# Failure of Unidirectionally Reinforced Composites with Frictional Matrix

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**Abstract:** A failure criterion for a composite with a frictional matrix and unidirectionally oriented tensile inclusions is presented. The frictional matrix is comprised of granular material with or without cohesion. An energy-based homogenization technique is used to derive the macroscopic stress state associated with the limit state of the composite. The failure condition for a composite with long inclusions is expressed as a piecewise function, and a closed-form representation is given. A numerical scheme is presented for finding the collapse criterion for a composite reinforced with short fibers. Conditions derived can be used, for instance, to describe collapse of reinforced soils. A solution to a boundary value problem is presented, using the method of characteristics for solving the set of hyperbolic-type partial differential equations. A peculiar type of stress discontinuity is found, characteristic of plastic stress fields in anisotropic materials.

## INTRODUCTION

Fiber-reinforced cementitious composites have found a very wide application from civil engineering to aerospace engineering. The composite material considered in the present paper is different in that its matrix is built of low or noncementitious, but frictional material (such as granular or cohesive soils). Only one aspect of the material behavior is considered here: failure. In particular, a failure criterion for a unidirectionally reinforced granular matrix with low or no cementation will be presented.

Elastic and elastoplastic behavior of composite materials has been described using various methods of homogenization (averaging), ranging from self-consistent schemes [see, for instance, Hill (1965), Budiansky (1965), and Mori and Tanaka (1973)] to the finite element approach [for instance, Dvorak et al. (1974)]. An excellent survey of techniques used for analysis of composite materials was presented by Hashin (1983), and, since then, a significant interest in fiber composites has been maintained. Little attention has been paid to composites with granular or low-cementitious matrices, however. Failure criteria of such materials must be known to evaluate the stability of structures such as, for instance, reinforced earth slopes. The existing literature includes only a handful of papers with experimental results or case histories, and few attempts have been made to theoretically describe the behavior of fiber-reinforced or continuous filament-reinforced granular composites (Sawicki 1983; de Buhan and Siad 1989; di Prisco and Nova 1993; Michalowski and Zhao 1994, 1995).

The present paper shows a failure criterion for a frictional and cohesive matrix reinforced with unidirectional inclusions, such as traditional long reinforcement strips or short fibers. The criterion can be made a part of the constitutive relations used in numerical techniques, such as the finite element method, for solving boundary value problems.

The scale of the constituents in the composite may vary significantly here. If a matrix of the composite is a granular material such as sand, then the length of fibers in the fiber-reinforced composite needs to be of the order of centimeters. Spacing of fibers needs to be large enough so that the interaction among fibers can be ignored (a common restriction in all self-consistent homogenization techniques). Here this spacing is considered to be at least several grain diameters, with the overall fiber content not exceeding 10% (for lack of experimental evidence this number is somewhat arbitrary and perhaps low, but practical cases of soil reinforcement are expected to be significantly less than 10%). However, the composite material considered here may also represent a reinforced soil mass where the reinforcing strips or grids are meters long (with, for instance, 0.5 m spacing). In the latter case, the failure criteria derived here can still be used in the analysis of global stability of structures, provided the dimensions of the structure are at least an order of magnitude larger than the spacing of the reinforcement.

The homogenization scheme is presented in the next section followed by derivation of the failure criteria. Implementation of the derived criterion in a soil-mechanics problem is shown next, and the paper concludes with some final remarks.

## HOMOGENIZATION SCHEME

The purpose of homogenization is to represent quantities such as stresses and strains or material properties (such as elastic moduli) as average quantities that take into account microstresses in the composite constituents, volumetric proportions of these constituents, their respective properties, and shape of inclusions. Here the objective is to represent the failure criterion of a pressure-dependent (frictional) material reinforced with longitudinal inclusions. Such a criterion is to be represented in terms of the macroscopic stress; that is, the stress averaged over the two solid constituents of the composite (matrix and fibers). The term macroscopic here pertains to the average stress, or properties, of the composite, as opposed to the stress or properties in the constituents (microscopic). The terms macroscopic and microscopic do not relate here to the size of the composite representative element.

The diameter of the inclusions is considered to be at least an order of magnitude larger than the diameter of the grains in the matrix, and the dry friction law is considered applicable on the soil-fiber interface. The aspect ratio of the inclusions is at least of the order of $10^1$ to $10^2$, and spacing is of at least one order of magnitude higher than the inclusions' thickness/diameter. Under such circumstances one can expect that, given sufficient confining stresses, a tensile force can be induced in longitudinal reinforcing elements that allows the macroscopic stress in the composite to increase beyond what would be considered a limit stress for the matrix alone.

A kinematics (or energy-based) approach to homogenization will be used in which a plastic velocity field for a representative composite element, such as in Fig. 1, is assumed, and

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Note. Associate Editor: Robert Y. Liang. Discussion open until April 1, 1997. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on September 26, 1994. This paper is part of the ***Journal of Engineering Mechanics***, Vol. 122, No. 11, November, 1996. ©ASCE, ISSN 0733-9399/96/0011-1086–1092/$4.00 + $.50 per page. Paper No. 9299.

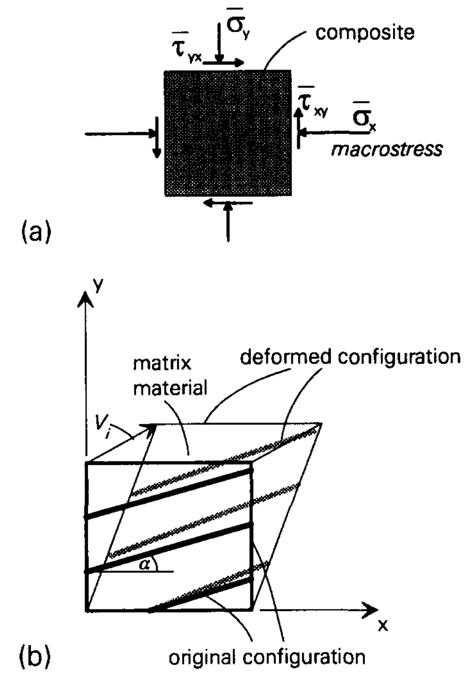


**FIG. 1. Composite Material: (a) Macrostress; (b) Deformation of Composite Element**

the energy dissipation rate in the constituents of the composite, $\dot{D}(\dot{\varepsilon}_{ij})$, is equated to the work performed by the macroscopic stress $\bar{\sigma}_{ij}$

$$\bar{\sigma}_{ij}\dot{\bar{\varepsilon}}_{ij} = \frac{1}{V}\int_V \dot{D}(\dot{\varepsilon}_{ij})\, dV \quad (1)$$

where $V$ = volume of representative element of composite; and $\dot{\bar{\varepsilon}}_{ij}$ = macroscopic (average) strain rate. Such a concept was explored earlier in the context of cementitious composites by Hashin (1964), Shu and Rosen (1967), and, for two-dimensional membranes, by McLaughlin and Batterman (1970). This homogenization procedure was also suggested for finding failure criteria for soils reinforced with randomly distributed fibers (Michalowski and Zhao 1996). Once the energy dissipation rate during the plastic deformation (collapse) of the element is calculated, the macroscopic stress $\bar{\sigma}_{ij}$ can be calculated from (1). This macroscopic stress then can be represented as a point on the failure surface in the macroscopic stress space.

Of interest are failure criteria associated with plane kinematics where the uniaxial reinforcement is contained in the plane of deformation. In this homogenization scheme we will assume a linear velocity field throughout the representative element in the form

$$v_i = a_{ij}x_j \quad (2)$$

where $v_i$ = velocity vector; $x_j$ = Cartesian coordinate; and $a_{ij}$ = matrix of coefficients subject to constraints imposed by dilatancy of base (matrix) material.

## FAILURE CRITERIA

To calculate the energy dissipation rate, $\dot{D}(\dot{\varepsilon}_{ij})$, the strain rate field, $\dot{\varepsilon}_{ij}$, inside the representative element needs to be known. As the velocity field throughout the representative element is assumed to be linear [see (2)], the strain rates are uniform, and the macroscopic strain rate $\dot{\bar{\varepsilon}}_{ij} = \dot{\varepsilon}_{ij}$. While such an assumption would not be realistic for elastic deformation or a hardening flow regime, it is a reasonable one when the composite failure is reached (zero hardening modulus).

The matrix of the composite is assumed here to conform to the Mohr-Coulomb failure criterion and the associative flow rule. Consequently, the strain rate field used in the kinematical approach must satisfy the relation (plane strain)

$$\frac{\dot{\varepsilon}_v}{\dot{\varepsilon}_1 - \dot{\varepsilon}_3} = -\sin\varphi \quad (3)$$

where $\dot{\varepsilon}_v = \dot{\varepsilon}_{ii}$ = volumetric strain rate; $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_3$ = maximum and minimum principal strain rates, respectively; and $\varphi$ = internal friction angle of matrix, which, for the associative flow rule, also indicates rate of dilation.

There is some controversy about using the normality rule since the laboratory results for sand indicate less dilation than that predicted by the associative flow law. Here, however, the associative flow rule is used only to select a virtual deformation in the homogenization scheme. It is convenient to use the associative rule, since the energy dissipation rate in the composite matrix becomes, in such case, independent of a particular stress state (zero for a noncementitious matrix). Discussion of the normality rule versus the nonassociative flow law, however, is beyond the scope of this paper.

The rate of energy dissipation per unit volume of the matrix during plastic deformation under plane strain conditions is

$$\dot{D}^m = \sigma^m_{ij}\dot{\varepsilon}_{ij} = (\dot{\varepsilon}_1 - \dot{\varepsilon}_3)c\cos\varphi \quad (4)$$

where $\sigma^m_{ij}$ satisfies Mohr-Coulomb failure function ($c$ = cohesion and $\varphi$ = internal friction angle of matrix material). The amount of reinforcement inclusions is characterized here by reinforcement concentration (volume density)

$$\rho = \frac{V_r}{V} \quad (5)$$

where $V_r$ = volume of inclusions; and $V$ = volume of entire representative composite element. We are considering composites where the volume of reinforcing inclusions (for instance, fibers) is small compared to the volume of the composite ($\rho << 1$); thus $\dot{D}^m$ in (4) can be interpreted as the dissipation rate in the matrix per unit volume of the composite.

## Long Inclusions (Strips/Bars)

First a composite with long reinforcing inclusions is considered, where no slip occurs. Such composite material is representative of ''traditional'' reinforced soil. The yield point of the reinforcement material is $\sigma_0$. Since the deformation of the reinforcing inclusions is assumed to be the same as that for the matrix, the dissipation rate in the inclusions per unit volume of the composite is

$$\dot{D}^f = \rho\sigma^f_{ij}\dot{\varepsilon}_{ij} = \langle\xi\rangle\rho\sigma_0 n_i n_j \dot{\varepsilon}_{ij} \quad (6)$$

where $n_i$ and $n_j$ = unit vectors in direction of reinforcement; and coefficient $\langle\xi\rangle$ depends on mobilization of tensile force in fibers

$$\langle\xi\rangle = \begin{cases} -1 & \text{when } \dot{\varepsilon}_{ij}n_i n_j < 0 \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

As most applications of the theory presented here are in soil mechanics, we assume that the tension is negative. Coefficient $\xi$ represents mobilization of the stress in the reinforcement, and, in general, it can vary in the range of $-1 \le \xi \le 0$, but the energy can be dissipated in fibers only when yield point $\sigma_0$ is reached ($\xi = \langle\xi\rangle = -1$). Note that, according to (6), the contribution of the longitudinal inclusions (fibers, bars, strips) in the compressive regime to the composite strength is neglected here (due to buckling and kinking).

Eq. (1) can now be written for plane strain conditions as

$$\bar{\sigma}_x\dot{\varepsilon}_x + \bar{\sigma}_y\dot{\varepsilon}_y + 2\bar{\tau}_{xy}\dot{\varepsilon}_{xy} = \dot{D}^m + \dot{D}^f \quad (8)$$

It is convenient to represent the macroscopic failure criterion for plane strain conditions in space $p$, $q$, $\bar{\tau}_{xy}$, where $p = (\bar{\sigma}_x + \bar{\sigma}_y)/2$ and $q = (\bar{\sigma}_x - \bar{\sigma}_y)/2$. Introducing the angle of inclination of the major principal macrostress to the $x$-axis $\psi$

$$\tan 2\psi = \frac{2\bar{\tau}_{xy}}{\bar{\sigma}_x - \bar{\sigma}_y} = \frac{\bar{\tau}_{xy}}{q} \tag{9}$$

Eq. (8) takes the form

$$(p + q)\dot{\varepsilon}_x + (p - q)\dot{\varepsilon}_y + 2q \tan 2\psi \dot{\varepsilon}_{xy} = \dot{D}^m + \dot{D}^f \tag{10}$$

It was found to be convenient to calculate points on the failure surface in space $p$, $q$, $\bar{\tau}_{xy}$ by calculating in-plane stress invariant $R$ for given values of $p$ and angle $\psi$

$$R = \frac{1}{2}\sqrt{(\bar{\sigma}_x - \bar{\sigma}_y)^2 + 4\bar{\tau}_{xy}^2} = \sqrt{q^2 + \bar{\tau}_{xy}^2} = |q|\sqrt{1 + \tan^2 2\psi} \tag{11}$$

Using (4) and (6), solving (10) for $q$, and using the relation in (11), one obtains

$$R(\dot{\varepsilon}_x, \dot{\varepsilon}_y, \dot{\varepsilon}_{xy}) = \left| \frac{c \cos\varphi\sqrt{(\dot{\varepsilon}_x - \dot{\varepsilon}_y)^2 + 4\dot{\varepsilon}_{xy}^2} + \langle\xi\rangle\rho\sigma_0\dot{\varepsilon}_\alpha - p(\dot{\varepsilon}_x + \dot{\varepsilon}_y)}{\dot{\varepsilon}_x - \dot{\varepsilon}_y + 2\dot{\varepsilon}_{xy}\tan 2\psi} \right| \cdot \sqrt{1 + \tan^2 2\psi} \tag{12}$$

where $\dot{\varepsilon}_\alpha$ = magnitude of rate of strain in direction of reinforcing inclusions

$$\dot{\varepsilon}_\alpha = \dot{\varepsilon}_{ij}n_i n_j = \dot{\varepsilon}_x \cos^2\alpha + \dot{\varepsilon}_y \sin^2\alpha + \dot{\varepsilon}_{xy} \sin 2\alpha \tag{13}$$

and $\alpha$ = angle of inclination of inclusions to $x$-axis. As this approach yields the upper bound to the macroscopic failure criterion, a minimum $R$ was sought from (12) in an optimization scheme where the strain rates were variable with restriction in (3) and with $\langle\xi\rangle$ determined in (7). Cross sections of the failure surface calculated ($p$ = constant) are shown in Fig. 2. The failure surface in space $p$, $q$, $\bar{\tau}_{xy}$ is presented in Fig. 3. It consists of two conical surfaces joined by two plane segments.

The matrix of the composite was assumed to obey the Mohr-Coulomb yield function and the associative flow rule. The resulting flow rule for the composite also conforms to the normality rule.

The failure criterion of an anisotropic pressure-dependent material under plain-strain conditions can be, in general, written as

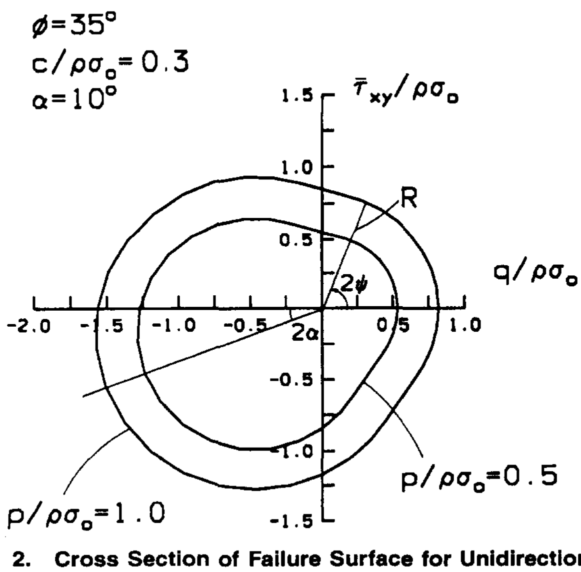


**FIG. 2. Cross Section of Failure Surface for Unidirectionally Reinforced Composite with Long Inclusions (No Slip)**

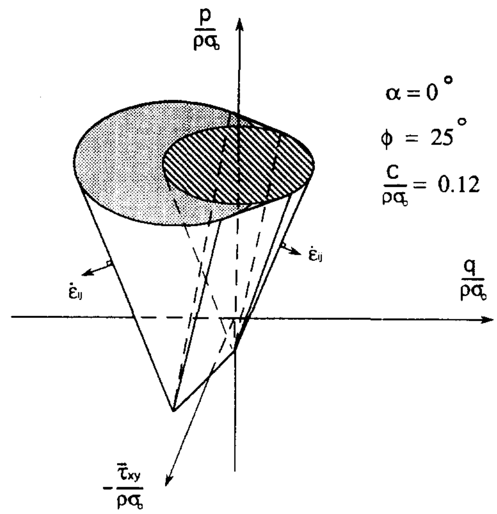


**FIG. 3. Failure Surface for Unidirectionally Reinforced Composite in Space $p$, $q$, $\bar{\tau}_{xy}$ (No Slip of Reinforcing Inclusions)**

$$f(\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy}) = R - F(p, \psi) = 0 \tag{14}$$

where $\psi$ and invariant $R$ are given in (9) and (11), respectively; and $p$ = another in-plane invariant [$p = (\bar{\sigma}_x + \bar{\sigma}_y)/2$]. When function $F$ is independent of $\psi$, (14) represents an isotropic yield criterion. Representation of failure surface $f(p, q, \bar{\tau}_{xy}) = 0$ can be given as a piecewise function. Analytical expressions are given next, in which angle $\alpha$ is the angle of inclination of the principal axis of anisotropy to the $x$-axis (inclination of the direction of reinforcing inclusions). For the part where no influence of fibers is present ($\xi = 0$), i.e., where

$$|2\psi - 2\alpha| \le \frac{\pi}{2} - \varphi \tag{15}$$

the analytic representation of the macroscopic yield function is

$$R = F(p) = p \sin\varphi + c \cos\varphi \tag{16}$$

For planar segments in Fig. 3 ($-1 < \xi < 0$), angle $2\psi$ remains in the range

$$\frac{\pi}{2} - \varphi < |2\psi - 2\alpha| \le \frac{\pi}{2} - \varphi + \tan^{-1}\left(\frac{0.5\rho\sigma_0}{p \tan\varphi + c}\right) \tag{17}$$

and the failure criterion is

$$R = F(p, \psi) = \frac{p \sin\varphi + c \cos\varphi}{\sin(2\psi - 2\alpha + \varphi)} \tag{18}$$

When the strength in the fibers is fully mobilized ($\xi = -1$), angle $2\psi$ remains in the range

$$\frac{\pi}{2} - \varphi + \tan^{-1}\left(\frac{0.5\rho\sigma_0}{p \tan\varphi + c}\right) < |2\psi - 2\alpha| \le \pi \tag{19}$$

and

$$R = F(p, \psi) = -0.5\rho\sigma_0 \cos 2(\psi - \alpha) + \sqrt{[(p + 0.5\rho\sigma_0)\sin\varphi + c\cos\varphi]^2 - [0.5\rho\sigma_0 \sin 2(\psi - \alpha)]^2} \tag{20}$$

A surface identical to that in Fig. 3 can be obtained through a purely static approach as in the theory of mixtures. This surface is an envelope to the sum of the limit stress in the matrix and stresses at or below failure in reinforcing strips/fibers. Such an approach, however, would be very cumbersome when applied to homogenization of the stress in a composite with short fibers, as presented in the next subsection.

The former was considered earlier (de Buhan and Siad 1989) for the case of a noncementitious matrix, but the failure surface was presented in a different stress space. A more convenient analytical description for this perfect case was found recently (Michalowski and Zhao 1995).

No lab test results on composite samples are available to verify the failure criterion derived. Later in this paper, a solution to a boundary value problem (collapse of a vertical slope) is compared to the result from a lab test on a physical model to indicate the rationality of the description proposed.

## Short Inclusions (Fibers)

A model of a composite with short inclusions is representative of fiber-reinforced soils. In some applications, such as rolled subgrades of airfields, the preferred orientation of the fibers is horizontal. If the plane of deformation is any vertical plane, then the failure criterion as presented here can be used to describe the strength of such composite (the effective length of fibers being the average of the fibers' projection on the plane of deformation).

No cohesive bond between the fibers and the matrix is considered, and the load transfer to the fibers has a frictional nature. During plastic deformation of the composite, fibers are expected to slip in the matrix at a low confining pressure, and to fail in tension at large mean stresses. In the latter case, the ends of fibers slip to a distance where the yield stress, $\sigma_0$, is mobilized in the fibers. Fig. 4 presents an expected distribution of shear at the fiber surface and axial stress during deformation of rigid-perfectly plastic fibers in a matrix subjected to the velocity field in (2). Assuming cylindrical fibers with radius $r$, the length of slip region $s$ (Fig. 4) is

$$s = \frac{r}{2} \frac{\sigma_0}{\sigma_n \tan \varphi_w} \tag{21}$$

where $\sigma_n$ = stress normal to fiber surface; and $\varphi_w$ = friction angle of matrix-fiber interface. In order for the tensile collapse of fiber to occur, the length $s$ must be smaller than half of the fiber length $l/2$, which occurs when

$$\eta > \frac{1}{2} \frac{\sigma_0}{\sigma_n \tan \varphi_w} \tag{22}$$

where $\eta$ = fiber aspect ratio

$$\eta = \frac{l}{2r} \tag{23}$$

We assume here that all fibers have the same aspect ratio $\eta$. The energy dissipation rate due to the plastic deformation of fibers (yielding) per unit volume of the composite can be calculated now as

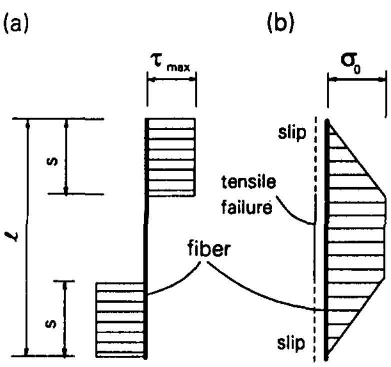


**FIG. 4. Expected Stress Distribution for Rigid-Perfectly Plastic Fiber in Uniformly Deforming Matrix: (a) Shear Stress at Fiber/Matrix Interface; (b) Axial Stress**

$$\dot{D}^f_1 = \langle\xi\rangle\rho\sigma_0 \left(1 - \frac{1}{2\eta} \frac{\sigma_0}{\sigma_n \tan \varphi_w}\right) \dot{\varepsilon}_\alpha \tag{24}$$

where $\langle\xi\rangle$ and $\dot{\varepsilon}_\alpha$ are given in (7) and in (13), respectively. The dissipation due to slip of the fiber ends (per unit volume of the composite) is

$$\dot{D}^f_2 = \langle\xi\rangle\rho\sigma_0 \frac{1}{4\eta} \frac{\sigma_0}{\sigma_n \tan \varphi_w} \dot{\varepsilon}_\alpha \tag{25}$$

Note that no dissipation is accounted for in the compressive regime ($\langle\xi\rangle = 0$). The total energy dissipation rate in fibers per unit volume of the composite now becomes

$$\dot{D}^f = \dot{D}^f_1 + \dot{D}^f_2 = \langle\xi\rangle\rho\sigma_0 \left(1 - \frac{1}{4\eta} \frac{\sigma_0}{\sigma_n \tan \varphi_w}\right) \dot{\varepsilon}_\alpha \tag{26}$$

when

$$\eta \le \frac{1}{2} \frac{\sigma_0}{\sigma_n \tan \varphi_w} \tag{27}$$

the fiber energy dissipation occurs in the slip mode only, and it becomes independent of yield point $\sigma_0$

$$\dot{D}^f = \dot{D}^f_2 = \langle\xi\rangle\rho\eta\sigma_n \tan \varphi_w \dot{\varepsilon}_\alpha \tag{28}$$

The failure criterion for a frictional matrix reinforced with short fibers can now be obtained by following the same procedure as in the preceding section, with the exception that the dissipation rate in the fibers, (6), needs to be replaced with (26) [or (28)]. Consequently, the failure condition can be obtained through minimizing function $R$ in (12) where the second term in the numerator is replaced with (26) [or (28)]. Notice that when $\eta \rightarrow \infty$ (12) is recovered. The influence of the fiber aspect ratio on the macroscopic failure criterion is illustrated in Fig. 5.

While limited laboratory results on random fiber soil reinforcement are available, no experimental data on uniaxially reinforced soils exist that could be presented in terms of a failure surface in the macroscopic stress space. However, in an early work by Yang (1972), one point on the failure surface was obtained for which the major principal stress was perpendicular to the direction of reinforcement. That result was for long reinforcement and it is identical to a particular case of (20) when $2\psi = \pi + 2\alpha$.

The length of the fiber slip section [refer to (21)] and the energy dissipation rate due to fibers [refer to (26)] depend on

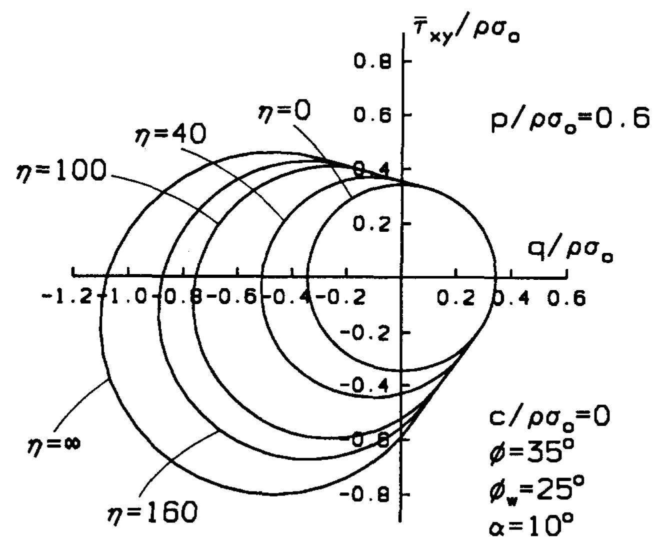


**FIG. 5. Cross Sections of Composite Failure Surfaces ($p$ = Constant) for Fibers of Different Aspect Ratios**

the stress normal to the fiber surface. While the normal stress in the plane of deformation can be represented as a function of the limit macrostress $\bar{\sigma}_{ij}$, distribution of the normal stress along the entire perimeter of the fibers cannot be found, since the resulting failure criterion is independent of the intermediate principal stress. Any attempt at calculation of $\sigma_n$ then will be approximate. Stress $\sigma_n$ was taken in calculations here as being equal to the mean of maximum and minimum macrostresses in the plane of deformation ($p$).

## IMPLEMENTATION

Two examples of application of the derived failure criterion are presented next. Both solutions are for a composite with very long fibers ($\eta \rightarrow \infty$), since the closed-form criterion was derived only for this case. The influence of the aspect ratio ($\eta$) on the bearing capacity in the first example can be ascertained, to a certain extent, from Fig. 5, where the effectiveness of reinforcement as a function of $\eta$ is illustrated.

A flat smooth punch indentation of a rigid-plastic half-space whose failure criterion is described by the surface in Fig. 3 [see also (15)–(20)] is considered. The limit load is found using the slip-line method. Eq. (14), along with the set of differential equilibrium equations, leads to a set of two hyperbolic-type partial differential equations that can be solved using the method of characteristics. The equations of characteristics can be expressed as (Booker and Davis 1972)

$$\frac{dy}{dx} = \tan(\psi - m - \nu), \quad s_1\text{-characteristic} \qquad (29a)$$

$$\frac{dy}{dx} = \tan(\psi - m + \nu), \quad s_2\text{-characteristic} \qquad (29b)$$

and the stress relations along the characteristics are

$$\sin[2(m - \nu)] \frac{\partial p}{\partial s_1} + 2F \frac{\partial \psi}{\partial s_1} + \gamma \cos(2m) \left[ \cos(2\nu) \frac{\partial x}{\partial s_1} - \sin(2\nu) \frac{\partial y}{\partial s_1} \right] = 0, \quad s_1 \qquad (30a)$$

$$\sin[2(m + \nu)] \frac{\partial p}{\partial s_2} + 2F \frac{\partial \psi}{\partial s_2} + \gamma \cos(2m) \left[ \cos(2\nu) \frac{\partial x}{\partial s_2} + \sin(2\nu) \frac{\partial y}{\partial s_2} \right] = 0, \quad s_2 \qquad (30b)$$

and

$$\tan(2m) = \frac{1}{2F} \frac{\partial F}{\partial \psi}; \quad \cos(2\nu) = \cos(2m) \frac{\partial F}{\partial p} \qquad (31a,b)$$

where $\gamma$ = unit weight of soil. The gravity acceleration is assumed here to be directed opposite of coordinate $y$.

Solutions to two boundary value problems are given next. The application in Fig. 6 is similar to the smooth punch-indentation problem considered by Hill (1950), with the exception that the half-space is now pressure-dependent and anisotropic. The failure criterion is represented by (15)–(20). Such a composite is representative of a soil subgrade reinforced with horizontal reinforcement strips or blankets of geosynthetic material. The matrix (fill) material is cohesionless with an internal friction angle $\varphi = 35°$. The geometry of the punch and the amount of reinforcement are characterized by coefficient $\gamma b/\rho\sigma_0 = 0.4$ ($\gamma$ = specific weight; $b$ = punch half-width). The stress boundary condition is given along AG (Fig. 6) as a vertical pressure, $q_0/\gamma b = 0.25$, and the direction of the limit pressure along AB is vertical (smooth punch).

Along AG we have $\psi = 0$, and the force in the reinforcement is not mobilized in the entire triangle AFG. The failure criterion is described here by the classical Mohr-Coulomb failure function, (16), and the Cauchy boundary value problem in AFG reduces to the isotropic case, as in Sokolovskii (1965). Fan of characteristics AFC represents the slip-line solution for a boundary value problem with a singular point at A. The case where no force is mobilized in the reinforcement extends from characteristic AF to AE. At point A of characteristic AE angle $2\psi = \pi/2 - \varphi$. Beyond characteristic AE a tensile force is mobilized in the reinforcement, but the reinforcement stress does not reach yield point $\sigma_0$ [see (17) and (18)].

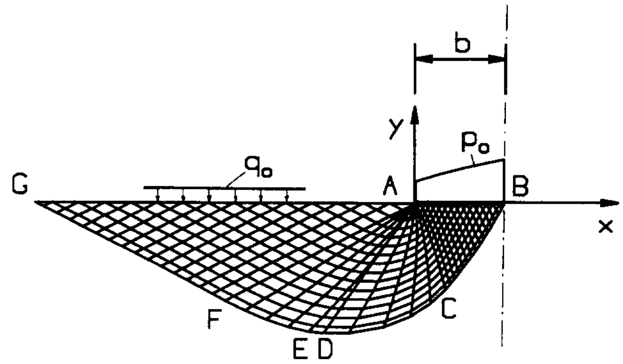


**FIG. 6. Stress Characteristics Field for Smooth Punch Indentation into Anisotropic Half-Space**

Using (31) and (18), expressions for $2m$ and $2\nu$ are found, and, after substituting these into (29), the equations of characteristics become

$$\frac{dy}{dx} = \tan \alpha, \quad s_1\text{-characteristic} \qquad (32a)$$

$$\frac{dy}{dx} = \tan \left( \frac{\pi}{2} - \varphi + \alpha \right), \quad s_2\text{-characteristic} \qquad (32b)$$

where $\alpha$ = angle of inclination of reinforcing inclusions to $x$-axis (inclination angle of axis of anisotropy). This comes as a surprise, since the inclination of the characteristic lines is now independent of the principal stress directions. Further increase in angle $\psi$ at singular point A does not generate more characteristics in fan AFC until angle $\psi$ reaches the range expressed in (19). The latter is indicative of the reinforcement reaching yield point $\sigma_0$, which occurs in the region to the right of line AE; line AE is a stress discontinuity. This is quite different from slip-line fields for isotropic materials where a stress characteristic cannot become a stress discontinuity. This peculiar type of discontinuity was first reported by Rice (1973) for a pressure-independent (nonfrictional) material. Such discontinuities are always associated with plane segments of failure criterion in the space presented in Fig. 3.

The stress state in the plastic region to the right of stress discontinuity AE satisfies (20) (yield point reached in the reinforcement). The average limit pressure along boundary AB is $\bar{p}_0/\gamma b = 39.93$, a 2.03 times increase above the limit pressure over an unreinforced half-space.

The practical application of the solution herein is for a relatively weak reinforcement where the subgrade failure is associated with plastic flow of inclusions or slip of fiber-like reinforcement. Although some lab test results on small footings over reinforced sand beds are available, the limit state in these tests is not associated with reinforcement yielding (the strength of reinforcement is not scaled), and the results are not comparable to the case calculated here. A structural approach, where the reinforcement is considered as an additional structural element, would be more appropriate to interpret these results.

It is important to note that an increase in the bearing capacity of a reinforced soil slab is influenced by the geometrical changes in the shape of the reinforcement (an effect often considered in techniques for design of unpaved roads). This effect can be accounted for using a structural approach for solving boundary value problems with reinforcement, and it cannot be addressed through the homogenization method as presented in this paper.

The second example of application is shown in Fig. 7. This is a simulation of a collapse of a 3 m tall vertical slope built of cohesive soil reinforced with 12 horizontal reinforcing layers. A physical model of the wall was laboratory tested, and the details can be found in Wu (1992). The material parameters [based on data reported by Wu (1992)] are cohesion $c$ = 82.7 kN/m$^2$, internal friction angle $\varphi$ = 12.6°, and unit weight $\gamma$ = 18.9 kN/m$^3$. The reinforced soil mass is homogenized here, and its failure condition is described by (15)–(20). There were 12 layers of geosynthetic reinforcement used with strength estimated from tests as 6 kN/m which, for a wall height of 3 m, yields a macroscopic strength of $\rho\sigma_0$ = 24 kN/m$^2$. The wall was loaded using air bags, and a pressure of 227 kN/m$^2$ was regarded as the failure load since it was associated with a disproportionately large increment of displacement.

The stress state at failure and the limit load are calculated here using the method of characteristics. The slip-line network is shown in Fig. 7. Zero traction is given at boundary AD, and the collapse load along AB is vertical. The tensile stress in the reinforcing inclusions is now mobilized everywhere in the composite mass, and the failure criterion is expressed by (20). The average collapse load at AB was calculated to be $\bar{p}_0$ = 216 kN/m$^2$, while the model of the slope collapsed at $\bar{p}_0$ = 227 kN/m$^2$. While such coincidence of results cannot be regarded as verification of the failure criterion derived, it indicates that the stability analyses based on such a criterion are reasonable. In an associated velocity field discontinuities could occur along slip lines. Shear bands examined at the side wall of the test tank, however, do not coincide with characteristics from the theoretical solution. This is probably caused by the small depth-to-height ratio of the wall backfill in the experiment, which did not allow for a full development of the field similar to that in Fig 7.

It needs to be mentioned that the model wall was loaded with air bags, and the structure used to brace the air bags probably restricted the freedom of horizontal displacement of the top boundary. A concentrated load cannot be included in the slip line solution, but this support condition could be simulated approximately with a distributed horizontal component of the load on boundary AB. The limit load calculated then would have increased. In such a case, point A would become a singular point in the stress field, giving rise to a fan of slip lines between the Cauchy stress region in triangle ADC and the mixed stress boundary value problem in ABC.

The quantity of the horizontal force at the support was not measured in the experiment. Without an accurate measurement of the horizontal load and without including it in the boundary condition, one cannot make an assessment as to whether the vertical component of the failure load calculated would become even closer to the actual collapse load or whether it would overestimate it.

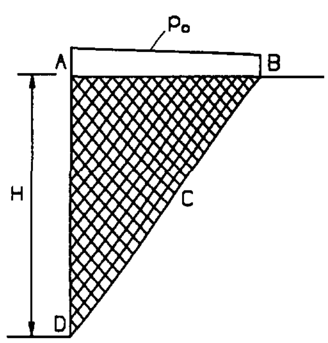


**FIG. 7. Stress Characteristics Field for Anisotropic Vertical Slope (Simulation of Reinforced Soil Slope)**

## FINAL REMARKS

A particular case of a composite with a frictional matrix was considered with unidirectionally placed reinforcing inclusions. Failure criteria were derived for cases where the plastic deformation process (collapse) takes place in a plane parallel to the longitudinal inclusions. The failure conditions can be used for describing fiber-reinforced soils. A predominantly horizontal orientation of fibers can be expected for fiber-granular fill mixtures compacted by rollers (the effective length of fibers here is approximately equal to the average of their projections on the plane of deformation). The mathematical description of the failure criterion developed can also represent the collapse condition of a "traditionally" reinforced soil with long strips or grids of geosynthetic material. This description goes beyond the concept of an artificial "anisotropic cohesion" (Schlosser et al. 1972).

The failure criterion derived here can be used for calculating limit loads on reinforced soil structures. A peculiar type of discontinuity appears in the plastic stress field of the homogenized continuum. This discontinuity is associated with the mobilization of the force in the reinforcing strips or fibers.

Further efforts will concentrate on predicting failure conditions for granular composites with a randomly distributed orientation of fibers.

## ACKNOWLEDGMENTS

The work presented in this paper was sponsored by the Air Force Office of Scientific Research, grant number F49620-93-1-0192, and the National Science Foundation, grant number CMS9301494. This support is greatly appreciated.

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