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ARCHING IN GRANULAR SOILS

Radoslaw L. Michalowski[1], Fellow, ASCE, and Namgyu Park[2]

ABSTRACT: Arching is characterized by a stress distribution where the load is transferred from softer to stiffer regions of a structure forming a stable configuration. Since it is a stable arrangement, arching is not a typical limit state problem. However, it can be formulated in a manner that allows using the static theorem of limit analysis to assess the likelihood of arching. Radial stress fields within piles of sand are constructed to search for stress distributions that promote arching. Governing equations are derived for radial stress distributions with regions varying from the yielding stress state to the elastic stress state. The stress fields with an elastic core promote arching, whereas the field where all material is in the active limit state does not support arching. However, if a passive limit state is induced in a sand heap, arching over the central part of the pile base becomes a distinct feature of the fully plastic stress field. Considerations are limited here to wedge-shaped sand heaps, though many of the conclusions are applicable to conical (axi-symmetrical) sand piles.

INTRODUCTION

Investigation of stress fields in sand piles indicates some characteristics of arching that do not confirm the stereotypical belief that arching is associated with a transfer of load from yielding parts of the structure to the stiffer parts. It appears that equally plausible arching stress fields are ones where the load is transferred through the sand in the stress state at or close to the limit, over the sand that has not reached the yield state or is significantly below the limit.

[1] Professor, Department of Civil and Env. Engrg., University of Michigan, 2340 G.G. Brown Bldg., Ann Arbor, MI 48109-2125, U.S.A., e-mail: rlmich@umich.edu
[2] Graduate Assistant, Texas A&M University; formerly University of Michigan, U.S.A.

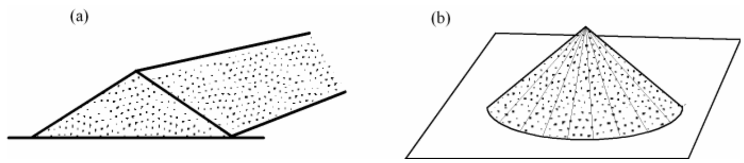

Figure 1. (a) Prismatic, and (b) conical sand piles.

Stress distribution in sand piles became a fashionable research area in the late 1990s with the focus on a counterintuitive observation that the stress at the base can exhibit a depression (or a 'dip') at the center of a conical or a wedge-shaped pile. The stress depression is a result of arching, but the predictions of the degree to which the soil will arch are not easily made. The significance of this phenomenon was probably overstated in the Science article of Watson (1996): "sand pile pressure dip is to granular mechanics what Fermat's Last Theorem was to number theory."

While the stress dip itself is a curiosity problem, the phenomenon of arching associated with it is one of interest and importance in engineering. A framework of plasticity analysis is used here to shed some light on the issue of arching in soils.

Continuum approaches to describing the stress 'dip' under sand piles can be found in Wittmer et al. (1997), Savage (1997) and Didwania et al. (2000), and a comprehensive review of continuum efforts was presented by Savage (1998). Here, we only focus on the limit analysis approach and admissible radial stress fields in prismatic sand piles. A more comprehensive consideration of this approach will appear elsewhere (Michalowski and Park 2004).

EXPERIMENTAL EVIDENCE OF ARCHING IN SAND PILES

Convincing evidence for arching in piles of sand can be found in an early paper by Hummel and Finnan (1920) who presented the measurements of the distribution of the interface stress under both conical and prismatic heaps of sand. These piles were constructed by discharging the sand from a point source (conical piles) or a line source (prismatic heaps). The sand was poured over a wooden platform of 8 by 9 ft. One set of results for a conical pile is shown in Fig. 2. The depression or the 'stress dip' at the center of the base is very distinct. The maximum stress occurs at a distance of about 1/3 of the base radius from the center, and the local stress minimum at the center is roughly equal to 50% of the maximum stress.

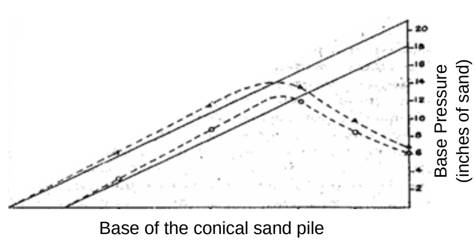

Figure 2. Measured distribution of the base stress under a conical sand pile (after Hummel and Finnan 1920).

The 'stress dip' is also present under a prismatic pile of sand, Fig. 3, but the magnitude of the depression is smaller, reaching about 5% of the maximum stress.

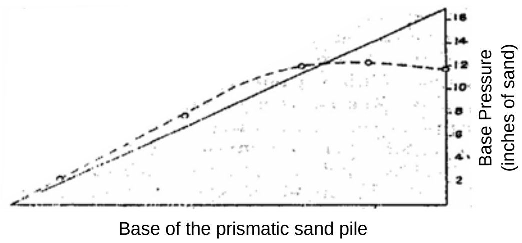

Figure 3. Measured distribution of the base stress under a prismatic sand pile (after Hummel and Finnan 1920).

Hummel and Finnan (1920) made an observation that directly leads to the conclusion that arching is causing the stress to drop down at the center of the pile sand: "When the sand was removed at the end of the experiment, it was found that about one-third the height of the cone could be taken away without altering the reading of the centre gauge."

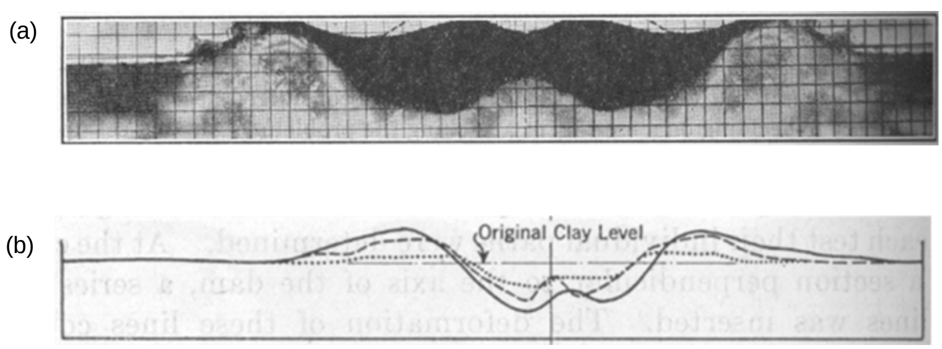

Figure 4. (a) X-ray picture of the model embankment (led shot) over soft base (gelatin), and (b) embankment-clay deflection line (after Hough 1938).

Hough (1938) modeled soft soil under an embankment using gelatin as a model material, with the embankment load simulated by lead shot. A symmetric deflection pattern of the gelatin was recorded, with a maximum settlement moved away from the center, Fig. 4, giving rise to the conjecture that the stress at the center may have a local minimum.

An interesting set of results was produced by Trollope (1956) indicating that the occurrence of the stress 'dip' is associated with the deflection of the base, Fig. 5. Numerical calculations of Savage (1998) also point to the fact that deflection of the base promotes arching in a sand pile.

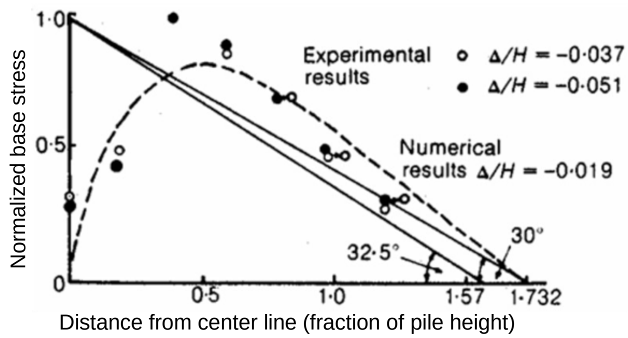

Figure 5. Experimental measurements of base stress under a prismatic sand pile (after Trollope 1956).

Experiments on conical piles of sand and granular fertilizer by Smid and Novosad (1981) confirmed the presence of a stress depression under the center of a conical mass. In addition to the distribution of the normal stress at the base, Smid and Novosad also measured the shear stress, and concluded that the base friction under the pile was not uniformly mobilized. The heaps were prepared by pouring the granular material from a funnel over a steel platform of 2 by 2 m in size.

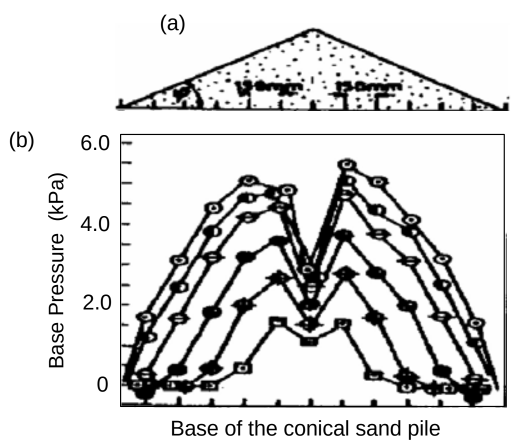

Figure 6. Base pressure under a conical sand pile during various stages of sand deposition (after Smid and Novosad 1981).

Finally, very recent experimental results from testing the base stress under both conical and prismatic sand piles by Vanel et al. (1999) clearly demonstrate that the occurrence of arching depends on the sand deposition process. Only the results for the prismatic pile are reproduced in Fig. 7. When the sand pile is constructed by a 'raining' technique, no 'stress dip' was detected beneath the center of the pile, whereas construction of the pile by dispensing the sand from a line source gives rise to stress distribution with a depression at the center, Fig. 7(a).

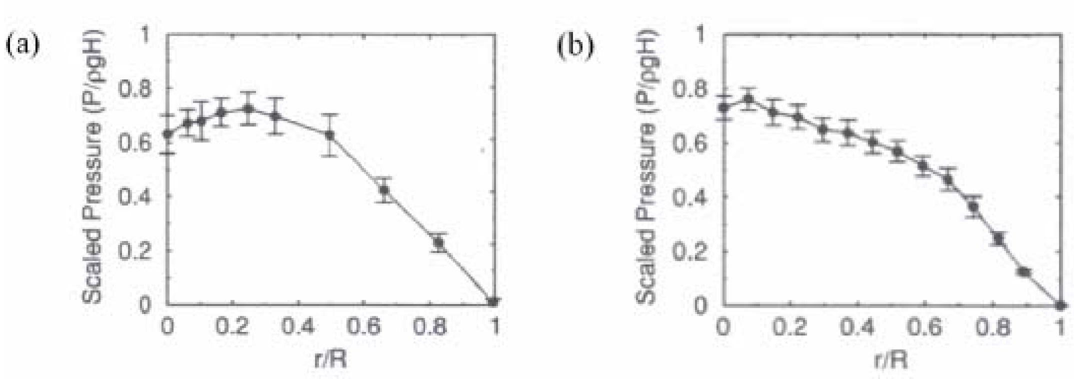

Figure 7. Base pressure under a prismatic sand heap (after Vanel et al. 1999): (a) sand pile deposited from a line source, and (b) sand pile constructed by uniform raining.

LIMIT ANALYSIS OF ARCHING

Arching is not a typical limit state problem because arching renders structures stable. Limit analysis theorems are based on considerations of the incipient collapse state, leading to lower or upper estimates of limit loads. We will make use of the static theorem only: *collapse will not occur if a safe admissible stress field can be found everywhere in the structure* (Drucker et al. 1952). The application of this theorem to sand piles is not straightforward, since sand piles with stress-free slopes at (or below) the angle of repose are inherently stable. Therefore, we consider a fictitious failure mechanism, Fig. 8, introduced here solely for the purpose of investigating arching in prismatic sand heaps.

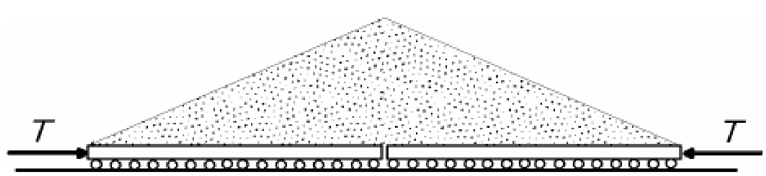

Figure 8. Fictitious "spreading" collapse mechanism of a prismatic sand heap.

The static theorem of limit analysis follows directly from the principle of maximum plastic work (Hill 1948), which states that the true stress field at collapse maximizes the plastic work

$$\sigma_{ij}\dot{\varepsilon}_{ij} \geq \sigma_{ij}^{s}\dot{\varepsilon}_{ij} \tag{1}$$

where σ_{ij} and $\dot{\varepsilon}_{ij}$ are the true stress and strain rate fields during collapse, and σ_{ij}^{s} is any statically admissible stress field. This principle holds true for materials with deformation governed by the normality rule with a convex yield condition. Integrating the inequality in (1), and applying the principle of virtual work to the right-hand side, one arrives at the following statement

$$\int_V \sigma_{ij}\,\dot{\varepsilon}_{ij}\,dV \;\geq\; \int_S T_i^s v_i\,dS + \int_V \gamma_i v_i\,dV \tag{2}$$

where v_i is the velocity vector in the plastic deformation field, T_i^s is the traction vector on boundary S of the mechanism and γ_i is the unit weight vector (T_i^s and σ_{ij}^s are in equilibrium by definition of the admissible stress field). Hence, the alternative statement of the static theorem is that the rate of internal work of the true stresses on the true (plastic) deformation is not less than the rate of work of surface traction and body forces in any statically admissible stress field. Since force $\boldsymbol{T}$ in Fig. 8 is equal to the reaction of the base, then the first integral on the right-hand-side of inequality (2) is negative, and this theorem leads to an upper estimate of $\int_S T_i\,dS$, v_i being constant along base S (Fig. 8). Consequently, one would expect that a prismatic pile of sand will be in a state closest to failure when force $\boldsymbol{T}$ is minimized, but it will be stable for larger $\boldsymbol{T}$. One might expect that stress fields with large $\boldsymbol{T}$ are ones that exhibit arching. In the next Section we will consider the radial class of stress fields and investigate whether such fields exhibit arching.

Another mechanism of incipient collapse is the one associated with deflection of the base, Fig. 9. As opposed to the spreading, deflection is not a fictitious mechanism (although the linear distribution of the displacement is, of course, an approximation). The deflection process occurs gradually during deposition of sand. Therefore, the deflection considered here should be understood as an incremental process related to an incremental increase in the sand pile mass during an interval of the sand deposition process.

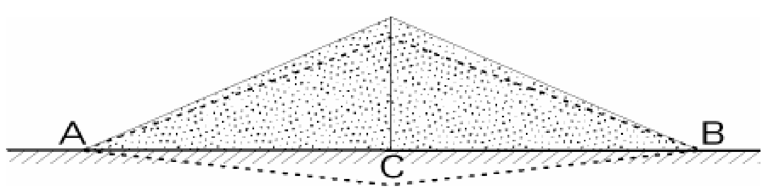

Figure 9. Deflection of the base under a prismatic sand pile.

Consider small rotation of the base segments AC and BC about toe points A and B, respectively. The true deformation pattern of the sand during this incipient process is not known, and it may occur with or without sliding along the base. The distribution of the base stress T_i that is closest to the true distribution at incipient failure maximizes the right-hand side of inequality (2). However, T_i is the reaction to the sand weight, and it does negative work during the process, *i.e.*, the first term on the right-hand side of (2) is negative. Hence, maximizing the right-hand side of (2) requires minimizing of the absolute rate of work $\left|\int_S T_i v_i \, dS\right|$. Because the displacement of the base decreases linearly toward the toe, minimizing of $\left|\int_S T_i v_i \, dS\right|$ requires that the resultant $\int_S T_i \, dS$ under each half of the pile be located as far away from the center of the pile as possible (as close to the toe as possible). This occurs for a distribution with a stress depression at the center. Hence, the deflection of the base induces arching.

RADIAL STRESS FIELD

Consider the Mohr-Coulomb yield condition for a purely frictional material, represented in a polar coordinate system

$$f(\sigma_r, \sigma_\theta, \tau_{r\theta}) = (\sigma_r + \sigma_\theta)\sin\phi - \sqrt{(\sigma_r - \sigma_\theta)^2 + 4\tau_{r\theta}^2} = 0 \qquad (3)$$

where ϕ is the internal friction angle. Introducing in-plane mean stress $p = (\sigma_r + \sigma_\theta)/2$, one can re-write the yield function in eq. (3) as

$$\begin{aligned} \sigma_r &= p\,(1 + \sin\phi\cos 2\psi') \\ \sigma_\theta &= p\,(1 - \sin\phi\cos 2\psi') \\ \tau_{r\theta} &= p\,\sin\phi\sin 2\psi' \end{aligned} \qquad (4)$$

where ψ' is the inclination angle of the major principal stress to radius r, as defined in Fig. 10.

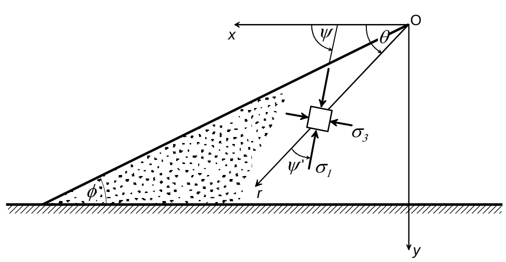

Figure 10. Polar coordinate system.

The stress components in eq. (4) must, of course, satisfy the equations of equilibrium

$$\begin{aligned} \frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} &= \gamma\sin\theta \\ \frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2\tau_{r\theta}}{r} &= \gamma\cos\theta \end{aligned} \qquad (5)$$

where γ is the unit weight of sand.

In purely frictional materials the principal stress directions are ill-defined along stress-free contours. Consequently, the stress boundary condition cannot be uniquely defined at such boundaries. To overcome this lack of uniqueness Sokolovskii (1965) suggested that for a wedge problem (or a practical case of soil pressure on retaining walls) some characteristics of the stress field be predetermined. More specifically, Sokolovskii suggested that the in-plane mean stress p be a function of angular coordinate θ and linear function of radius r, and the principal directions be independent of r

$$p = \gamma r\, \chi(\theta)\,, \qquad \psi' = \psi'(\theta) \tag{6}$$

Substituting equations in (4) into (5), and using (6), Sokolovskii obtained the following set of ordinary differential equations for the two unknown functions $\chi(\theta)$ and $\psi'(\theta)$

$$\begin{aligned} \frac{d\chi}{d\theta} &= \frac{\cos(2\psi' + \theta) + \chi \sin 2\psi'}{\cos 2\psi' - \sin\phi} \\ \frac{d\psi'}{d\theta} &= \frac{\sin\theta - \sin\phi \sin(2\psi' + \theta) - \chi\cos^2\phi}{2\chi\sin\phi(\cos 2\psi' - \sin\phi)} - 1 \end{aligned} \tag{7}$$

Equations (7) describe the plastic *radial* stress field. These equations were used earlier to describe the stresses in granular media stored or being discharged from containers (e.g., Jenike 1961, Michalowski 1984, Drescher 1991). Such a stress field can be useful in describing the stresses in a prismatic sand pile, but the stress in a sand pile may not necessarily be in the limit state everywhere. Therefore, we seek a radial stress distribution that allows for some part of the sand in the pile to be in the elastic state.

Assume that the stress distribution in the radial field conforms to a function that is similar to the yield condition, but with the internal friction angle replaced by angle ϕ^* that is a function of angular coordinate θ and is not larger than ϕ anywhere in the field

$$(\sigma_r + \sigma_\theta)\sin\phi^* - \sqrt{(\sigma_r - \sigma_\theta)^2 + 4\tau_{r\theta}^2} = 0\,, \qquad \phi^* = \phi^*(\theta) \le \phi \tag{8}$$

We emphasize that the function in eq. (8) is not a material property, whereas eq. (3) is. The governing equations for the stress field that conforms to the function in eq. (8) take the form (Michalowski and Park 2004)

$$\frac{d\chi}{d\theta}=\frac{\cos(2\psi'+\theta)+\chi(\sin 2\psi'+\dfrac{d\phi^*}{d\theta}\cos\phi^*)}{\cos 2\psi'-\sin\phi^*}$$
$$\frac{d\psi'}{d\theta}=\frac{\sin\theta-\sin\phi^*\sin(2\psi'+\theta)-\chi\cos\phi\,(\cos\phi^*+\dfrac{d\phi^*}{d\theta}\sin 2\psi')}{2\chi\sin\phi^*(\cos 2\psi'-\sin\phi^*)}-1 \tag{9}$$

Equations (9) reduce to (7) when $\phi^*=\phi=\text{const.}$

STRESS DISTRIBUTION UNDER A PRISMATIC SAND HEAP

First, consider that the entire sand heap is in the limit state. Two cases need to be distinguished

$$\chi=0 \quad \text{at} \quad \theta=\phi$$
$$\psi'=0 \quad \text{at} \quad \theta=\pi/2 \tag{10}$$

and

$$\chi=0 \quad \text{at} \quad \theta=\phi,$$
$$\psi'=\pi/2 \quad \text{at} \quad \theta=\pi/2 \tag{11}$$

The first set of boundary conditions defines the active state where the major principal stress is vertical at the symmetry line, and the second set of boundary conditions relates to the passive case. The equations in (7) were solved using the finite difference method. The internal friction angle was taken as 30°, and the solutions for both sets of boundary conditions are presented in Fig. 11.

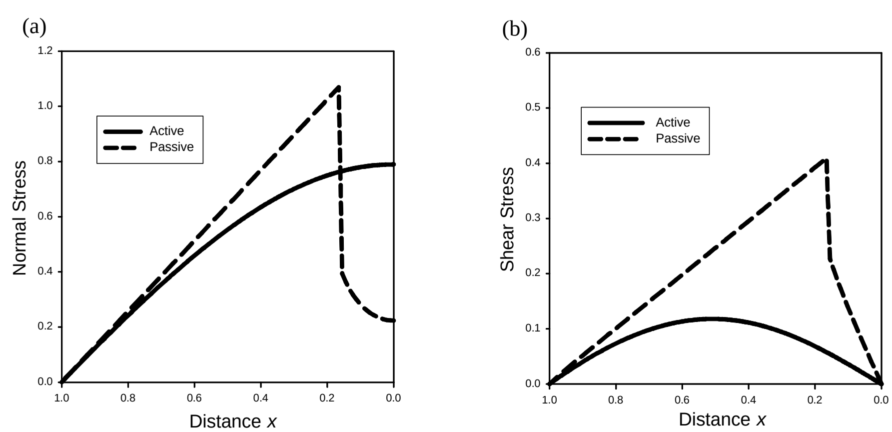

Figure 11. Distribution of the base stress for a fully plastic stress field in a prismatic pile: (a) normal stress, and (b) shear stress.

The stresses indicated in Fig. 11 are the ratio of the actual stresses to γH (γ is the unit weight of sand, and H is the pile height). Distance x is the normalized distance from the center of the pile. Similar solutions were obtained earlier by others (*e.g.*, Wittmer 1997, Savage 1998). It appears that a fully plastic active stress field does not produce a 'stress dip' underneath the center of the sand pile. The passive solution, on the other hand, has a discontinuity in the stress distribution and a distinct local minimum at the center. Such a stress field is plausible if the base of the pile deflects, as considered in the mechanism schematically illustrated in Fig. 9. However, it would not be admissible for the "spreading" mechanism, Fig. 8, as it would produce energy during incipient failure (negative internal work).

Based on the theorem in eq. (2) we expect that distributions with a large integral reaction $\boldsymbol{T}$ may exhibit arching. Therefore, we now consider a case where not all sand is in the limit state, with the stress state satisfying eq. (8) and ϕ^* described by

$$\phi^* = \phi \ \frac{\cos^n \theta}{\cos^n \phi} \tag{12}$$

where n is a constant. This function renders the sand in the immediate neighborhood of the sloping surfaces to be in the limit state, and angle ϕ^* drops down to zero at the symmetry line. This is illustrated by ratio ϕ^*/ϕ in Fig. 12(a). The condition: $\phi^* = 0$ at $\theta = \pi/2$ forces the stress state to be isotropic at the symmetry line.

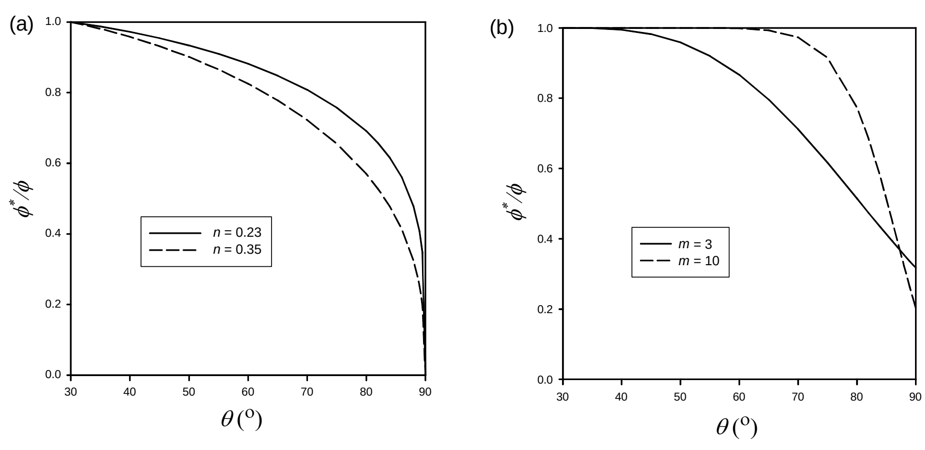

Figure 12. Angle ϕ^* as a function of angular coordinate θ: (a) cosine distribution, eq. (12), and (b) exponential distribution, eq. (13).

Equations (9) were used to solve for the stress state, and the "active" solution is shown in Fig. 13 along with the fully plastic distribution. Arching is now a distinct feature of the stress distribution in the pile, resulting in a 'dip' at the center of the base stress distribution.

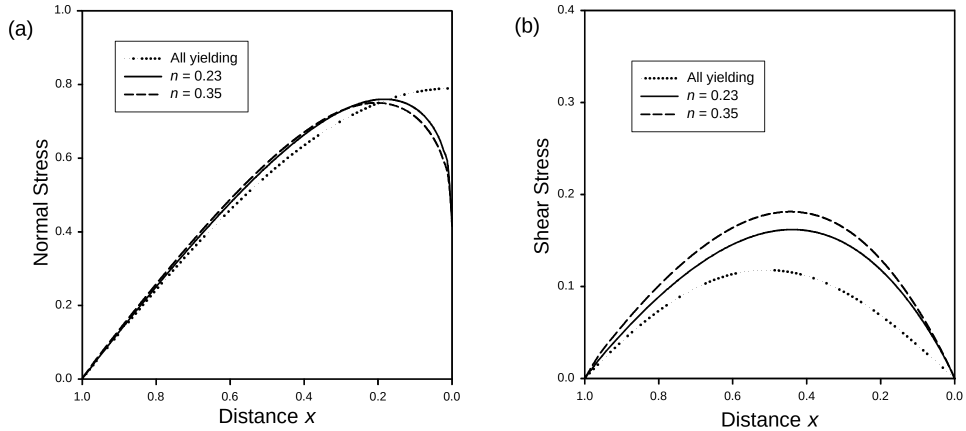

Figure 13. Distribution of stress beneath a prismatic sand pile with isotropic stress at the center: (a) normal stress, and (b) shear stress.

Consistent with limit analysis considerations, the horizontal component of the base reaction under one half of the pile (integral of the shear stress at the base) is now larger than that associated with the all-in-failure stress state.

The function in eq. (13) is another distribution of ϕ^*, for which the ratio ϕ^*/ϕ is shown in Fig. 12(b)

$$\phi^* = \phi\, e^{-(\theta-\phi)^m} \tag{13}$$

The distribution of the normal and shear stresses under the sand pile, associated with the function ϕ^* in eq. (13) is illustrated in Fig. 14.

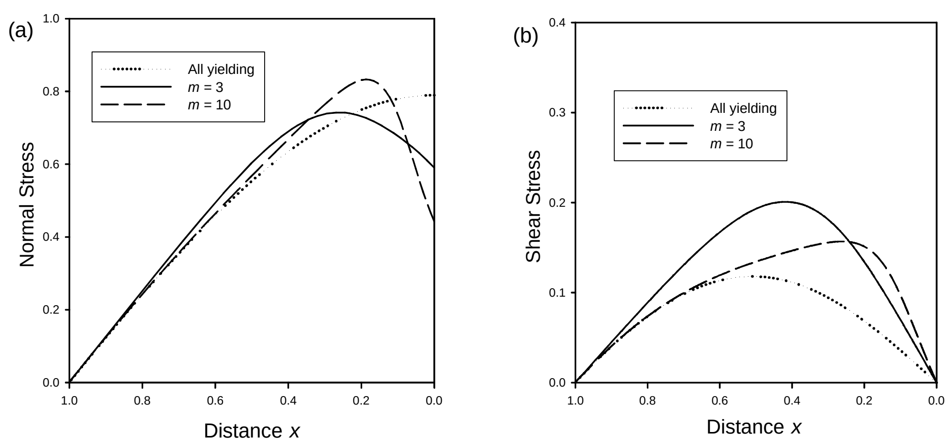

Figure 14. Distribution of stress under a prismatic sand pile; exponential distribution of ϕ^*: (a) normal stress, and (b) shear stress.

The sand in a large portion of the heap adjacent to the slopes is now in the limit state, whereas the material in the core of the pile is in the elastic state, but it does not drop down to the isotropic state at the symmetry axis.

DISCUSSION AND CONCLUSIONS

Arching continues to be an elusive problem in soil mechanics with no methods capable of predicting its presence. Arching is not a material property, but rather a response of a structure to the loading process and the ability of the soil structure to adapt itself to loads. In this, it is similar to shakedown or structural adaptation (Melan 1938, Koiter 1960), with the difference that shakedown is associated with a cyclic loading, whereas arching is typically induced during a monotonic process.

Shakedown is a phenomenon where a structure subjected to cyclic loading responds in an elasto-plastic manner in the first, or first few cycles, but the response becomes elastic in the subsequent cycles (adaptation). This occurs if a residual (self-equilibrated and time-independent) stress field is induced in the structure during the first cycles such that, in superposition with the loading in subsequent cycles, the stress state does not exceed the yield limit anywhere in the structure. Essentially, this is the Melan shakedown theorem (Melan 1938). The structure must, of course, be elasto-plastic for the residual stress state to be induced. Occurrence of arching requires that a statically admissible stress supporting arching be induced. This can be viewed as a special case of shakedown.

The plasticity approach based on the theorems of limit analysis can be used to shed some light on the occurrence of arching. In particular, the static theorem can be rephrased to indicate that *arching may occur and failure will not take place if a statically admissible stress field supporting arching can be found*. Of course, proving that an admissible arching stress field exists does not exclude "non-arching" stress fields from occurring. The second theorem then states that *arching will not occur and collapse is imminent if a kinematically admissible mechanism can be found in which the work rate of external loads exceeds the rate of internal work.*

Arching in homogeneous and isotropic granular media is promoted in stress fields with varied mobilization of strength. Granular materials governed by the Mohr-Coulomb yield condition cannot resist uniaxial compression, so, even when arching occurs, the arching regions must be supported, in some manner, by low-stressed regions. Hence, arching within granular media is not clearly manifested by the geometry of the structure, but it is associated with some special properties of the stress field. In the case of sand piles, arching is linked directly to an increase in the shear reaction at the base. In materials with cohesion, unsupported arches can form, much like those over a cavity in moist sand or in clay. When the size of grains becomes large compared to the size of the structure (a blocky system), a stable geometric alignment of the grains is possible, giving rise to an arch formation. This, however, is no longer a continuum problem.

In his classical text Terzaghi (1943) defined arching as the "transfer of pressure from a yielding mass of soil onto adjoining stationary parts." This description holds true when applied to the supporting structure, but the arching observed within sand piles appears to defy this definition: the sand in the yielding state appears to arch over the center core where the stress level is well below yielding.

Acknowledgements
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