach has an advantage of being effective in both static and kinematic limit analysis.

2 BACKGROUND

A method that essentially makes use of the kinematic theorem of limit analysis can be traced back to the XVIII century when Coulomb, in his 1773 *Essai*, intuitively used the “maximum and minimum rules” to arrive at solutions to forces associated with structural collapse. The modern use of this method is found more than 150 years later in the work of Gvozdev (1936), and the theoretical justification in terms of the formal theorems was presented later by Hill (1951) and Drucker, Prager and Greenberg (1952). The theorems of limit analysis can be considered as special cases of shakedown theorems (Melan 1938, Koiter 1960).

The theorems of limit analysis can be proved for materials that conform to perfect plasticity with a convex yield criterion, $f(\sigma_{ij}) = 0$, and with deformation governed by the normality rule. The rate of work during true plastic deformation for such materials is not smaller than the rate of work of any statically admissible stress field, σ_{ij}^s , during such deformation

$$\sigma_{ij} \dot{\epsilon}_{ij} \geq \sigma_{ij}^s \dot{\epsilon}_{ij} \quad (1)$$

This inequality is often referred to as the *principle of maximum work*. For brevity, we dropped the superscript *pl* in the plastic

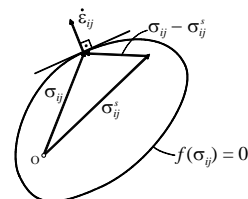


Figure 1. Illustration of the principle of maximum plastic work.

strain rate. Now, integrating inequality (1) and replacing the right-hand-side with the work rate of external forces using the principle of virtual work (in the rate form, power), we obtain

$$\int_V \sigma_{ij} \dot{\epsilon}_{ij} dV \geq \int_S T_i^s v_i dS + \int_V X_i v_i dV \quad (2)$$

Hence, the rate of work of external loads compatible with any statically admissible stress field on true plastic deformation is not larger than the true rate of internal plastic work. If T_i is the load on S causing collapse of the structure (active load: $\int T_i v_i dS > 0$), then the inequality in (2) can be rephrased to read:

Theorem I: If an equilibrium distribution of stress σ_{ij} exists that satisfies stress boundary conditions and is everywhere below yielding ($f < 0$), the structure will not collapse. As this theorem

allows bounding the limit load from below, it is often called the lower bound theorem. However, this static approach of limit analysis will yield an upper bound to a reaction (passive force). For passive load $\int T_i v_i dS < 0$ and from eq. (2) one concludes that the estimate of the work rate $\int T_i v_i dS$ is an upper bound. The use of this theorem requires construction of admissible stress fields, which becomes cumbersome, particularly when the boundaries are complicated, and an extension of the stress field into the half-space needs to be found.

The kinematic approach is based on constructing admissible mechanisms of failure, and the kinematic theorem is used more often, as reasonable collapse mechanisms are easier to construct than admissible stress fields. Introducing kinematically admissible strain rate field $\dot{\epsilon}_{ij}^k$ and associated stress σ_{ij}^k , one can prove that, in any kinematically admissible mechanism, the rate of plastic internal work is not less than the rate of work of true external loads

$$\int_V \sigma_{ij}^k \dot{\epsilon}_{ij}^k dV \geq \int_S T_i v_i dS + \int_V X_i v_i^k dV \quad (3)$$

Theorem II: If a kinematically admissible collapse mechanism can be found for which the rate of work of external forces exceeds the rate of internal work, the structure will collapse.

This theorem is usually referred to as the upper bound theorem, because it allows one to calculate an upper bound to the limit force causing failure (a lower bound if a passive force is sought). It needs to be clarified that the theorem indicates whether the work rate of the unknown boundary force is an upper or lower bound; in order to pass judgment on the force itself, the boundary velocity needs to be constant, so that the work rate can be written as $v_i \int T_i dS$.

Limit analysis theorems are most often used to estimate a limit load, but they can be used to find an estimate of the critical height of a slope, the strength of reinforcement needed to avoid collapse, or the critical acceleration of a structure subjected to seismic shaking. Developments of the last two decades are described in the next sections; these developments make limit analysis useful in geotechnical engineering.

3 NONASSOCIATIVE FLOW RULE IN LIMIT ANALYSIS

Because the proof of limit analysis theorems requires normality of the flow (associativity), the theorems are not valid for soils that are governed by the nonassociative flow rule. It can be shown, however, that the true solution to active limit loads on structures built of nonassociative material is bounded from above by the solution for associative material, and it is bounded from below by an estimate made for a fictitious material with the yield condition $G(\sigma_{ij}) = 0$. Surface $G = 0$ is convex in the stress space and it is inscribed into the yield surface $f = 0$ (surface $G = 0$ is constructed so that, for any σ_{ij} such that $f(\sigma_{ij}) = 0$, there is a corresponding stress σ_{ij}'' on the G -surface such that $\dot{\epsilon}_{ij}$ is normal to $G = 0$ at σ_{ij}''). The two statements were given by Radenkovic (1962), and they are often referred to as the limit analysis theorems for a *nonstandard* material. Other early contributions to limit analysis for materials that are not governed by the normality rule include de Josselin de Jong (1964), Palmer (1966), Collins (1969) and Mróz and Drescher (1969).

While the two theorems of Radenkovic indicate how the results of limit analysis should be interpreted when the material does not conform to the normality rule, they do not indicate a practical manner in which to calculate the overestimation of the *associative* results versus those for the nonstandard material. Such a method was suggested by Drescher and Detournay (1993), and, even though no proof is available that it produces a rigorous bound, it is briefly described next, as it is a development useful for applications in geotechnical engineering.

This method was derived for rigid-block mechanisms and materials that obey the Mohr-Coulomb yield condition, and it stems from consideration of the traction on the velocity discontinuities in nonassociative materials. Since the velocity discontinuities must be velocity characteristics, the inclination of traction on them differs for associative and nonassociative materials. Consequently, the rate of internal work (dissipation) along velocity discontinuities is different for the two materials. For kinematically admissible translational mechanisms there is a unique solution to the traction forces on the velocity discontinuities (Michalowski 1989). As this solution is statically determinate, it must be independent of the inclination of the velocity discontinuity vectors. Consequently, one can calculate the limit load for nonassociative material by replacing the true medium with a substitution associative material, but with the strength parameters replaced as follows

$$\tan \phi^* = \frac{\cos \psi \sin \phi}{1 - \sin \psi \sin \phi}, \quad c^* = c \frac{\cos \psi \cos \phi}{1 - \sin \psi \sin \phi} \quad (4)$$

where ψ is the dilation angle, and ϕ^* and c^* are the strength parameters of the substitution material. This approach is used today often in the kinematic approach, and it yields quite reasonable results (Michalowski and Shi 1995).

4 FRICTIONAL BOUNDARIES

Calculations of work along frictional boundaries where sliding occurs are hindered due to unknown distribution of stress. Such calculations are possible if sliding occurs with a uniform discontinuity vector, but if the velocity discontinuity vector varies, the integral dissipation rate on a sliding interface cannot be found directly. This issue was addressed independently by Mróz and Drescher (1969) and Collins (1969) by introducing the normality sliding rule at frictional interfaces. Such a sliding rule leads to perpendicular stress and velocity jump vectors on interfaces. Consequently, the rate of work during interface sliding becomes zero. However, the normality rule on the interface leads to "interface separation." While such an incipient separation is not realistic, it does not contradict the assumptions used to prove the kinematic theorem of limit analysis, and active loads calculated are still rigorous upper bounds to true limit loads. This technique of accounting for frictional boundaries was implemented by Mróz and Drescher (1969) in the problem of hopper flow, and by Collins (1969) in an analysis of a block squeezed between dies. If this method is used in a case where the rate of work on the interface can be calculated directly, the technique suggested and the direct computations produce identical results in terms of limit loads.

5 PORE WATER PRESSURE IN LIMIT ANALYSIS: SEEPAGE AND BUOYANCY FORCES

Including water in the kinematic approach of limit analysis requires that the pore water pressure be considered as an external (but distributed) load acting on the soil. For instance, the work of pore water on soil particle B in Fig. 2 on virtual displacement δu is exactly equal to the work of the buoyancy force acting on particle B and the seepage force. The work of both must be included in limit analysis. It is demonstrated here that this can be done conveniently by considering the work of the pore water pressure on the expansion (dilatancy) of the skeleton, much like the air pressure acting on the shell of a balloon. If the balloon membrane expands, the air pressure inside the balloon does work on this expansion. In the same manner the pore water pressure in the soil will do work on the volumetric strain of the skeleton. Consequently, the work of the pore water pressure

can be included as an additional term on the right-hand-side in theorem (3).

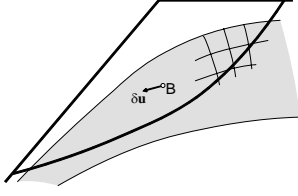


Figure 2. Particle moving during failure in saturated soil.

To derive the terms in the work rate balance equation that account for the presence of water, consider derivative $\partial/\partial x_i$ of the product uv_i , where u is the pore water pressure and v_i is the velocity vector

$$\frac{\partial}{\partial x_i}(uv_i) = \frac{\partial u}{\partial x_i}v_i + u \frac{\partial v_i}{\partial x_i} \quad \text{where} \quad \frac{\partial v_i}{\partial x_i} = -\dot{\epsilon}_{ii} \quad (5)$$

$\dot{\epsilon}_{ii}$ is the volumetric strain rate (the minus sign appears because of the compression-positive sign convention). The water pressure u in eq. (5) can be represented as a function of the hydraulic head h . With the omission of the kinetic part, the hydraulic head is

$$h = \frac{u}{\gamma_w} + Z \quad (6)$$

where γ_w is the unit weight of water, and Z is the elevation head. Substituting the pore pressure u from eq. (6) into eq. (5), and after some transformations, one obtains

$$-\int_V u \dot{\epsilon}_{ii} dV = \int_S un_i v_i dS - \gamma_w \int_V \frac{\partial h}{\partial x_i} v_i dV + \gamma_w \int_V \frac{\partial Z}{\partial x_i} v_i dV \quad (7)$$

The second term on the right-hand side represents the work of the seepage force ($-\gamma_w \partial h / \partial x_i$) in the entire mechanism, and the last term is the work of the buoyancy force. To include the influence of the water on the stability of an earth structure, both the work of seepage and buoyancy forces must be included in the analysis. These terms can be included explicitly, or, based on eq. (7), one can write

$$\dot{W}_u = -\int_V u \dot{\epsilon}_{ii} dV - \int_S un_i v_i dS \quad (8)$$

The first integral is the work of the pore pressure on the volumetric strain of the skeleton, and the second one is the work of the water pressure on boundary S of the structure. The first term on the right-hand side is positive in a field with dilating soil and compressive pore pressure. Examples of application of this technique can be found in Michalowski (1995, 1999).

6 LIMIT ANALYSIS WITH NONLINEAR YIELD CONDITIONS

A benefit of using linear Mohr-Coulomb yield conditions in limit analysis stems from convenient calculations of the internal work rate (dissipation) both on discontinuities and within continually deforming regions. For materials with linear yield conditions and the associative flow rule, this rate is independent of the stress state in the soil. This is not true, however, for materials with nonlinear yield criteria. Early calculations of the dissipation

rate for a non-linear yield condition are those in Chen and Drucker (1969), where the stress vector was uniquely related to the magnitude of an angle of dilatancy through the flow rule. A series of papers appeared on the subject in the late 1980s, one with a suggestion of approaching the problem using variational technique (Zhang and Chen 1987), and others with a proposal of replacing the nonlinear yield condition with one, or a series of straight-line segments (Drescher and Christopoulos 1988, Collins et al. 1988).

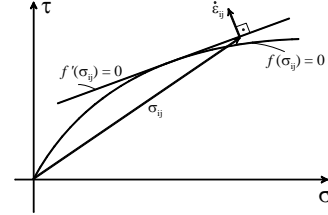


Figure 3. Limit analysis with a non-linear yield condition.

As the non-linear yield condition is inscribed into a fictitious linear or piece-wise linear yield function, calculated limit loads remain rigorous upper bounds. The central issue in this technique is the choice of the linear segment(s), so that the best estimate of the limit load is obtained. The method suggested by Drescher and Christopoulos (1988) is particularly convenient, as it allows one to utilize existing solutions for the linear Mohr-Coulomb criterion to assess limit loads for non-linear yield functions.

7 ACCOUNTING FOR SEISMIC EFFECTS USING LIMIT ANALYSIS

Seismic effects are traditionally accounted for in stability considerations of geotechnical structures by including a quasi-static force associated with some magnitude of seismic acceleration. Including such a force in the kinematic approach of limit analysis is straightforward, through the inclusion of additional work-rate term on the right-hand side of inequality (3). However, such an approach ignores the seismic process (acceleration history) and does not give any insight into the behavior of the structure. The kinematic approach of limit analysis lends itself to seismic analysis consistent with the 'sliding block' concept (Newmark 1965), allowing for calculations of permanent displacements of the structure. Both rotational (Chang et al. 1984) and multi-block translational mechanisms have been successfully used within the limit analysis framework (Michalowski and You 2000) and presented in a form suitable for practical calculations of displacements due to seismic shaking.

8 REINFORCED SOIL STRUCTURES

One of the more recent geotechnical applications of limit analysis is soil reinforcement. It was first suggested in the late 1980s with two approaches: (a) the continuum approach, where the soil and reinforcement are first homogenized, and the anisotropic continuum is considered (e.g., de Buhan et al. 1989, Sawicki 1983, 2000), and (b) the structural approach, where reinforcement is considered as separate structural members (Anthoine 1989, de Buhan and Salençon 1993, Michalowski and Zhao 1995). The latter is often called a mixed approach, since the reinforcement is considered as structural members and the soil is considered as a continuum.

The kinematic approach of limit analysis requires that, when reinforcement is used, the rate of work in reinforcement during failure be added to the left-hand side of inequality (3). There are two modes of reinforcement collapse: tensile failure (or rupture), and pull out of reinforcement from the soil. The former lends

itself to rigorous limit analysis (strict bound) whereas the latter yields an approximate result. This is because the distribution of the normal stress acting on reinforcement is needed for calculations of the pull out force, while this stress can only be

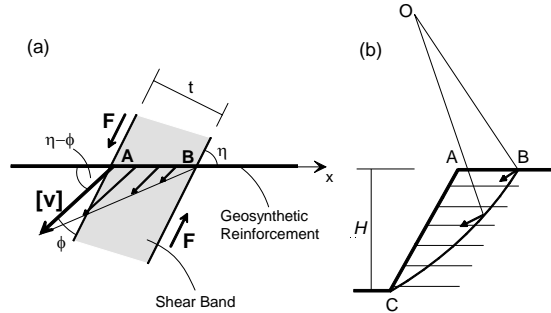


Figure 4. Reinforced soil: (a) Schematic for calculating rate of internal work, and (b) application to reinforced slopes.

approximately estimated without taking on numerical elasto-plastic analysis. Denoting the tensile strength of reinforcement as T_r , the rate of dissipated internal work during failure can be calculated as an integral of the work in the reinforcement segment within the shear band of thickness t , Fig. 4

$$D = \int_0^{t/\sin\eta} T_r |\dot{\epsilon}_r| dx = T_r [v] \cos(\eta - \phi) \quad (9)$$

The work that is associated with shear and bending of soil nails during incipient failure can be calculated in a similar manner (de Buhan and Salençon 1993). Including the internal work of reinforcement in inequality (3) allows one to calculate the necessary strength of the reinforcement to prevent collapse of the structure (or the number of layers, the strength being given). Consideration of the pull-out failure then leads to estimating the necessary length of reinforcement (e.g., Michalowski 1997).

9 THREE-DIMENSIONAL PROBLEMS

Calculations of 3D problems using limit analysis are similar to those in 2D analysis. However, as the method is based on “guessing” the stress distribution or the most realistic failure mechanism, the process becomes cumbersome. For incompressible materials (such as those conforming to the Tresca yield condition and associative flow rule), the solutions require somewhat less effort (Shield and Drucker 1953), but for dilatant materials construction of 3D mechanisms is elaborate (Leca and Dormieux 1990, Michalowski 1985, 2001). These mechanisms typically consist of plane-strain sectors (Chen 1975) or axisymmetric regions (Drescher 1983). Because of the constraints imposed by the dilatancy on the kinematical admissibility, the 3D mechanisms considered are usually restrictive, despite their complexities, and the limit analysis numerical approach to 3D problems (Salgado et al. 2004), or other numerical methods (Zhu and Michalowski 2005) are likely to yield more accurate results.

10 GENERALIZED STRESS SPACE

For problems where multi-parameter limit loads are sought, it is convenient to introduce a generalized load space, analogous to the generalized stress space suggested by Prager (1955, 1959). This concept was adopted by Nova and Montrasio (1991) to develop a constitutive law at the “structural” level with the forces and moments, and displacements and rotations being analogous to Prager’s generalized stresses and generalized strains. This manner of representing both the experimental and limit analysis findings was found particularly convenient for footings (Got-

tardi and Butterfield 1993, Salençon and Pecker 1995, and Michalowski and You 1998), Fig. 5.

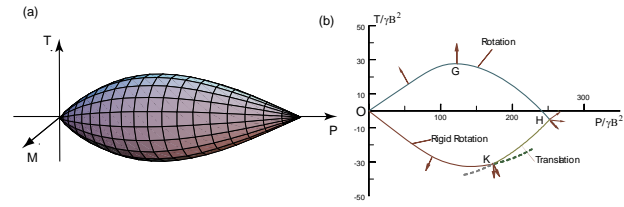


Figure 5. Failure criterion for strip footings: (a) schematic in force-moment space, and (b) cross section (Michalowski and You 1998).

Different segments on the failure surface are associated with different modes of failure of the soil under the footing. Representation of the failure state in the generalized stress space makes it possible to perform realistic soil-structure interaction analyses.

11 NUMERICAL APPROACH

Geotechnical problems that have been treated by limit analysis successfully are those that can be well represented by plane-strain mechanisms. Three-dimensional mechanisms, particularly those for dilatant materials, often become complicated in their analytical description, leading to untractable solutions. The same is true when complicated geometry or nonhomogeneous soils are involved. A numerical approach to limit analysis was found more efficient in these applications. A significant advantage that the numerical approach brings is the ability to yield both upper and lower-bound solutions, whereas the traditional (semi-analytical) approach is efficient only in the kinematic approach. A steady development of the numerical method, and its applications, has been noted in the last two decades, with the earlier contributions by Sloan (1988, 1989), and, more recently, with a team of coworkers (Salgado et al. 2004, Hjjaj et al. 2005).

ACKNOWLEDGEMENT

This report was written while the author was supported by the Army Research Office, grant No. DAAD19-03-1-0063. This support is greatly appreciated.

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