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Admissible stress fields and arching in piles of sand

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Stress distributions under conical and prismatic heaps of sand can exhibit a local minimum at the centre of the base. Occurrence of the 'dip' in the stress distribution is affected by the sand deposition history and the deflection of the base. The dip is due to arching of sand over the centre core of the heap. Radial stress fields in prismatic piles of sand are constructed, and distributions with a dip indicative of arching are found. Arching is promoted in stress fields where the sections adjacent to the sloping surfaces are in the yielding state whereas the inner core of the heap is in the elastic state. The static approach of limit analysis is used to find that the tendency to arching increases with an increase in the horizontal component of the reaction under half of the mound.

KEYWORDS: limit state design/analysis; plasticity; sands

La répartition de la contrainte sous des tas de sable conique ou prismatiques peut montrer un minimum local au centre de la base. L'occurrence de ce 'creux' de répartition de contrainte dépend de l'historique de déposition du sable et du fléchissement de la base. Le creux est dû à la courbure du sable sur le noyau central du tas. Nous avons construit des champs de contrainte radiale dans les tas de sable prismatiques et nous avons trouvé les répartitions avec un creux indicatif de la courbure. La courbure est facilitée dans les champs de contrainte où les sections adjacentes aux surfaces en pente sont dans l'état d'écoulement alors que le noyau interne du tas est dans un état élastique. Nous avons utilisé l'approche statique de l'analyse limite pour trouver que la tendance à la courbure augmente avec une augmentation du composant horizontal de la réaction sous la moitié du tas.

INTRODUCTION

The term *sand pile* is used in this paper to describe a mound or a heap of sand, and not a sand column, as is often used in foundation engineering.

Stress distribution in sand piles became a fashionable research area in the late 1990s, with a focus on a counter-intuitive observation that the stress at the base can exhibit a depression (or a 'dip') at the centre of a conical pile, or at the symmetry plane of a prismatic wedge-shaped sand heap. The pressure 'dip' can be predicted using classical methods of plasticity analysis. We point out that most of the earlier solutions can be classified as statically admissible stress fields that are sought in the static approach of limit analysis. A multitude of admissible stress distributions with a characteristic 'dip' in the centre can be found using methods of continuum mechanics, a point made earlier by Savage (1997) and Didwania *et al.* (2000). While the stress dip itself is a curious problem, the phenomenon of arching associated with it is one of interest and importance in engineering.

Considerations are limited here to wedge-shaped (or prismatic) sand heaps, though many of the conclusions are applicable to conical (axisymmetrical) piles. We introduce the problem with an example of an embankment, and a justification of why the stresses under embankments have a non-linear distribution.

It is customary to assume in design that the load under embankments follows the trapezoidal shape of the fill (e.g. Osterberg, 1957). Such an approximation, although not unreasonable in practical design, leads to violation of force equilibrium (and, of course, stress equilibrium). A simple explanation of why the distribution of stress under embankments cannot follow the shape of the fill is illustrated in Fig. 1 for the special case of a long prismatic sand heap (or an embankment with a zero-width crown). As the symmetry axis must be a principal stress direction, the total 'thrust' **H** is a horizontal force at some height above the base. This force is equilibrated by force **T** (an integral of the shear stress over the base half-length *l*). The total weight **W** of the fill must be balanced by the vertical reaction **N**. But, in order to balance the moment of the force couple **H-T**, the resultant vertical reaction **N** must not be collinear with **W**; rather, the moment of force couple **W-N** must balance the moment of the couple **H-T**. Hence the distribution of the vertical stress at the base cannot follow the triangular shape of the heap (as that would imply that forces **W** and **N** are collinear). The same can be proved for trapezoidal shape of the heap, or for an embankment.

Following the previous argument, one would expect that the distribution of the vertical stress at the heap base is non-linear, with a centroid moved somewhere beyond $l/3$ measured from the symmetry plane. Early experiments by Hough (1938) on lead shot model embankments over a gelatine model of a foundation soil indicated a symmetric deflection pattern with a maximum settlement moved away from the centre, giving rise to a conjecture that the stress at the centre may have a local minimum. Soon after, this became textbook information (Krynin, 1941).

The early tests on conical and prismatic piles conducted by Hummel & Finnan (1920) revealed very distinct pressure dips under sand piles of both geometries. In these experiments the sand was poured through a funnel over a wooden platform of dimensions 8 ft by 9 ft (2·4 m × 2·7 m). Hummel & Finnan reported a greater relative stress depression for conical piles than that for prismatic mounds (a drop in stress of about 50% at the centre, relative to maximum pressure, for a conical pile, compared with less than 10% depression for a triangular prism). Hummel & Finnan also made an observation that clearly links the depression in the base stress to arching: 'When the sand was removed at the end of the experiment, it was found that about one-third [of] the height of the cone could be taken away without altering the reading of the centre gauge.'

An admissible distribution of forces in a structure assembled of elastic discs, considered by Trollope (1957),

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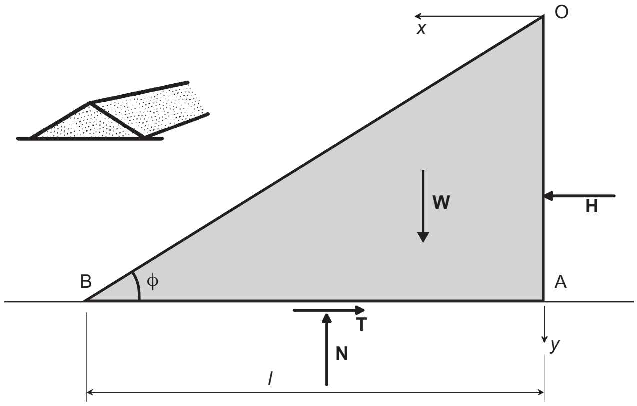

Fig. 1. Schematic of a prismatic sand heap

indicated that arching can occur in a clastic structure. Trollope's earlier measurements of stresses under 'wedges' of sand (Trollope, 1956; reproduced in Trollope & Burman, 1980) indicated that the occurrence of the stress depression is greatly affected by the deflection of the base: the larger the deflection, the more distinct the pressure 'dip'.

A set of results with measurements of both vertical (normal) and shear stresses at the base underneath triangular and trapezoidal embankments was presented by Wiesner (2000). Although presented only recently, these results date back to about three decades ago. The passive and active states were induced by opening and closing a gap at the centreline of the model embankment. To achieve this, two boards, hinged together at the bottom, were placed at the centreline prior to constructing the sand heap. The instrumented platform had dimensions of 2·18 m by 1·31 m. The measurements in the passive state indicated a very distinct stress depression at the base centre for both triangular and trapezoidal embankments, whereas no apparent 'dip' was present in the active state.

Experiments on conical piles of sand and granular fertilizer by Smid & Novosad (1981) confirmed the presence of a stress depression under the centre of a conical mass. In addition to the distribution of the normal stress at the base, Smid and Novosad also measured the shear stress, and they concluded that the base friction under the pile was not uniformly mobilised. The heaps were prepared by pouring the granular material through a funnel over a steel platform 2 m by 2 m in size.

A set of recent laboratory measurements of stress distributions by Vanel *et al.* (1999) beneath both the conical and the prismatic heaps indicated the influence of the construction process (or history of sand deposition) on the formation of a pressure dip. The heaps formed by pouring the sand from a point source (conical piles) or a line source (prismatic piles) exhibited a dip, whereas the piles produced by uniform 'raining' did not.

Analytical and numerical modelling efforts toward description of the distribution of stress in granular piles include both the continuum (e.g. Booker, 1969; Wittmer *et al.*, 1997; Didwania *et al.*, 2000) and discrete approaches (Trollope & Burman, 1980), and a comprehensive review of efforts in modelling was presented earlier by Savage (1997, 1998). Savage has also presented numerical solutions of his own, indicating that the appearance of stress dips under piles over rough surfaces can be a result of elastic anisotropy of the material, or the deflection of the pile base.

The authors' interest in the subject is primarily in the context of limit state analysis. Solving for stresses under sand heaps is not a typical problem of limit analysis, as sand piles are inherently stable, but we are making the argument that the static theorem is useful in assessing fields that promote arching. When this approach is applied to 'standard' problems, admissible stress fields are sought to provide a lower bound on an active force causing incipient failure (or an upper bound on a reaction). Application of this theorem to the arching problem in sand heaps is discussed later in the section 'Limit analysis approach to arching in sand piles'. Although this approach does not guarantee finding the true stress distribution, it gives a quantitative evaluation of the arching phenomenom.

This paper invokes the concept of the plastic radial stress field (Sokolovskii, 1965), and we use it as a means of constructing admissible stress distributions in a long and symmetric sand heap of a triangular cross-section (prismatic sand pile). The plastic radial stress field is reviewed first, and a modification is made to allow below-limit stress states. We then proceed to formulate the problem in terms of limit analysis, and show realistic stress distributions with stress depressions under the centre of sand piles. The paper is concluded with a discussion of the results, and final remarks.

THE RADIAL STRESS FIELD

Irreversible deformation of granular rate-independent materials becomes possible once the plastic stress state is reached. When the incipient failure is of interest (rather than an advanced process of plastic deformation), perfect plasticity is a common assumption, with the Mohr–Coulomb yield condition describing the stress state at failure. This criterion for sand is presented here in terms of stress components σ_r, σ_θ and $\tau_{r\theta}$ in the polar coordinate system r,θ (compression is taken as positive):

$$f(\sigma_r, \sigma_\theta, \tau_{r\theta}) = (\sigma_r + \sigma_\theta)\sin\phi - \sqrt{(\sigma_r - \sigma_\theta)^2 + 4\tau_{r\theta}^2} = 0 \tag{1}$$

where ϕ is the internal friction angle. Alternatively, the stress components that identically satisfy the yield condition can be written in terms of in-plane mean stress $p = (\sigma_r + \sigma_\theta)/2$ and angle ψ' of inclination of the major principal stress to radius r (Fig. 2) as

$$\begin{aligned} \sigma_r &= p(1 + \sin\phi\cos 2\psi') \\ \sigma_\theta &= p\,(1 - \sin\phi\cos 2\psi') \\ \tau_{r\theta} &= p\,\sin\phi\sin 2\psi' \end{aligned} \tag{2}$$

These stress components must, of course, satisfy equilibrium equations

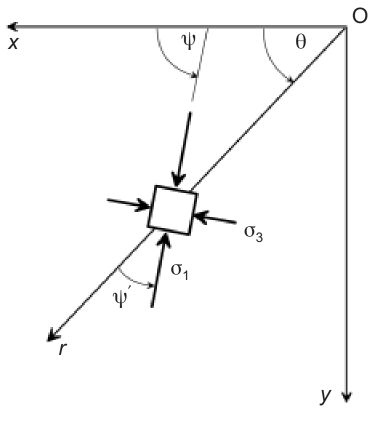

Fig. 2. Polar coordinate system

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = \gamma \sin\theta$$
$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = \gamma \cos\theta \qquad (3)$$

where γ is the unit weight of the sand. The radial stress field (Sokolovskii, 1965) is postulated by requiring that ψ' is a function of θ but not of r. It is also postulated that p is proportional to γ, r, and to an unknown dimensionless function $\chi(\theta)$:

$$p = \gamma r \chi(\theta), \qquad \psi' = \psi'(\theta) \qquad (4)$$

Substituting equations (2) into equation (3), and using equation (4), a set of ordinary differential equations is obtained with unknown functions $\chi(\theta)$ and $\psi'(\theta)$; the solution to the derivatives of the unknown functions was found by Sokolovskii (1965) in the following form:

$$\frac{\mathrm{d}\chi}{\mathrm{d}\theta} = \frac{\cos(2\psi' + \theta) + \chi \sin 2\psi'}{\cos 2\psi' - \sin\phi}$$
$$\frac{\mathrm{d}\psi'}{\mathrm{d}\theta} = \frac{\sin\theta - \sin\phi \sin(2\psi' + \theta) - \chi \cos^2\phi}{2\chi \sin\phi(\cos 2\psi' - \sin\phi)} - 1 \qquad (5)$$

This result was also given in an earlier edition of Sokolovskii's book (1960), and it was probably available already in the first (Russian) edition of his book in 1942. This solution has been used in solving limit state problems for walls retaining granular soils (Sokolovskii, 1965), where the boundary condition in terms of the principal stress direction cannot be formulated on a stress-free surface. It also was applied to predict stress fields in prismatic embankments by Booker (1969), and it was utilised extensively in problems of flow of bulk materials through hoppers (e.g. Jenike, 1961; Michalowski, 1984; Drescher, 1991). Here, the radial stress field will be used as a means of constructing admissible stress distributions in sand heaps on rough surfaces.

Statically admissible stress fields satisfy the stress boundary conditions, are in equilibrium, and do not violate the yield condition. The last requirement indicates that the stress field can be at the limit, as prescribed by equation (1) ($f = 0$), or the stress state can be below the yielding level ($f < 0$). Therefore, to widen the class of admissible stress fields in our considerations, we introduce a stress function similar to that in equation (1), but with angle ϕ replaced with variable ϕ^* (dependent only on θ) that is less than or equal to ϕ. In terms of principal stresses, the new stress function can be written as

$$\frac{\sigma_1}{\sigma_3} = \frac{1 + \sin\phi^*}{1 - \sin\phi^*}, \qquad \phi^* = \phi^*(\theta) \leqslant \phi \qquad (6)$$

and it is illustrated in Fig. 3. As opposed to equation (1), the function in equation (6) is not a material property, but a function describing an admissible combination of stress components. To use the terminology employed in earlier related work (e.g. Wittmer *et al*., 1996; Savage, 1998), equation (6) provides closure to the set of equations (3). As angle ϕ^* can vary with a change in θ, the governing equations for the stress field have to be re-derived (see Appendix 1). The solution to the derivatives of functions $\chi(\theta)$ and $\psi'(\theta)$ is now found in the following form:

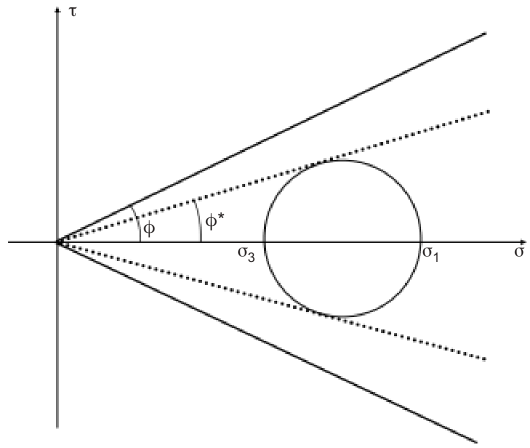

Fig. 3. Admissible stress state characterised by ϕ^*

$$\frac{\mathrm{d}\chi}{\mathrm{d}\theta} = \frac{\cos(2\psi' + \theta) + \chi\left[\sin 2\psi + \left(\dfrac{d\phi^*}{d\theta}\right)\cos\phi^*\right]}{\cos 2\psi' - \sin\phi^*}$$

$$\frac{\mathrm{d}\psi'}{\mathrm{d}\theta} = \left\{\sin\theta - \sin\phi^* \sin(2\psi' + \theta) - \chi\cos\phi \left[\cos\phi^* + \left(\frac{\mathrm{d}\phi^*}{\mathrm{d}\theta}\right)\sin 2\psi'\right]\right\} \Big/ [2\chi \sin\phi^*(\cos 2\psi' - \sin\phi^*)] - 1 \qquad (7)$$

The radial stress field described in equations (15) (Appendix 1) does not violate the yield condition for as long as $\phi^* \leqslant \phi$. It will become statically admissible for a sand pile if the base is perfectly rough (friction angle at the base interface is not less than angle ϕ^*), and if it satisfies the boundary conditions. Selecting ϕ^* that varies throughout the sand is equivalent to assuming that the sand is in different stages of 'mobilisation' of internal friction. Therefore this might be an effective method for investigating a variety of conditions that are likely to be dependent on the history of the construction sequence of a sand heap and, possibly, the conditions that arise due to deflection of the base.

LIMIT ANALYSIS APPROACH TO ARCHING IN SAND PILES

The static and kinematic approaches of limit analysis have been used extensively in structural engineering, plastic forming of metals, and geotechnical engineering. The method applies to perfectly plastic solids with convex yield functions, and with deformation governed by the normality rule. The static approach of limit analysis is based on a theorem that *collapse will not occur if an admissible stress field can be found for which $f < 0$* (see equation (1)) *everywhere in the structure* (Drucker *et al*., 1952). Typically, this theorem is used to estimate a limit load acting on a structure (a load causing failure), and it provides a lower bound to an active load or an upper bound to a reaction. However, application of this theorem to sand piles is not straightforward, as sand piles with stress-free slopes at (or below) the angle of repose are inherently stable. Therefore we consider a fictitious failure mechanism (Fig. 4), introduced here solely for the purpose of investigating arching in prismatic sand heaps.

Arching in a sand pile affects the distribution of the stress within the granular mass, but the integrated reaction at the base is exactly equal to the weight of the pile, independent of whether or not arching has occurred. However, arching

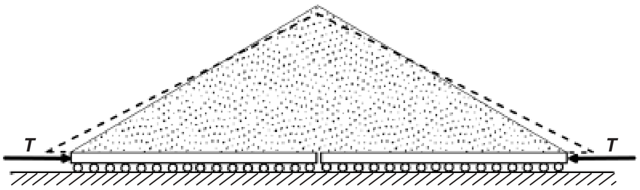

Fig. 4. Schematic of the base support for limit analysis of a spreading mechanism

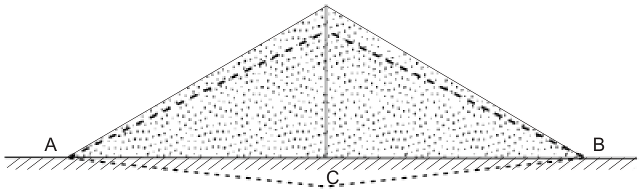

Fig. 5. Deflection of the base under a prismatic sand pile

does affect the distribution of the horizontal component of the reaction, even though its integral under the entire pile is equal to zero. We now ask the following question: What is the magnitude of the horizontal reaction $\boldsymbol{T}$ at the instant of 'spreading' failure of the pile shown in Fig. 4? A useful statement of the static theorem of limit analysis follows directly from the principle of maximum plastic work (Hill, 1948):

$$\sigma_{ij}\dot{\varepsilon}_{ij} \geqslant \sigma^{s}_{ij}\dot{\varepsilon}_{ij} \tag{8}$$

where $\dot{\varepsilon}_{ij}$ and σ_{ij} are the true plastic strain rate and stress tensors, and σ^{s}_{ij} is the stress state in any statically admissible field. By principle of virtual work we can write directly

$$\int_V \sigma_{ij}\dot{\varepsilon}_{ij}\,\mathrm{d}V \geqslant \int_S T^{s}_{i}v_i\,\mathrm{d}S + \int_V \gamma_i v_i\,\mathrm{d}V \tag{9}$$

where v_i is the velocity vector in the plastic deformation field, T^{s}_{i} is the traction vector (with the integral of the horizontal component equal to reaction $\boldsymbol{T}$ at the pile base, Fig. 4) and γ_i is the unit weight vector (T^{s}_{i} and σ^{s}_{ij} are in equilibrium by definition of the admissible stress field). Hence the alternative statement of the static theorem is that the rate of internal work of the true stresses on the true (plastic) deformation is not less than the rate of work of surface traction and body forces of any statically admissible stress field on the true (plastic) deformation. As v_i is constant along the base of the pile (Fig. 4), one can use the theorem in equation (9) to interpret the horizontal component of the reaction $\int_l T^{s}_{i}\mathrm{d}S$ under one half of the pile base in any statically admissible stress field. Because the first term on the right-hand-side of equation (9) is negative for the problem in Fig. 4, the reaction calculated from an admissible stress field is an upper bound to the true reaction at failure.

One could argue now that a rough base does not allow the kinematic mechanism ('spreading') to take place. However, according to the static theorem of limit analysis, the horizontal reaction at the base calculated from the stress field with a perfectly rough interface is still the rigorous upper bound on the true reaction at failure. One can attempt to estimate its magnitude by finding a stress field in the sand pile that minimises this reaction. However, admissible stress states that produce a reaction larger than the minimum may promote arching. Hence the magnitude of the reaction ($\boldsymbol{T}$) will be considered a measure of the tendency to arching (the larger the $\boldsymbol{T}$, the larger the tendency to arching). This also implies that low roughness of the base would inhibit the tendency to arching.

Using limit analysis again, we shall demonstrate next that the deflection of the base promotes arching in sand piles. A collapse mechanism associated with the deflection of the base is illustrated schematically in Fig. 5. This is a realistic mechanism, although the linear distribution of the base displacement associated with the rotation of the two halves of the base support is, of course, an approximation. The deflection process occurs gradually during deposition of sand. Therefore the deflection considered here should be understood as an incipient deformation at some stage of the sand deposition process.

The true deformation pattern of the sand is not known, and it may occur with or without sliding along the base. However, the distribution of the base stress T_i that is closest to the true distribution at incipient failure maximises the right-hand side of inequality (9). T_i is the reaction to the sand weight, and it does negative work during the process: that is, the first term on the right-hand side of equation (9) is negative. Hence maximising the right-hand side of inequality (9) requires minimising the absolute rate of work $|\int T_i v_i \mathrm{d}S|$. Because the displacement of the base decreases linearly toward the toe, minimising $|\int T_i v_i \mathrm{d}S|$ requires that the resultant $\int T_i \mathrm{d}S$ under each half of the pile be located as far away from the centre of the pile as possible. This is equivalent to minimising the absolute value of the work rate of the reaction moment about point A, on incipient base rotation about A. This minimum occurs for a distribution with a stress depression at the centre. Hence the deflection of the base will induce arching.

Admissible stress fields will be constructed in order to search for arching stress distributions. Equation (7) will be used here as a means of constructing such stress distributions, with ϕ^* being variable. In searching through these fields one might expect to encounter 'safe' or below-failure states that could be associated with the characteristic feature of a 'dip' in the distribution of the stress at the base. Although admissible, radial stress fields are not necessarily the true ones; nevertheless, they give an indication of plausible stress distributions.

STRESS DISTRIBUTIONS UNDER SAND PILES

The boundary conditions required to solve for the stress field in a sand pile (Fig. 2) are:

$$\psi' = 0 \text{ (active) or } \psi' = \pi/2 \text{ (passive) on OA (or } \theta = \pi/2)$$

$$\chi = 0 \text{ on OB (or } \theta = \phi) \tag{10}$$

When the stress state is assumed to be at the limit everywhere, the outcome reduces to the classical Sokolovskii solution that follows from equations (5). Such a solution was found earlier by Booker (1969), and it was termed *incipient failure everywhere* (IFE) by Wittmer *et al.* (1997). Fully plastic solutions for both active and passive states were also referred to by Savage (1998). These are special cases ($\phi^* = \phi = \text{const}$) of the radial solution presented here with a more general closure in equation (6). The distributions of the base stress for fully plastic sand piles are shown in Fig. 6 (the stress is normalised by γH, H being the height of the pile). The active case does not have a dip at the symmetry axis; rather, the distribution reaches its maximum under the centre of the sand pile. This stress state was obtained without consideration of kinematics, but the incipient failure mechanism consistent with this stress field is a 'spreading' mode (Fig. 4), which ensures non-negative internal work everywhere in the pile.

For the passive case, the stress at the base has a local

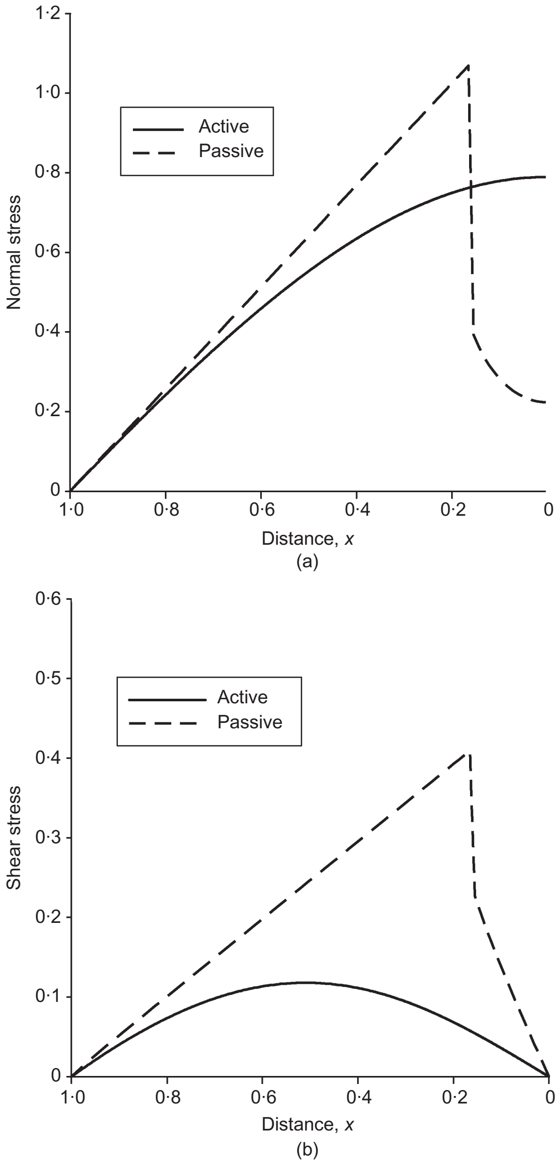

Fig. 6. Distribution of normalised stress under fully plastic sand heaps, ϕ = 30°: (a) normal stress; (b) shear stress

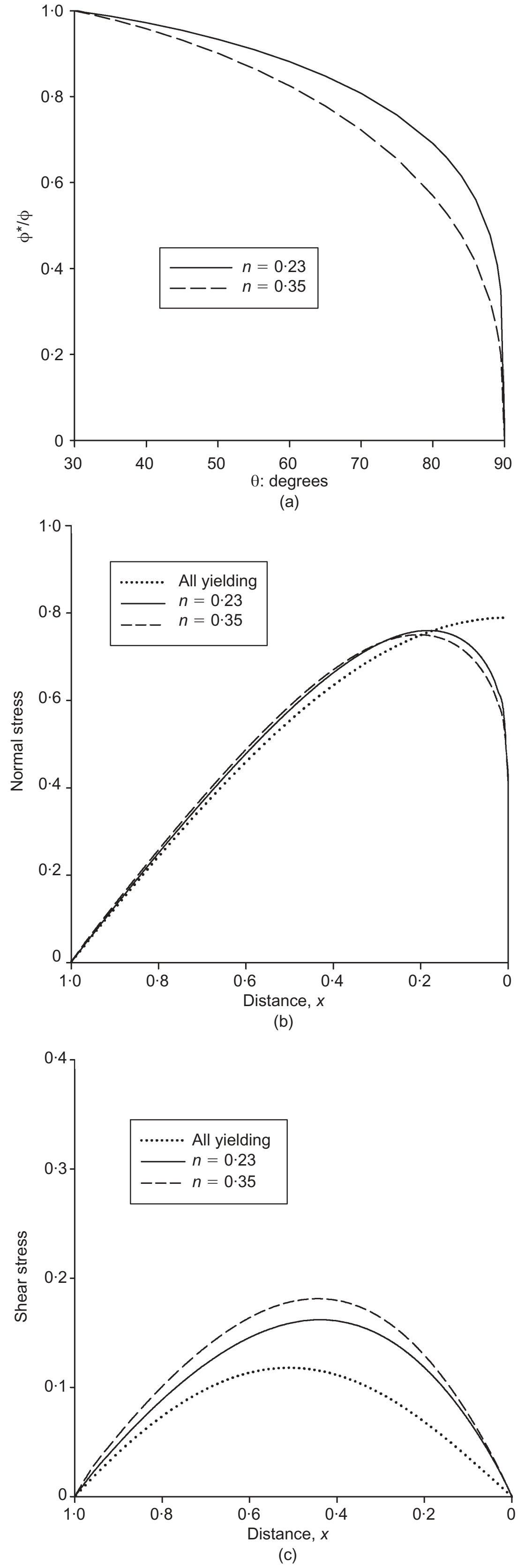

Fig. 7. Admissible normalised stress under sand pile with a cosine distribution of angle ϕ^* and ϕ = 30°: (a) distribution of ϕ^*; (b) normal stress; (c) shear stress

minimum at the centre, and a discontinuity at some distance away from the centre. This stress state is statically admissible, but the kinematics of an incipient failure is associated now with a deflection of the base (Fig. 5). This allows for extension along the symmetry axis, assuring non-negative internal work. (The passive state and a 'spreading' mechanism would lead to a negative rate of internal work in the neighbourhood of the symmetry axis, a case that must be rejected on grounds of thermodynamics.)

We now consider stress fields where not all of the material is at the yielding state. A special function ϕ^* in equation (6) is chosen here in the following form:

$$\phi^* = \phi \frac{\cos^n \theta}{\cos^n \phi} \tag{11}$$

This function has a value of ϕ at the pile surface (where $\theta = \phi$), and the value of zero at the symmetry plane (where $\theta = \pi/2$). Angle ϕ^* gradually drops down from ϕ at the slope to zero at the centre (Fig. 7(a)). The distribution of the base stress now has a distinct dip (Fig. 7(b)). The distributions of the normal stress at the base for both $n = 0{\cdot}23$ and $n = 0{\cdot}35$ are quite close to one another, but the total shear force (integral of stress distributions in Fig. 7(c)) is distinctly different for both cases, and, in both cases, larger than that for the all-in-failure active case. Invoking the static theorem of limit analysis, it was argued that the failure state in the sand pile is related to the stress distribution with the minimum integral of the shear reaction, and the magnitude of the shear integral at the base is indicative of the tendency to arching. Hence, out of the two 'dip' distributions shown in Fig. 7(b), arching is more distinct for the one with $n = 0{\cdot}35$.

A set of results for another function ϕ^*

$$\phi^* = \phi e^{-(\theta-\phi)^m} \quad (12)$$

is presented in Fig. 8. For large m function ϕ^* stays nearly equal to ϕ (fully mobilised strength) in a large region adjacent to the slope surface, and it drops down in the core of the pile to reach some fraction of ϕ at the centre. Again, the distribution of the base stress has a distinct depression at the pile centre. Although the larger dip is predicted by the distribution with $m = 10$, arching, as implied by the magnitude of the shear at the base, is more distinct when $m = 3$ (compare stress distributions in Fig. 8(c)). Assuming less than full mobilisation of the sand strength in the core of the pile appears to have caused the depression at the centre of the stress distribution.

For both distributions of ϕ^* considered above (equations (11) and (12)), the sand has been assumed to be uniform, but the stress was less than the yielding level in much of the pile. In order to draw some conclusions from these distributions, let us pretend for a moment that the sand is not uniform, but that the stress states are at their limits, with ϕ^* representing the angle of internal friction. The stronger sand now carries much of the load promoting arching over the weaker core of the pile. This phenomenon is well known in structural engineering, where stiffer structural members in statically indeterminate structures have a tendency to 'attract' more load than the softer members. Here the material is rigid-plastic, but the (pretended) non-uniform yield condition limits the stress level in the core of the pile. In uniform sand, one would expect such a behaviour when the base of the pile is subjected to differential settlement. Larger settlement at the centre of the base forms a 'softer' support, causing transfer of the load to the outer region of the pile through the arching mechanism. Experimental measurements by Trollope (1956) and finite element simulations by Savage (1998) showed a similar effect, where a stress depression would not occur over a rigid base, but was a distinct feature of the distribution over compliant bases (the larger the deflection at the centre, the larger the dip). The distribution of stress is likely to be affected also by the construction sequence (sand deposition process), and we discuss this in the next section.

INFLUENCE OF THE PILE CONSTRUCTION PROCESS ON ARCHING

The influence of the construction technique on the stress distribution under sand piles was recently considered by Vanel *et al.* (1999) in laboratory experiments. Vanel *et al.* measured the normal stresses under conical and wedge-shaped sand piles constructed by 'raining' and by dispensing the sand from a funnel (cone) or a slot source (prismatic heap). The piles constructed by the 'raining' technique did not exhibit a dip, whereas the ones formed using localised dispensing had a distinct local minimum in the normal stress distribution at the centre of the base. We will try to explain why, of the two techniques, the funnel deposition of sand is more predisposed to arching.

A schematic of construction is depicted in Fig. 9. The pile is built 'from the ground up' in a 'raining' process (Fig. 9(a)), but when the sand is dispensed from the local source, the pile 'grows' in a geometrically similar manner (Fig. 9(b)). At stage H_1 the shear stress at the base of the rained prismatic sand mass is very small, as the reaction to force **P** is distributed over a large area. During the second construction process, however, the reaction to **P** is distributed over a

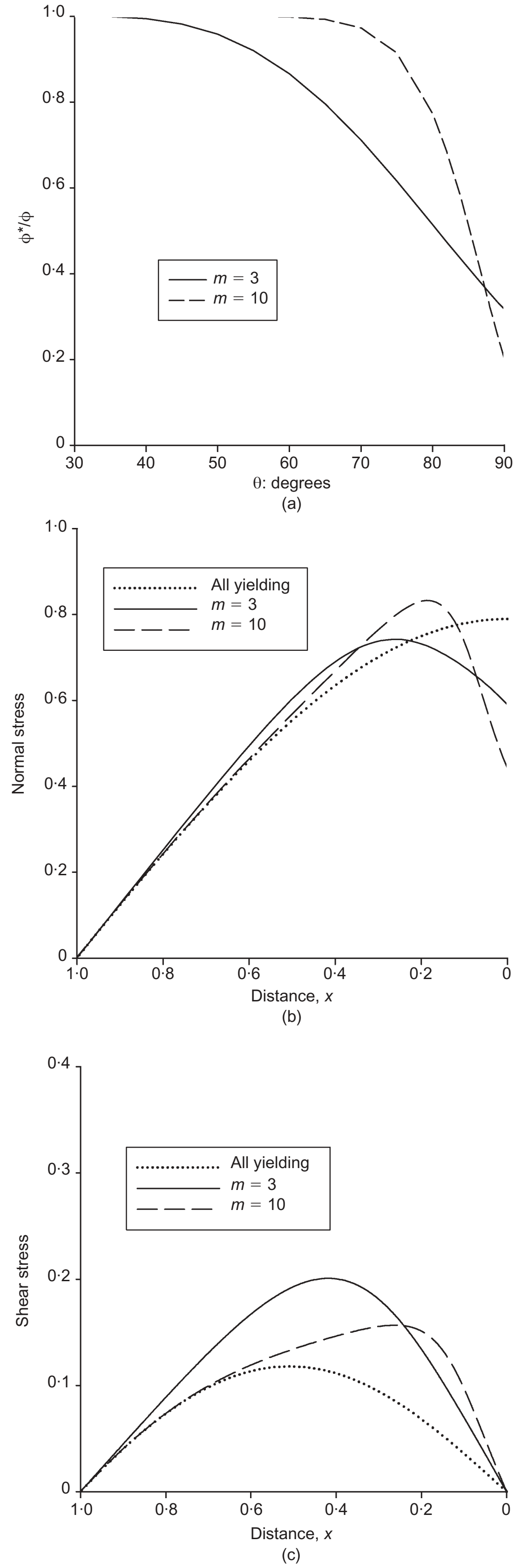

Fig. 8. Admissible normalised stress under sand pile with an exponential distribution of angle ϕ^* and $\phi = 30°$: (a) distribution of ϕ^*; (b) normal stress; (c) shear stress

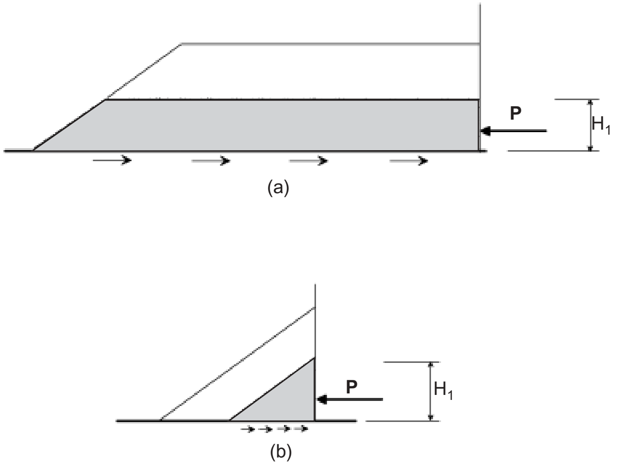

Fig. 9. Construction sequence: (a) raining process; (b) funnel deposition

much smaller area, giving rise to a larger intensity of shear (this is true even though forces **P** are not necessarily equal at stage H_1 for the two cases). During the raining process the normal stress gradually increases while the unbound sand is deposited at yielding stresses, and the shear stress steadily increases over the entire surface. During the sand deposition using the funnel method the sand slides down the slope, at every stage, and rests at the yielding stress state. The subsequent stages of construction cause the deviatoric stress in the inner core of the pile to increase at a pace slower than the isotropic stress, causing the stress state to drop below yielding. Consequently, arching occurs over the pile core at every stage of sand deposition. This conclusion follows directly from limit analysis: since the funnel construction process in Fig. 9(b) leads to shear stresses at the base larger than those for the process in Fig. 9(a), the former is more likely to exhibit arching (the larger the horizontal reaction **T**, the larger the tendency to arching).

We focus our attention on the funnel construction method. Once the grains are deposited in the pile, the geometric features of packing may cease to evolve, but the load is being continuously redistributed through the contacts between grains, giving rise to changing macroscopic stresses on material elements. This is illustrated in Fig. 10.

The diagram in Fig. 10 shows a sand pile scaled to a unit height. It is implied that the construction process is quasi-steady (geometrically similar or 'self-similar'), and the scaled stresses are independent of the construction stage. A change in location of a particle relative to the pile growing during the construction process can easily be traced on the unit diagram (much like location of particles relative to a wedge penetrating a half-space; Hill *et al.*, 1947). Consider particle P located at the sand pile tip at a stage when the height of the pile is *h*. If the construction process is advanced so that the height increases to 2*h*, the particle on the unit diagram travels from point P to P_l. In a similar manner, a particle at the toe of the slope will travel on the unit diagram from C to C_1, as will a particle travel from point E to E_1. The distribution of scaled stress within the sand pile remains constant in the unit diagram space, but the stresses on material elements evolve as these elements move from the outer regions of the pile toward the inner core. As the pile of sand 'grows', the true stresses on material elements increase. In addition, in radial stress fields, the principal directions of the stress tensor on a material element undergo rotation during construction (except for the element moving along PO, Fig. 10).

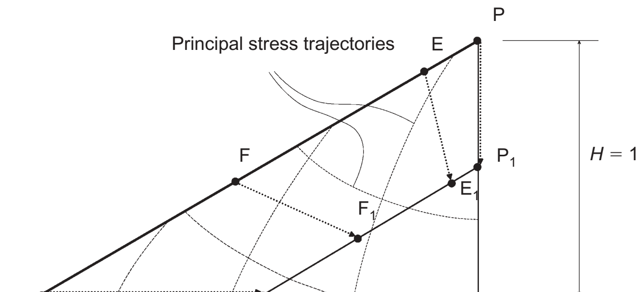

Fig. 10. Unit diagram

Based on this interpretation of the construction process we conclude that the (scaled) stresses acting on a material element are not 'frozen in' at the time of deposition, but that they evolve (including rotation of the principal directions of the stress tensor) as the element changes its relative location within the pile.

DISCUSSION OF AN EARLIER SOLUTION

The stress distributions presented in the previous section can be classified as statically admissible stress states within the framework of plasticity modelling. It was argued earlier that the sand pile in the failure state associated with a 'spreading' mechanism is characterised by a minimum shear force at the base, whereas an increasing base shear is indicative of arching and moving further away from the collapse state. The minimum of the shear stress integral under one half of the pile was found for the case where the sand was in the yielding state in the entire prism (Fig. 6). This solution was termed IFE (*incipient failure everywhere*; Wittmer *et al.*, 1997). Other solutions with different closures (Figs 7 and 8) induce larger shear at the base, and are associated with arching where the stress state in the outer shell of the pile is at or near yielding, while the inner core is in the elastic state.

Some of the solutions suggested earlier (Cantelaube *et al.*, 1998; Wittmer *et al.*, 1997; Didwania *et al.*, 1998) fall under the same category of admissible stress fields. These distributions have been referred to by Cates *et al.* (1998) as 'local rules of stress propagation' or 'local constitutive relations'. We shall avoid this terminology, as in solid mechanics the term *constitutive relations* refers to material property functions (involving stresses, strains, or their rates) independent of boundary conditions, whereas the 'stress propagation' is dependent on boundary conditions of the problem. In that sense, the function in equation (1) is the constitutive function, whereas the function in equation (6) is a closure function, but not a material (or constitutive) property.

Among admissible stress states, the one described by Wittmer *et al.* (1996) and termed *fixed principal axis* (FPA) was given prominence in an article by Watson (1996). The FPA distribution has been referred to in several papers (e.g. Wittmer *et al.*, 1997; Cates *et al.*, 1998; Savage, 1998). We discuss this model here, as applied to a 2D (prismatic) pile. The stress state adopted in the FPA 'model' is identical to that for infinite slopes proposed by Rankine (1857) in his seminal paper on *stability of loose earth*. Although Rankine considered slopes of arbitrary inclination, we discuss the case of a slope at the angle of repose, which is the special case used in the FPA model. This stress field is best interpreted using the Mohr stress circle (Fig. 11(b)). However, Rankine presented his paper to the Royal Society 10 years before Culmann published the concept that was later further developed and became widely known as the Mohr circle (Culmann, 1866). Rankine invoked the lemma that *if the pressure on a given plane at a given point be parallel to*

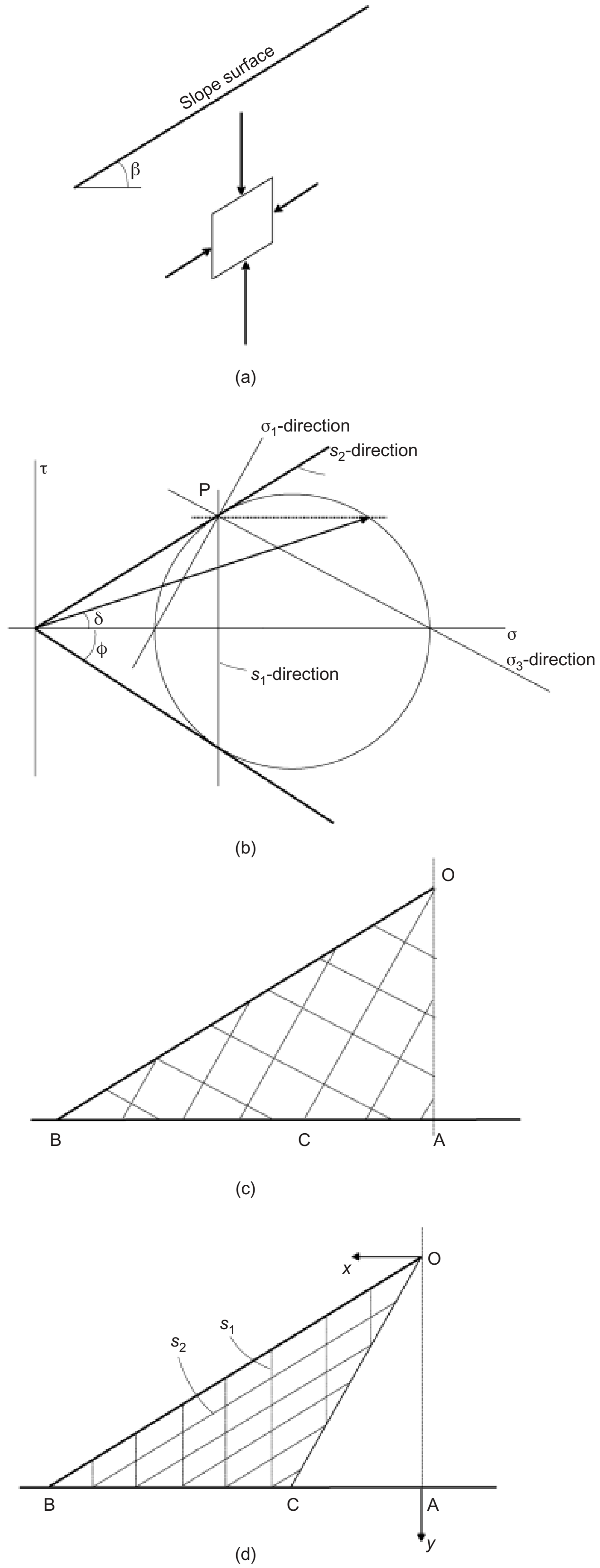

Fig. 11. Rankine infinite slope: (a) conjugate stresses; (b) Mohr circle; (c) uniform distribution of principal directions (FPA); (d) stress characteristics in the plastic zone

a second plane, the pressure on the second plane at the same point must be parallel to the first plane (conjugate planes, and conjugate stresses). The conjugate stresses are not orthogonal in general (Fig. 11(a)). The slope angle β is equal here to the angle of repose ϕ (we make no distinction between the angle of repose and the internal friction angle for a loose dry granular material).

The limit stress state is described by a set of hyperbolic-type differential equations that can be solved using the method of characteristics. The FPA model is a special case of the stress distribution where the principal stress directions are fixed, as indicated in Fig. 11(c). One family of characteristics (s_1) is vertical, and the other one (s_2) is parallel to the slope (Fig. 11(d)). The stress relations along the characteristics were given by Kötter (1903) (see also, e.g., Drescher, 1991), and the relation along s_1 is

$$\mathrm{d}p + 2p\tan\phi\,\mathrm{d}\psi = \gamma(\mathrm{d}y + \tan\phi\ \mathrm{d}x) \tag{13}$$

where ψ is the angle that the major principal stress makes with axis x (Fig. 2). For a straight-line characteristic ($\mathrm{d}\psi = 0$) in the vertical direction ($\mathrm{d}x = 0$), equation (13) reduces to

$$\mathrm{d}p = \gamma\mathrm{d}y \tag{14}$$

Hence the stresses in the slope increase linearly with the distance from the slope surface (p is the in-plane mean stress, and γ is the unit weight of the sand). Such a stress state, originally conceived for infinite slopes, can be adopted for a limited region without loss of admissibility, provided the boundaries of the region are rough.

This limit stress state was used by Wittmer *et al.* (1997) to describe the stress field in region OBC of the sand pile (Fig. 11(d)). The principal directions were assumed by Wittmer *et al.* to be fixed to the same angle in the remaining part of the field (OCA). Thus equilibrium requires that the stress state becomes isotropic at the symmetry axis. The stress field in region OCA is linear in p, and it can be constructed directly from the equations of equilibrium. The line at which the two regions are joined (OC) is a weak discontinuity in p (discontinuity in the derivative), whereas the symmetry plane contains singularity in ψ. The distribution of the normal and tangential base stress for the FPA model is illustrated in Fig. 12 (identical to that in Wittmer *et al.*, 1997). The FPA model predicts quite a distinct stress depression under the centre of a prismatic sand heap.

If the FPA model were to be illustrated in the unit diagram (Fig. 10), the material elements would gradually move from the incipient failure region BCO (Fig. 11(d)) into elastic area ACO (in the physical space, of course, it is the interface OC between the two regions that propagates outwards during the construction process).

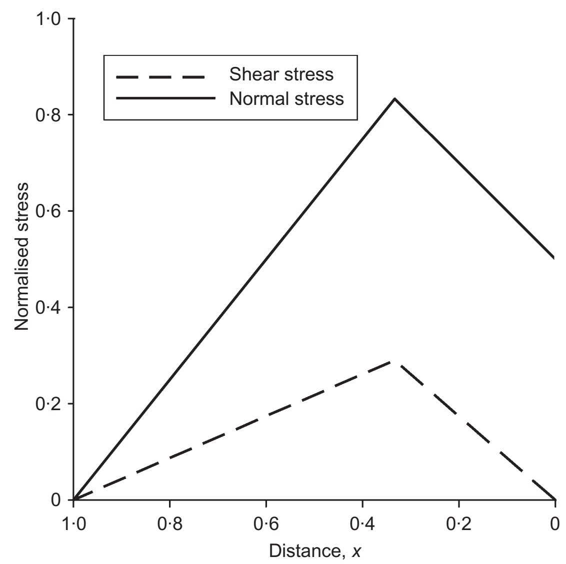

Fig. 12. Stress distribution under a pile of sand with uniform principal directions (as in Wittmer *et al.*, 1997)

The following comment relates to the base traction in the FPA model. Because the stress state satisfies the Mohr–Coulomb yield condition in region OBC (Fig. 11(d)), and the principal directions are constant, the inclination of traction along BC (Fig. 11(c)) must be constant (i.e. it is independent of the stress level). This inclination is shown as δ in Fig. 11(b). The measurements under conical piles clearly indicate that the traction inclination (or mobilisation of friction at the base interface) is not constant (see Figs 8 and 9 of Smid & Novosad, 1981). The sand in the 3D FPA stress field in conical piles suggested by Wittmer *et al.* (1997) is in incipient failure only at the outer ridges of the conical pile, but it is quite close to failure for more than half of the base radius (Fig. 8 of the reference by Wittmer *et al.*, 1997). Although this stress state is statically admissible, the measurements by Smid & Novosad (1981) do not confirm the validity of the *fixed principal axis* assumption for conical sand piles, as it directly contradicts the measured distribution of traction inclination at the base of conical piles. However, the measurements of normal and shear stresses for prismatic piles reported by Wiesner (2000) indicate that the direction of the base stress may be approximately constant under a portion of the heap.

The FPA model predicts a dip in the base stress distribution, because fixing the principal axes necessarily requires that the stress state become isotropic at the symmetry plane. This causes the stress state in the centre core of the pile to drop below yielding, and it promotes arching in the outer region (much like a strong material arching over a weaker 'inner core'). The FPA model belongs to the same category of admissible stress fields as other stress fields presented in this paper.

It may be of some interest that the stress state identical to that in region OBC (Fig. 11(d)) was postulated by Jaky (1944) in a stress analysis in a wedge-shaped sand heap that led to derivation of his well-known at-rest coefficient of earth pressure. Jaky assumed that the shear stress in region OCA drops down to zero at the symmetry plane according to a parabolic function. This led to a peculiar, though statically admissible, stress field with a 'dip' in the base stress at some distance from the symmetry plane, and with a maximum below the apex of the sand pile. Although somewhat unrealistic, this stress distribution gave rise to the Jaky K_0 coefficient that describes the principal stress ratio for loose deposits at rest.

REMARKS ON ARCHING IN GRANULAR MATERIALS

Arching is not a material property, but a response of a structure to the loading process. In that sense it is similar to another phenomenon: shakedown, or adaptation (Melan, 1938; Koiter, 1960). However, there are obvious differences, as shakedown is associated with cyclic loading, whereas arching is typically induced in a monotonic loading process. Nevertheless, it is useful to include shakedown as a class of structural response that has some similarities to arching, at least in the sense of structural adaptation. Shakedown is a phenomenon where a structure subjected to cyclic loading responds in an elasto-plastic manner in the first, or first few cycles, but the response becomes elastic in the subsequent cycles (adaptation). This is possible if a residual (self-equilibrated and time-independent) stress field is induced in the structure during the first cycles such that, in superposition with the loading in subsequent cycles, the stress state does not exceed the yielding limit anywhere in the structure. Essentially, this is the Melan shakedown theorem (Melan, 1938). The structure must, of course, be elasto-plastic for the residual stress state to be induced. Occurrence of arching requires that a statically admissible stress field be found that supports arching. This can be viewed as a special application case of the Melan theorem with zero residual stress (or the static theorem of limit analysis).

The theorems of limit analysis can be rephrased to assess the likelihood of arching occurrence. The static theorem then indicates that *arching may occur and failure will not take place if a statically admissible stress field supporting arching can be found*. Of course, proving that an admissible arching stress field exists does not exclude 'non-arching' stress fields from occurring. The second theorem then states that *arching will not occur and collapse is imminent if a kinematically admissible mechanism can be found where the work rate of external loads exceeds the rate of internal work.*

Arching in homogeneous and isotropic granular media is promoted in stress fields with varied mobilisation of strength. Granular materials governed by the Mohr–Coulomb yield condition cannot resist uniaxial compression, so, even when arching occurs, the arching regions must be supported, in some manner, by below-yield regions. Hence arching within granular media is not clearly manifested by the geometry of the structure, but it is associated with some special properties of the stress field. In the case of sand piles, arching is linked directly to an increase in the shear reaction at the base. In materials with cohesion, unsupported arches can form, much like those over a cavity in moist sand or in clay. When the size of grains becomes large compared with the size of the problem (a blocky system), a stable geometric alignment of the grains is possible, giving rise to an arch formation. This, however, is no longer a continuum problem.

FINAL REMARKS

Construction of statically admissible stress fields is a standard method in limit analysis for finding a lower bound to an active limit load, or an upper bound to a reaction. Here, the method was modified to indicate the tendency of the stress distribution within the structure to arching. A pile of sand reaches the state closest to failure (consistent with a 'spreading' mechanism) when the horizontal component of the reaction under a symmetric half of the pile reaches its minimum. Of different radial stress fields considered, the one with fully plastic stress appeared to yield the minimum of the shear force at the base. Once the stress fields were modified to include below-yielding regions, the horizontal reaction increased, and arching stress fields were found, distinguished by a characteristic 'dip' in the distribution of the normal stress at the base.

At least two reasons causing arching in sand piles can be indicated: a construction sequence, and the deflection of the base. Both have been previously pointed out by others as factors causing arching. Comparison of the raining and the funnel construction methods reveals that the latter produces larger mobilisation of the shear at the base: thus, according to the limit analysis considerations in this paper, it is more prone to arching. The static theorem of limit analysis implies also that the fully plastic active solution cannot exhibit arching (because it minimises the shear at the base of the sand pile), whereas the passive one must cause arching (because the distribution with the stress 'dip' minimises the moment of the reaction stress with respect to the heap's toe).

The unit diagram presented in this paper makes it easier to interpret the evolution of the stress state in the material during a funnel deposition process. The stress state in the material elements undergoes evolution where the limit stress immediately after deposition drops gradually below the level of yielding during a continued construction process. Therefore the scaled stress state in material elements is not 'frozen in' at the instant of deposition, in contrast to the assumption embedded in the FPA model of Wittmer *et al.*

(1996). The magnitude of the true stress components increases, of course, during the 'growth' of a sand heap, in addition to the evolution of principal directions and a relative drop in comparison with the yielding stress state.

Terzaghi (1943) defined arching as the 'transfer of pressure from a yielding mass of soil onto adjoining stationary parts'. This description holds true when applied to the supporting structure, but arching observed within sand piles appears to defy this definition, where the sand in the yielding state appears to arch over the centre core where the stress level is well below yielding.

ACKNOWLEDGEMENTS

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APPENDIX 1

To find the governing equations in the radial stress field with a varying mobilisation of strength, ϕ in equations (2) is replaced by angle $\phi^*(\theta)$, and equations (2) are substituted into equilibrium equations (3) to yield

$$\sin\phi^*\sin 2\psi'\left(\frac{d\chi}{d\theta}\right)+2\chi\sin\phi^*\cos 2\psi'\left(1+\frac{d\psi'}{d\theta}\right)$$
$$+\chi\left(1+\sin\phi^*\cos 2\psi'\right)=\sin\theta-\chi\left(\frac{d\phi^*}{d\theta}\right)\cos\phi^*\sin 2\psi'$$
$$\left(1-\sin\phi^*\cos 2\psi'\right)\left(\frac{d\psi'}{d\theta}\right)+2\chi\sin\phi^*\sin 2\psi'\left(1+\frac{d\psi'}{d\theta}\right)$$
$$+\chi\sin\phi^*\sin 2\psi'=\cos\theta+\chi\left(\frac{d\phi^*}{d\theta}\right)\cos\phi^*\cos 2\psi' \qquad (15)$$

The derivatives of functions $\chi(\theta)$ and $\psi'(\theta)$ now can be found by solving the set in equation (15), and the solution to these derivatives is given in equations (7).

NOTATION

H height of the sand prism
p in-plane mean stress
T_i stress vector
v_i velocity vector
y unit weight
$\dot{\varepsilon}$ strain rate tensor
θ polar coordinate
σ_{ij} stress tensor
ϕ internal friction angle
χ stress function
ψ angle of inclination of the major principal stress to axis x
ψ' angle of inclination of the major principal stress to radius r

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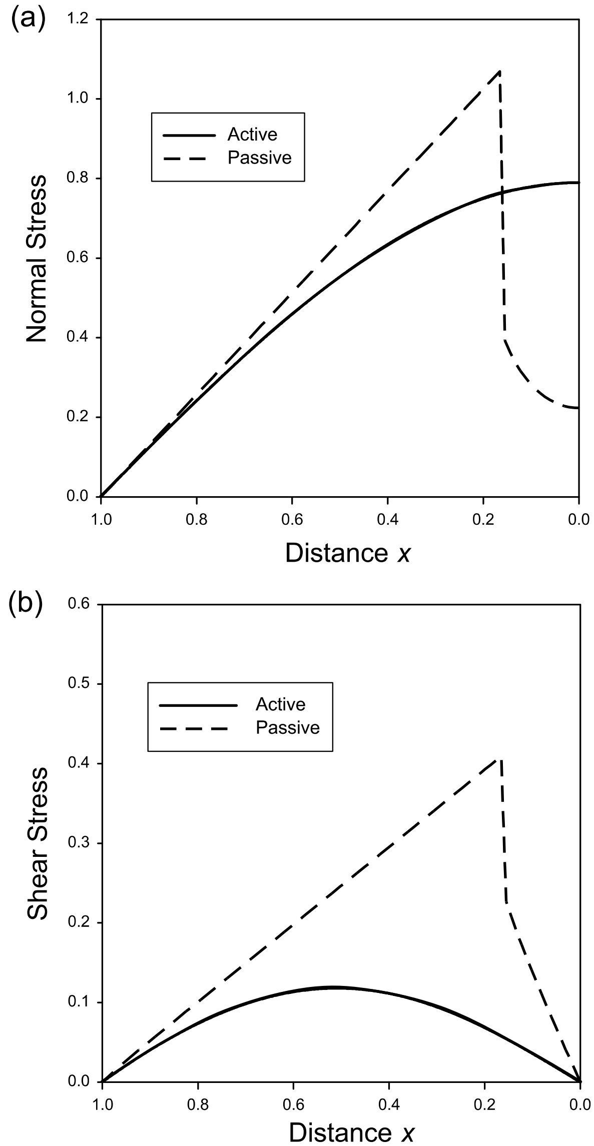

Figure 6 (Michalowski & Park)

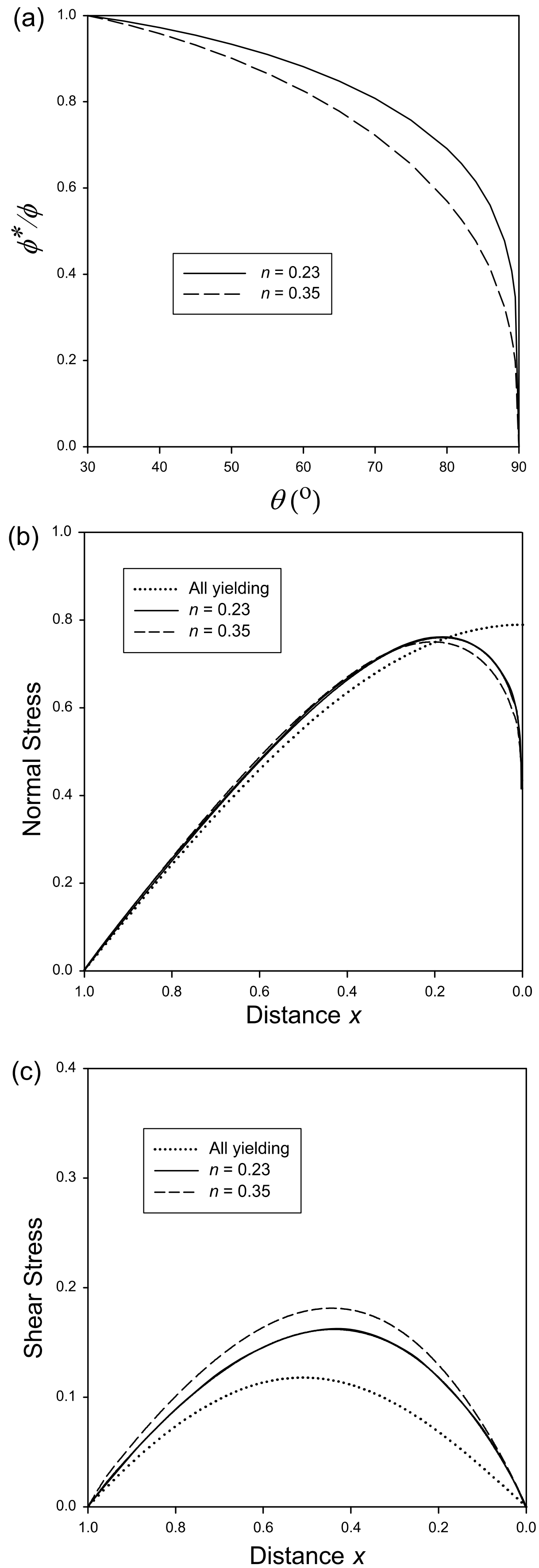

Figure 7 (Michalowski & Park)

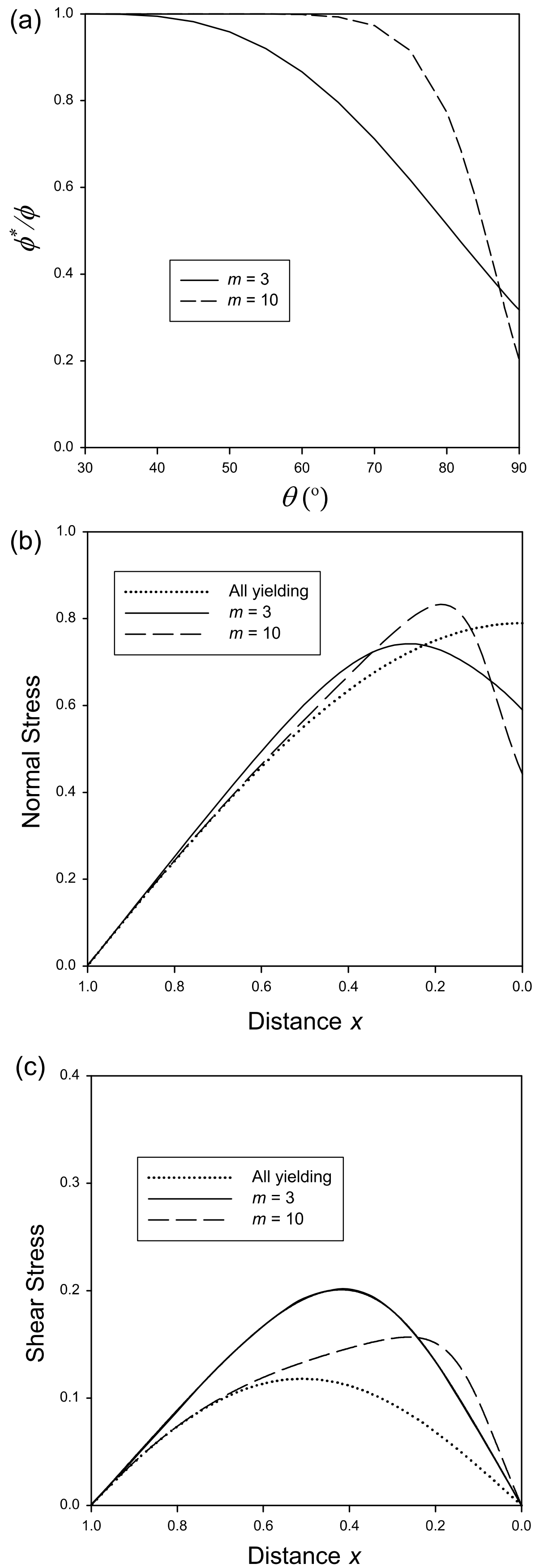

Figure 8 (Michalowski & Park)